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First order partial functional differential inequalities

Elżbieta Puźniakowska - Gałuch

Institute of Mathematics, University of Gdańsk, Wit Stwosz Street 57, 80-952 Gdańsk, Poland

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Abstract. Theorems on functional differential inequalities generated by initial boundary value problems are presented. A comparison theorem given in the paper shows that classical solutions of nonlinear functional inequalities can be estimated by maximal solutions of suitable initial problems for ordinary functional differential equations. Uniqueness criteria for mixed problems are obtained as applications of the comparison result.

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1. Introduction

Differential inequalities find numerous applications in the theory of first order partial differential or functional differential equations. The basic examples of such applications are estimates of solutions of differential equations, estimates of the domain of the existence of solutions, criteria of uniqueness and estimates of the error of approximate solutions. Discrete version of differential inequalities, the so called difference or functional difference inequalities, are frequently used to prove convergence of approximation methods. The classical theory of first order partial differential inequalities has been described extensively in the monographs [6], [7], [9]. Hyperbolic functional inequalities corresponding to initial or initial boundary value problems have been studied in the papers [1] - [4] and the monograph [5].

The aim of the paper is to obtain general comparison theorems on functional differential inequalities with initial boundary conditions. There are two different types of results on the comparison theorems of functional differential inequalities

E-mail address: epuzniak@inf.univ.gda.pl (E. Puźniakowska - Gałuch)

in the literature. The first type of theorems allow to estimate a function of several variables, while the second one, the so called comparison theorems, give estimates for functions of one variable. In Section 2 we deal with the first kind of results, while the comparison theorem is presented in Section 3.

We formulate our functional differential problems. For any metric spaces X, Y we denote by $C(X, Y)$ the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

Let $a > 0$, $b_0 \in \mathbb{R}_+$, $\mathbb{R}_+ = [0, +\infty)$, be fixed and suppose that the functions $\alpha, \beta \colon [0, a) \to \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, satisfy the conditions: α and β are of class C^1 on $[0, a)$ and $\alpha(t) < \beta(t)$ for $t \in [0, a)$. Let E be a generalized Haar pyramid

$$E = \{(t, x) \in \mathbb{R}^{1+n} : t \in (0, a), x \in [\alpha(t), \beta(t)]\} \tag{1.1}$$

and $E_0 = [-b_0, 0] \times [b_*, b^*]$, $b_* = \alpha(0)$, $b^* = \beta(0)$. For $(t, x) \in E$ we define the set $D[t, x]$ as follows

$$D[t, x] = \{(\tau, y) \in \mathbb{R}^{1+n} : \tau \leq 0, (t + \tau, x + y) \in E_0 \cup E\}.$$

It is clear that $D[t, x] = D_0[t, x] \cup D_*[t, x]$, where

$$D_0[t, x] = [-b_0 - t, -t] \times [b_*, b^*],$$

$$D_*[t, x] = \{(\tau, y) \in \mathbb{R}^{1+n} : -t \leq \tau \leq 0, -x + \alpha(t + \tau) \leq y \leq -x + \beta(t + \tau)\}.$$

Write $B = [-b_0 - a, 0] \times [c - d, d - c]$, where $c = (c_1, \ldots, c_n)$, $d = (d_1, \ldots, d_n)$ and

$$c_i = \inf\{\alpha_i(t) : t \in [0, a)\},\ d_i = \sup\{\beta_i(t) : t \in [0, a)\},\ i = 1, \ldots, n.$$

Then we have $D[t, x] \subset B$ for $(t, x) \in E$. For a function $z \colon E_0 \cup E \to \mathbb{R}$ and for a point $(t, x) \in E$, we define a function $z_{(t,x)} \colon D[t, x] \to \mathbb{R}$ as follows:

$$z_{(t,x)}(\tau, y) = z(t + \tau, x + y),\ (\tau, y) \in D[t, x].$$

Then $z_{(t,x)}$ is the restriction of z to the set $(E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n)$ and this restriction is shifted to the set $D[t, x]$. Write $\partial_0 E = \partial E \cap ((0, a) \times \mathbb{R}^n)$, where ∂E is the boundary of E. Suppose that the sets Δ_0, $\Delta \subset \partial E$ satisfy the conditions:

