Lapl+probafv.tex, 2008.05.01.

$$\boxed{\varphi'' - 3\varphi' - 10\varphi = x^2 e^{-2x}, \quad \varphi(0) = 7,\ \varphi'(0) = 2}$$

PRÓBAFÜGGVÉNNYEL

A karakterisztikus egyenlet:

$\lambda^2 - 3\lambda - 10 = 0, \quad \lambda_1 = -2\ , \quad \lambda_2 = 5\ , \quad m = 1$ ($\gamma = -2$ multiplicitása)

A próbafüggvény:

$$\hat{\varphi}(x) = xe^{-2x}\left(ax^2 + bx + c\right) = e^{-2x}\left(ax^3 + bx^2 + cx\right)\ ,$$

$$\frac{d}{dx}\hat{\varphi}(x) = -2e^{-2x}\left(ax^3 + bx^2 + cx\right) + e^{-2x}\left(3ax^2 + 2bx + c\right) =$$
$$= e^{-2x}\left(-2ax^3 + (3a - 2b)\,x^2 + (2b - 2c)\,x + c\right)$$

$$\frac{d^2}{dx^2}\hat{\varphi}(x) = -2e^{-2x}\left(-2ax^3 + (3a - 2b)\,x^2 + (2b - 2c)\,x + c\right) +$$
$$+e^{-2x}\left(-6ax^2 + 2\,(3a - 2b)\,x + 2b - 2c\right) =$$
$$= 2e^{-2x}\left(2ax^3 + (2b - 6a)\,x^2 + (-4b + 2c + 3a)\,x - 2c + b\right)\ .$$

Visszahelyettesítve az $\quad \hat{\varphi}'' - 3\hat{\varphi}' - 10\hat{\varphi} = x^2 e^{-2x} \quad (\forall x \in \mathbb{R})$ egyenletbe:

$$2e^{-2x}\left(2ax^3 + (2b - 6a)\,x^2 + (-4b + 2c + 3a)\,x - 2c + b\right) -$$
$$-3e^{-2x}\left(-2ax^3 + (3a - 2b)\,x^2 + (2b - 2c)\,x + c\right) -$$
$$-10e^{-2x}\left(ax^3 + bx^2 + cx\right) \quad = \quad x^2 e^{-2x} \qquad / : e^{-2x}$$

$$(-21a - 1)\,x^2 + (-14b + 6a)\,x + 2b - 7c = 0 \qquad (\forall x \in \mathbb{R})$$

Az együtthatók összehasonlítása:

$$\left.\begin{array}{r} -21a - 1 = 0 \\ -14b + 6a = 0 \\ 2b - 7c = 0 \end{array}\right\}, \quad \text{ahonnan } a = -\tfrac{1}{21}\ , \quad c = -\tfrac{2}{343}\ , \quad b = -\tfrac{1}{49}\ .$$

Tehát az inhomogén *egyik* megoldása:

$$\hat{\varphi}(x) = xe^{-2x}\left(-\frac{1}{21}x^2 - \frac{1}{49}x - \frac{2}{343}\right) \quad .$$

Mivel a homogén egyenlet általános megoldása

$$\varphi_{\text{hom}}^{ált}(x) = c_1 e^{-2x} + c_2 e^{5x}$$

ezért az inhomogén *összes* megoldása:

$$\begin{aligned}\varphi_{in\,\mathrm{hom}}^{ált}(x) &= \varphi_{\mathrm{hom}}^{ált}(x) + \hat{\varphi}(x) = \\ &= c_1e^{-2x} + c_2e^{5x} + xe^{-2x}\left(-\frac{1}{21}x^2 - \frac{1}{49}x - \frac{2}{343}\right) .\end{aligned}$$

A kezdetiérték probléma:

$\varphi(0) = 7 = c_1e^0 + c_2e^0 + 0 = c_1 + c_2$,

$$\begin{aligned}\varphi'(x) &= \tfrac{d}{dx}\left(c_1e^{-2x} + c_2e^{5x} + e^{-2x}\left(-\tfrac{1}{21}x^3 - \tfrac{1}{49}x^2 - \tfrac{2}{343}x\right)\right) = \\ &= -2c_1e^{-2x} + 5c_2e^{5x} - 2e^{-2x}\left(-\tfrac{1}{21}x^3 - \tfrac{1}{49}x^2 - \tfrac{2}{343}x\right) + e^{-2x}\left(-\tfrac{3}{21}x^2 - \tfrac{2}{49}x - \tfrac{2}{343}\right)\end{aligned}$$

$\varphi'(0) = 2 = -2c_1e^0 + 5c_2e^0 - 2e^0 \cdot 0 + e^0\left(0 - \tfrac{2}{343}\right) = -2c_1 + 5c_2 - \tfrac{2}{343}$,

vagyis az egyenletrendszer:

$$\begin{cases} 7 = c_1 + c_2 \\ 2 = -2c_1 + 5c_2 - \tfrac{2}{343} \end{cases} \quad \text{és megoldása:} \quad \begin{cases} c_1 = \tfrac{11\,317}{2401} \approx 4.7135 \\ c_2 = \tfrac{5490}{2401} \approx 2.2865 \end{cases} .$$

A feladat megoldása:

$$\begin{aligned}\varphi_{kép}(x) &= \tfrac{11\,317}{2401}e^{-2x} + \tfrac{785}{343}e^{5x} + xe^{-2x}\left(-\tfrac{1}{21}x^2 - \tfrac{1}{49}x - \tfrac{2}{343}\right) \\ &= \tfrac{5490}{2401}e^{5x} + e^{-2x}\left(-\tfrac{1}{21}x^3 - \tfrac{1}{49}x^2 - \tfrac{2}{343}x + \tfrac{11\,317}{2401}\right) \\ &\approx 2.2865e^{5x} + e^{-2x}\left(-0.0476x^3 - 0.0204x^2 - 0.0058x + 4.7135\right) . \qquad \square\end{aligned}$$