$$\Delta_0 \cup \Delta = \partial_0 E,\ \Delta_0 \cap \Delta = \emptyset.$$

The cases $\Delta_0 = \emptyset$ or $\Delta = \emptyset$ are not excluded. We will assume that differential functional equations or inequalities are satisfied on $Int\, E \cup \Delta$ and initial boundary conditions hold on $E_0 \cup \Delta_0$. Suppose that $\kappa \colon [0, a) \to \mathbb{R}$ and $\psi_* = (\psi_1, \ldots, \psi_n) \colon E \to \mathbb{R}^n$ are given functions. The requirements on κ and ψ_* are that $0 \leq \kappa(t) \leq t$ and $(\kappa(t), \psi_*(t, x)) \in E_0 \cup E$ for $(t, x) \in E$. Set $\psi(t, x) = (\kappa(t), \psi_*(t, x))$ for $(t, x) \in E$. Write $\Omega = E \times \mathbb{R} \times C(B, \mathbb{R}) \times \mathbb{R}^n$ and suppose that $f \colon \Omega \to \mathbb{R}$ and $\varphi \colon E_0 \cup \Delta_0 \to \mathbb{R}$ are given functions. We will say that the function f satisfies the condition (V) if for each $(t, x, p, q) \in E \times \mathbb{R} \times \mathbb{R}^n$ and for $w, \tilde{w} \in C(B, \mathbb{R})$ such that $w(\tau, y) = \tilde{w}(\tau, y)$ for

$(\tau,y)\in D[\psi(t,x)]$ we have $f(t,x,p,w,q)=f(t,x,p,\tilde{w},q)$. Note that the condition (V) means that the value of f at the point $(t,x,p,w,q)\in\Omega$ depends on (t,x,p,q) and on the restriction of w to the set $D[\psi(t,x)]$ only.

Let z be the unknown function of the variables (t,x) with $x=(x_1,\dots,x_n)$. We consider the following problem consisting of the differential functional equation

$$\partial_t z(t,x)=f(t,x,z(t,x),z_{\psi(t,x)},\partial_x z(t,x)) \tag{1.2}$$

with initial boundary condition

$$z(t,x)=\varphi(t,x)\quad \text{on } E_0\cup\Delta_0, \tag{1.3}$$

where $\partial_x z=(\partial_{x_1}z,\dots,\partial_{x_n}z)$. We assume that f satisfies the condition (V) and we consider the classical solutions of (1.2), (1.3).

Now we give examples of equations which can be obtained from (1.2) by specializing the operator f.

Example 1.1. *Suppose that $\tilde{f}\colon E\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ is a given function. Set $f(t,x,p,w,q)=\tilde{f}(t,x,p,w(0,\theta),q)$ where $\theta=(0,\dots,0)\in\mathbb{R}^n$. Then (1.2) becomes the differential equation with deviated variables*

$$\partial_t z(t,x)=\tilde{f}(t,x,z(t,x),z(\psi(t,x)),\partial_x z(t,x)). \tag{1.4}$$

Example 1.2. *Suppose that $\psi(t,x)=(t,x)$, $(t,x)\in E$. For the above $\tilde{f}$ we put*

$$f(t,x,p,w,q)=\tilde{f}(t,x,p,\int_{D[t,x]}w(\tau,y)d\tau dy,q).$$

Then (1.2) is equivalent to the differential integral equation

$$\partial_t z(t,x)=\tilde{f}(t,x,z(t,x),\int_{D[t,x]}z_{(t,x)}(\tau,y)d\tau dy,\partial_x z(t,x)). \tag{1.5}$$

It is clear that more complicated equations with deviated variables and differential integral equations can be obtained from (1.2).

Sufficient conditions for the existence of solutions to (1.2), (1.3) can be found in [8].

2. Functional differential inequalities

For each $(t,x)\in E$ there exist sets of integers $I_0[t,x],I_-[t,x],I_+[t,x]$ such that $I_-[t,x]\cap I_+[t,x]=\emptyset$ and $I_0[t,x]\cup I_-[t,x]\cup I_+[t,x]=\{1,2,\dots,n\}$ and

$$\alpha_i(t)<x_i<\beta_i(t)\quad\text{for } i\in I_0[t,x],$$

$$x_i=\alpha_i(t)\quad\text{for } i\in I_-[t,x],\qquad x_i=\beta_i(t,x)\quad\text{for } i\in I_+[t,x].$$

Of course, the cases $I_0[t,x] = \emptyset$ or $I_-[t,x] = \emptyset$ or $I_+[t,x] = \emptyset$ are possible. For $(t,x) \in \partial_0 E$ and $i \in I_-[t,x]$, $j \in I_+[t,x]$ we write

$$x[i,\alpha]=(x_1,\dots,x_{i-1},\alpha_i(t),x_{i+1},\dots,x_n),\ x[j,\beta]=(x_1,\dots,x_{j-1},\beta_j(t),x_{j+1},\dots,x_n).$$

A function $z\colon E_0 \cup E \to \mathbb{R}$ will be called a function of class C^* if z is continuous on $E_0 \cup E$, it has first order partial derivatives in an interior of E and z possesses the total differential on Δ. Write $e_i = (0,\dots,0,1,0,\dots,0) \in \mathbb{R}^n$ with 1 standing on the i -th place and $i = 1,\dots,n$. We formulate the main assumption on f, ψ and E as follows.

Assumption $H[f,\psi,E]$. The functions $\alpha,\beta\colon [0,a) \to \mathbb{R}^n$, $\kappa\colon [0,a) \to \mathbb{R}$, $\psi_*\colon E \to \mathbb{R}^n$ and $f\colon \Omega \to \mathbb{R}$ satisfy the conditions

1) α and β are of class C^1 on $[0,a)$ and $\alpha(t) < \beta(t)$ for $t \in [0,a)$;

2) $0 \le \kappa(t) \le t$ for $t \in (0,a)$ and $\alpha(t) \le \psi_*(t,x) \le \beta(t)$ for $(t,x) \in E$;

3) the function f of the variables (t,x,p,w,q), $q = (q_1,\dots,q_n)$, satisfies the condition (V) and the following monotonicity conditions holds: if $(t,x,p,w,q) \in \Omega$, $\tilde{w} \in C(B,\mathbb{R})$ and $w(\tau,y) \le \tilde{w}(\tau,y)$ for $(\tau,y) \in B$ then $f(t,x,p,w,q) \le f(t,x,\tilde{p},\tilde{w},q)$;

4) for each $(t,x) \in \Delta$, $i \in I_-[t,x]$, $h < 0$, we have

$$\alpha_i'(t) \ge -\frac{1}{h}[f(t,x[i,\alpha],p,w,q) - f(t,x[i,\alpha],p,w,q - e_i h)],$$

where $p \in \mathbb{R}$, $q \in \mathbb{R}^n$, $w \in C(B,\mathbb{R})$;

5) for each $(t,x) \in \Delta$, $i \in I_+[t,x]$, $h > 0$, we have

$$\beta_i'(t) \le -\frac{1}{h}[f(t,x[i,\beta],p,w,q) - f(t,x[i,\beta],p,w,q - e_i h)],$$

where $p \in \mathbb{R}$, $q \in \mathbb{R}^n$, $w \in C(B,\mathbb{R})$.

Write

$$F[z](t,x) = f(t,x,z(t,x),z_{\psi(t,x)},\partial_x z(t,x)).$$

We prove a theorem on strong functional differential inequalities.