LAPLACE TRANSZFORMÁCIÓVAL:

$$\boxed{\varphi'' - 3\varphi' - 10\varphi = x^2 e^{-2x}, \quad \varphi(0) = 7,\ \varphi'(0) = 2}$$

$$\mathcal{L}\left(\varphi'' - 3\varphi' - 10\varphi\right) = \mathcal{L}\left(x^2 e^{-2x}\right)\ , \quad \mathcal{L}\left(\varphi\right) = \Phi$$

$$\mathcal{L}\left(\varphi''\right) - 3\mathcal{L}\left(\varphi'\right) - 10\mathcal{L}\left(\varphi\right) = \mathcal{L}\left(x^2 e^{-2x}\right)$$

$$\left[s\left(s\Phi - 7\right) - 2\right] - 3\left(s\Phi - 7\right) - 10\Phi = \tfrac{2}{(s+2)^3}$$

$$\left(s^2 - 3s - 10\right)\Phi - 7s + 19 = \tfrac{2}{(s+2)^3}$$

$$\left(s^2 - 3s - 10\right)\Phi = \tfrac{2}{(s+2)^3} + 7s - 19$$

$$\begin{aligned}\Phi &= \frac{\frac{2}{(s+2)^3} + 7s - 19}{\left(s^2 - 3s - 10\right)} = \frac{2 + (7s - 19)(s+2)^3}{\left(s^2 - 3s - 10\right)(s+2)^3} = \\ &= \frac{7s^4 + 23s^3 - 30s^2 - 172s - 150}{(s+2)^4 (s-5)}\end{aligned}$$

mert $\quad s^2 - 3s - 10 = 0 \Longrightarrow s_1 = -2\ ,\ s_2 = 5$.

Parciális törtekre bontás:

$$\frac{7s^4 + 23s^3 - 30s^2 - 172s - 150}{(s+2)^4 (s-5)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3} + \frac{D}{(s+2)^4} + \frac{E}{s-5}$$

Közös nevezőre hozva

$$7s^4 + 23s^3 - 30s^2 - 172s - 150 =$$
$$= A(s+2)^3(s-5) + B(s+2)^2(s-5) + C(s+2)(s-5) + D(s-5) + E(s+2)^4$$

s helyébe néhány értéket behelyettesítve:

$s = 5 \quad \Longrightarrow \quad 5490 = E \cdot 7^4 \quad \Longrightarrow \quad E = \frac{5490}{2401} \approx 2.2865$,

$s = -2 \quad \Longrightarrow \quad 2 = D \cdot (-7) \quad \Longrightarrow \quad D = \frac{-2}{7} \approx -0.2857$,

$$\left\{\begin{array}{l} s = 0 \\ \\ s = 1 \\ \\ s = -1 \end{array}\right. \quad \Longrightarrow \quad \left\{\begin{array}{l} -150 = -40A - 20B - 10C + \frac{10}{7} + 2^4 \cdot \frac{5490}{2401} \\ \\ -322 = -108A - 36B - 12C + \frac{8}{7} + 3^4 \cdot \frac{5490}{2401} \\ \\ -24 = -6A - 6B - 6C + \frac{12}{7} + \frac{5490}{2401} \end{array}\right. ,$$

az egyenletrendszer megoldása:

$A = \frac{11\,317}{2401} \approx 4.7135\ , \quad B = -\frac{2}{343} \approx -0.0058\ , \quad C = -\frac{2}{49} \approx -0.0408$.

Tehát

$$\Phi = \frac{11\,317}{2401} \cdot \frac{1}{s+2} - \frac{2}{343} \cdot \frac{1}{(s+2)^2} - \frac{2}{49} \cdot \frac{1}{(s+2)^3} - \frac{2}{7} \cdot \frac{1}{(s+2)^4} + \frac{5490}{2401} \cdot \frac{1}{s-5}$$

vagyis

$$\begin{aligned}
\varphi &= \mathcal{L}^{-1}(\Phi) \\
&= \tfrac{11\,317}{2401}\mathcal{L}^{-1}\left(\tfrac{1}{s+2}\right) - \tfrac{2}{343}\mathcal{L}^{-1}\left(\tfrac{1}{(s+2)^2}\right) - \tfrac{2}{49}\mathcal{L}^{-1}\left(\tfrac{1}{(s+2)^3}\right) \\
&\quad -\tfrac{2}{7}\mathcal{L}^{-1}\left(\tfrac{1}{(s+2)^4}\right) + \tfrac{5490}{2401}\mathcal{L}^{-1}\left(\tfrac{1}{s-5}\right) = \\
&= \tfrac{11\,317}{2401}e^{-2t} - \tfrac{2}{343}te^{-2t} - \tfrac{2}{49} \cdot \tfrac{1}{2}t^2e^{-2t} - \tfrac{2}{7} \cdot \tfrac{1}{6}t^3e^{-2t} + \tfrac{5490}{2401}e^{5t} = \\
&= \tfrac{5490}{2401}e^{5t} + e^{-2t}\left(-\tfrac{1}{21}t^3 - \tfrac{2}{98}t^2 - \tfrac{2}{343}t + \tfrac{11\,317}{2401}\right) \approx \\
&\approx 2.2865e^{5t} + e^{-2t}\left(-0.0476t^3 - 0.0204t^2 - 0.0058t + 4.7135\right). \quad \square
\end{aligned}$$