Theorem 2.1. *Suppose that Assumption $H[f,\psi,E]$ is satisfied and*
1) the functions $u,v\colon E_0 \cup E \to \mathbb{R}$ are of the class C^* and the following initial boundary inequalities hold:

$$u(t,x) \le v(t,x) \quad \text{on } E_0 \quad \text{and} \quad u(0,x) < v(0,x) \quad \text{for } x \in [b_*,b^*]$$

and

$$u(t,x) < v(t,x) \quad \text{for } (t,x) \in \Delta_0;$$

2) denoted by

$$\Sigma = \{(t,x) \in E\backslash\Delta_0 : u(\tau,y) < v(\tau,y) \text{ on } E \cap ((0,t) \times \mathbb{R}^n) \text{ and } u(t,x) = v(t,x)\}$$

we assume that

$$\partial_t u(t,x) - F[u](t,x) < \partial_t v(t,x) - F[v](t,x) \quad \text{for } (t,x) \in \Sigma.$$

Then we have

$$u(t,x) < v(t,x) \quad \text{for } (t,x) \in E. \tag{2.1}$$

Proof. Suppose, by contradiction, that inequality (2.1) fails to be true. Let

$$T_+ = \{t \in [0,a) : u(\tau,y) < v(\tau,y) \text{ on } E \cap ([0,t] \times \mathbb{R}^n) \text{ and } u(t,x) = v(t,x)\}.$$

Then the set T_+ is not empty. Let $\tilde{t} = \min T_+$. From the assumption 1) it is clear that $\tilde{t} > 0$ and there exists $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$ such that $u(\tilde{t},\tilde{x}) = v(\tilde{t},\tilde{x})$ and $u(t,x) < v(t,x)$ on $E \cap ([0,\tilde{t}) \times \mathbb{R}^n)$. The function $u - v$ satisfies the conditions

$$\partial_{x_i}(u-v)(\tilde{t},\tilde{x}) \geq 0 \quad \text{for } i \in I_+[\tilde{t},\tilde{x}], \qquad \partial_{x_i}(u-v)(\tilde{t},\tilde{x}) \leq 0 \quad \text{for } i \in I_-[\tilde{t},\tilde{x}] \tag{2.2}$$

and

$$\partial_{x_i}(u-v)(\tilde{t},\tilde{x}) = 0 \quad \text{for } i \in I_0[\tilde{t},\tilde{x}]. \tag{2.3}$$

Consider the function $\gamma = (\gamma_1, \dots, \gamma_n) \colon [0,\tilde{t}] \to \mathbb{R}^n$ given by:

$$\gamma_i(t) = \tilde{x}_i \quad \text{for } i \in I_0[\tilde{t},\tilde{x}], \tag{2.4}$$

$$\gamma_i(t) = \beta_i(t) \quad \text{for } i \in I_+[\tilde{t},\tilde{x}], \quad \gamma_i(t) = \alpha_i(t) \quad \text{for } i \in I_-[\tilde{t},\tilde{x}], \tag{2.5}$$

and the function $\xi(t) = (u-v)(t,\gamma(t))$, $t \in [0,\tilde{t}]$. Because $\xi(t) < 0$ for $t \in [0,\tilde{t})$ and $\xi(\tilde{t}) = 0$ then $\xi'(\tilde{t}) \geq 0$. Since $u - v$ is of class C^* we have

$$0 \leq \xi'(t) = \partial_t(u-v)(\tilde{t},\tilde{x}) + \sum_{i=1}^{n} \gamma_i'(\tilde{t}) \partial_{x_i}(u-v)(\tilde{t},\tilde{x}),$$

and consequently

$$0 \leq \partial_t(u-v)(\tilde{t},\tilde{x}) + \sum_{i \in I_-[\tilde{t},\tilde{x}]} \alpha_i'(\tilde{t}) \partial_{x_i}(u-v)(\tilde{t},\tilde{x}) + \sum_{i \in I_+[\tilde{t},\tilde{x}]} \beta_i'(\tilde{t}) \partial_{x_i}(u-v)(\tilde{t},\tilde{x}). \tag{2.6}$$

We also have that $(\tilde{t},\tilde{x}) \in \Sigma$. From Assumption $H[f,\psi,E]$ and (2.2) and (2.3), we deduce that

$$\begin{aligned} \partial_t(u-v)(\tilde{t},\tilde{x}) &< F[u](\tilde{t},\tilde{x}) - F[v](\tilde{t},\tilde{x}) \\ &\leq -\sum_{i \in I_-[\tilde{t},\tilde{x}]} \alpha_i'(\tilde{t}) \partial_{x_i}(u-v)(\tilde{t},\tilde{x}) - \sum_{i \in I_+[\tilde{t},\tilde{x}]} \beta_i'(\tilde{t}) \partial_{x_i}(u-v)(\tilde{t},\tilde{x}), \end{aligned}$$

which contradicts (2.6). Therefore, the set T_+ is empty and the statement (2.1) follows. □

Remark 2.1. *In Theorem 2.1 we can assume instead of condition 2) that*

$$\partial_t u(t,x) \le F[u](t,x) \quad \text{and} \quad \partial_t v(t,x) \ge F[v](t,x),$$

where $(t,x) \in E\backslash\Delta_0$ *and for each* $(t,x) \in E\backslash\Delta_0$ *equality holds in at most one place.*

Now we consider weak functional differential inequalities. Write $\Gamma = [-b_0 - a, 0]$ and $\Gamma[t] = [-b_0 - t, 0]$, where $t \in [0,a]$. Then, $\Gamma[t] \subset \Gamma$ for $t \in [0,a]$. We need the operator $W\colon C(B,\mathbb{R}) \to C(\Gamma,\mathbb{R}_+)$ defined by

$$W[w](t) = \max\{|w(t,y)| : y \in [c-d, d-c]\},\ t \in \Gamma.$$

For $w \in C(B,\mathbb{R})$ and $\xi \in C(\Gamma,\mathbb{R})$, we define $w + \xi\colon B \to \mathbb{R}$ in the following way: $(w+\xi)(\tau,y) = w(\tau,y) + \xi(\tau)$, $(\tau,y) \in B$. For a function $\eta\colon \Gamma \cup [0,a) \to \mathbb{R}$ and for a point $t \in [0,a]$, we define $\eta_t\colon \Gamma \to \mathbb{R}$ by $\eta_t(\tau) = \eta(t+\tau)$, $\tau \in \Gamma$.
Suppose that $\sigma\colon [0,a) \times \mathbb{R}_+ \times C(\Gamma,\mathbb{R}_+) \to \mathbb{R}_+$ is a given function. We will say that σ satisfies the condition (V_0) if for each $(t,p,\xi) \in [0,a) \times \mathbb{R}_+ \times C(\Gamma,\mathbb{R}_+)$, $\tilde{\xi} \in C(\Gamma,\mathbb{R}_+)$ such that $\xi(\tau) = \tilde{\xi}(\tau)$ for $\tau \in \Gamma[\kappa(t)]$ we have $\sigma(t,p,\xi) = \sigma(t,p,\tilde{\xi})$

Assumption $H_*[\sigma]$. The function $\sigma\colon [0,a) \times \mathbb{R}_+ \times C(\Gamma,\mathbb{R}_+) \to \mathbb{R}_+$ satisfies the condition (V_0) and

1) σ is continuous and $\sigma(t,0,\mathbb{O}) = 0$ for $t \in [0,a)$ where $\mathbb{O} \in C(\Gamma,\mathbb{R}_+)$ is given by $\mathbb{O}(\tau) = 0$ for $\tau \in \Gamma$;
2) the following monotonicity condition is satisfied: if $p \in \mathbb{R}$, $\zeta, \tilde{\zeta} \in C(\Gamma,\mathbb{R}_+)$ and $\zeta(\tau) \le \tilde{\zeta}(\tau)$ for $\tau \in \Gamma$ then $\sigma(t,p,\zeta) \le \sigma(t,p,\tilde{\zeta})$;
3) the function $\tilde{\eta}(t) = 0$ for $t \in [-b_0, a)$ is the unique solution of the Cauchy problem

$$\eta'(t) = \sigma(t,\eta(t),\eta_{\kappa(t)}), \quad \eta(t) = 0 \quad \text{for } t \in [-b_0, 0].$$

Theorem 2.2. *Suppose that Assumptions* $H[f,\psi,E]$, $H_*[\sigma]$ *are satisfied and*
1) the functions $u, v\colon E_0 \cup E \to \mathbb{R}$ *are of the class* C^* *and*

$$u(t,x) \le v(t,x) \quad \text{for } (t,x) \in E_0 \cup \Delta_0;$$

2) the estimate

$$f(t,x,p,w,q) - f(t,x,\tilde{p},w-\xi,q) \le \sigma(t,p-\tilde{p},\xi)$$

is satisfied for $(t,x) \in E\backslash\Delta_0$, $w \in C(B,\mathbb{R})$, $\xi \in C(\Gamma,\mathbb{R}_+)$, $p,\tilde{p} \in \mathbb{R}$ *and* $p \ge \tilde{p}$;
3) denoted by

$$\Sigma^* = \{(t,x) \in E\backslash\Delta_0 : u(\tau,y) > v(\tau,y)\}$$

we assume that

$$\partial_t u(t,x) - F[u](t,x) \le \partial_t v(t,x) - F[v](t,x) \quad \text{for } (t,x) \in \Sigma^*.$$

Then we have

$$u(t,x) \le v(t,x) \quad \text{for } (t,x) \in E. \tag{2.7}$$

Proof. Let $0 < \tilde{a} < a$ be fixed. We prove that $u(t,x) \le v(t,x)$ on $E \cap ([0,\tilde{a}) \times \mathbb{R}^n)$. Let us denote by $\omega(\,\cdot, \epsilon\,)$ the maximal solution of the Cauchy problem

$$\eta'(t) = \sigma(t, \eta(t), \eta_{\kappa(t)}) + \epsilon, \quad \eta(t) = \epsilon \quad \text{for } t \in [-b_0, 0].$$

There exists $\tilde{\epsilon} > 0$ such that for every $0 < \epsilon < \tilde{\epsilon}$ the solution $\omega(\,\cdot, \epsilon\,)$ is defined on $[-b_0, \tilde{a})$ and

$$\lim_{\epsilon \to \infty} \omega(t, \epsilon) = 0 \quad \text{uniformly on } [-b_0, \tilde{a}).$$

Write

$$u_\epsilon(t,x) = u(t,x) - \omega(t,\epsilon), \quad (t,x) \in E_0 \cup \big[E \cap ([0,\tilde{a}) \times \mathbb{R}^n)\big].$$

We will show that

$$u_\epsilon(t,x) < v(t,x) \quad \text{for } (t,x) \in E \cap ([0,\tilde{a}) \times \mathbb{R}^n). \tag{2.8}$$

It is clear that the following initial boundary inequality is satisfied

$$u_\epsilon(t,x) < v(t,x) \quad \text{for } E_0 \cup \Delta_0.$$

Write

$$\begin{aligned}\tilde{\Sigma} \;\; = \;\; &\{(t,x) \in (E \backslash \Delta_0) \cap ([0,\tilde{a}) \times \mathbb{R}^n) : u_\epsilon(\tau,y) < v(\tau,y) \text{ on } E \cap ((0,t) \times \mathbb{R}^n) \\ &\text{and } u_\epsilon(t,x) = v(t,x)\}.\end{aligned}$$

We prove that

$$\partial_t u_\epsilon(t,x) - F[u_\epsilon](t,x) < \partial_t v(t,x) - F[v](t,x) \quad \text{for } (t,x) \in \tilde{\Sigma}.$$

Suppose that $(t,x) \in \tilde{\Sigma}$. Then $(t,x) \in \Sigma^*$ and

$$\begin{aligned}\partial_t u_\epsilon(t,x) - F[u_\epsilon](t,x) &\le \partial_t v(t,x) - F[v](t,x) - \omega'(t,\epsilon) - F[u_\epsilon](t,x) + F[u](t,x) \\ &\le \partial_t v(t,x) - F[v](t,x) - \omega'(t,x) + \sigma(t, \omega(t,\epsilon), \omega_{\kappa(t)}(\cdot,\epsilon)) \\ &< \partial_t v(t,x) - F[v](t,x),\end{aligned}$$

which completes the proof of (2.6). □

Then we deduce (2.8) from Theorem 2.1. From the above inequality we obtain inequality $u(t,x) \le v(t,x)$ on $E \cap ([0,\tilde{a}) \times \mathbb{R}^n)$. By the arbitrariness of $0 < \tilde{a} < a$, the assertion follows.

Remark 2.2. *Condition 2) in Theorem 2.2 can be replaced by the following assumption:*

$$f(t,x,p,w,q) - f(t,x,\tilde{p},\tilde{w},q) \le \sigma(t, p - \tilde{p}, W[w - \tilde{w}]),$$

where $(t,x) \in E \backslash \Delta_0$, $q \in \mathbb{R}^n$, $p, \tilde{p} \in \mathbb{R}$, $w, \tilde{w} \in C(B,\mathbb{R})$ *and* $p \ge \tilde{p}$, $w(\tau,y) \ge \tilde{w}(\tau,y)$ *for* $(\tau,y) \in B$.

Now we prove a theorem in which strong differential functional inequalities and weak initial boundary inequalities for $u, v\colon E_0 \cup E \to \mathbb{R}$ imply the estimate $u(t,x) < v(t,x)$ for $(t,x) \in E\backslash\Delta_0$.

Theorem 2.3. *Suppose that Assumption $H[\,f,\psi,E\,]$, $H_*[\,\sigma\,]$ are satisfied and*
1) the functions $u, v\colon E_0 \cup E \to \mathbb{R}$ are of the class C^* and

$$u(t,x) \le v(t,x) \quad \textit{for } (t,x) \in E_0 \cup \Delta_0;$$

2) the estimate

$$f(t,x,p,w,q) - f(t,x,\tilde{p},w-\xi,q) \le \sigma(t,p-\tilde{p},\xi)$$

is satisfied for $(t,x) \in E\backslash\Delta_0$, $w \in C(B,\mathbb{R})$, $\xi \in C(\Gamma,\mathbb{R}_+)$, $p,\tilde{p} \in \mathbb{R}$ and $p \ge \tilde{p}$;
3) denoted by

$$\Sigma_* = \{(t,x) \in E\backslash\Delta_0 : u(\tau,y) > v(\tau,y)\}$$

we assume that

$$\partial_t u(t,x) - F[u](t,x) < \partial_t v(t,x) - F[v](t,x) \quad \textit{for } (t,x) \in \Sigma^*.$$

Then

$$u(t,x) < v(t,x) \quad \textit{for } (t,x) \in E. \tag{2.9}$$

This assertion can be proved by applying Theorem 2.3 and then repeating the argument used in the proof of Theorem 2.1.
In the next theorem we assume that $\Delta_0 = \emptyset$ and we prove that weak functional differential inequalities and strong initial inequality for $u, v\colon E_0 \cup E \to \mathbb{R}$ imply the estimate $u(t,x) < v(t,x)$ on E.

Assumption $H_0[\,\sigma\,]$. The function $\sigma\colon [0,a) \times \mathbb{R}_- \to \mathbb{R}_+$, $\mathbb{R}_- = (-\infty, 0]$, satisfies the conditions:

1) σ is continuous and $\sigma(t,0) = 0$ for $t \in [0,a)$;

2) the left hand minimal solution of the problem

$$\eta'(t) = \sigma(t,\eta(t)), \qquad \lim_{t \to a_-} \eta(t) = 0 \tag{2.10}$$

is $\tilde{\eta}(t) = 0$, $t \in [0,a)$.

Theorem 2.4. *Suppose that Assumptions $H[\,f,\psi,E\,]$ and $H_0[\sigma]$ are satisfied, $\Delta_0 = \emptyset$ and*
1) the functions $u, v\colon E_0 \cup E \to \mathbb{R}$ are of class C^* and

$$\partial_t u(t,x) - F[u](t,x) \le \partial_t v(t,x) - F[v](t,x) \quad \textit{for } (t,x) \in E; \tag{2.11}$$

2) for $(t,x) \in E_0$ we have $u(t,x) \le v(t,x)$ and $u(0,x) < v(0,x)$ for $x \in [b_*, b^*]$;
3) if $(t,x,p,w,q) \in \Omega$, $h > 0$ and $\xi \in C(\Gamma, \mathbb{R}_+)$, then

$$f(t,x,p,w,q) - f(t,x,p+h,w+\xi,q) \le \sigma(t, -\min\{h, \|\xi\|_\Gamma\}).$$

Under these assumptions, we have

$$u(t,x) < v(t,x) \quad \text{for } (t,x) \in E. \tag{2.12}$$

Proof. We will prove (2.12) for $(t,x) \in E \cap ([0,a-\epsilon) \times \mathbb{R}^n)$, where $0 < \epsilon < a$. Write

$$0 < p_0 < \min\{v(0,x) - u(0,x) \,:\, x \in [b_*, b^*]\}.$$

For $\delta > 0$, we denote by $\omega(\,\cdot\,,\delta)$ the right hand side minimal solution of the Cauchy problem

$$\eta'(t) = -\sigma(t, -\eta(t)) - \delta, \quad \eta(0) = p_0. \tag{2.13}$$

For fixed $p_0 > 0$ and $\epsilon > 0$ the solution $\omega(\,\cdot\,,\delta)$ exists on $[0, a-\epsilon)$ and $\omega(t,\delta) > 0$ for $t \in [0, a-\epsilon)$. Let $\delta > 0$ be such a small constant that $\omega(\,\cdot\,,\delta)$ satisfies the above conditions. Let us denote by $\tilde z\colon E_0 \to \mathbb{R}$ a continuous function such that

$$u(t,x) \le \tilde z(t,x) \le v(t,x) \text{ for } (t,x) \in E_0 \text{ and } \tilde z(0,x) = u(0,x) + p_0 \text{ for } x \in [b^*, b_*].$$

Write $\tilde u(t,x) = \tilde z(t,x)$ for $(t,x) \in E_0$, $\tilde u(t,x) = u(t,x) + \omega(t,\delta)$ for $(t,x) \in E \cap ([0, a-\epsilon) \times \mathbb{R}^n)$. We will show that

$$\tilde u(t,x) < v(t,x) \quad \text{for } (t,x) \in E \cap ([0,a-\epsilon) \times \mathbb{R}^n). \tag{2.14}$$

It follows from Assumption $H_0[\sigma]$ and from (2.11) that

$$\begin{aligned}
\partial_t \tilde u(t,x) \;-\; & F[\tilde u](t,x) \\
&= \partial_t u(t,x) + \omega'(t,\delta) \\
&\le \partial_t v(t,x) - F[v](t,x) + F[u](t,x) - F[\tilde u](t,x) + \omega'(t,\delta) \\
&\le \partial_t v(t,x) - F[v](t,x) + f(t,x,u(t,x),u_{\psi(t,x)},\partial_x u(t,x)) \\
&\qquad\quad - f(t,x,(u+\omega(\,\cdot\,,\delta))(t,x),(u+\omega(\,\cdot\,,\delta))_{\psi(t,x)},\partial_x u(t,x)) + \omega'(t,\delta) \\
&\le \partial_t v(t,x) - F[v](t,x) + \sigma(t, -\omega(t,\delta)) + \omega'(t,\delta) \\
&\le \partial_t v(t,x) - F[v](t,x) - \delta,\ (t,x) \in E \cap ((0,a-\epsilon) \times \mathbb{R}^n),
\end{aligned}$$

and consequently

$$\partial_t \tilde u(t,x) - F[\tilde u](t,x) < \partial_t v(t,x) - F[v](t,x) \quad \text{for } (t,x) \in E \cap ((0,a-\epsilon) \times \mathbb{R}^n).$$

It follows from Theorem 2.1 that estimate (2.14) is satisfied and consequently, $u(t,x) \le v(t,x)$ for $(t,x) \in E \cap ((0,a-\epsilon) \times \mathbb{R}^n)$. Since $0 < \epsilon < a$ is arbitrary, inequality (2.12) holds true. □

It is clear that conditions 4), 5) of Assumption $H[f,\psi,E]$ are important in theorems on functional differential inequalities. We give examples of the sets E, Δ, and Δ_0 and we formulate suitable assumptions on Δ.

Remark 2.3. *Write*

$$E = \{(t,x) \in \mathbb{R}^{1+n} \ : \ t \in (0,a), x \in [-b+Mt, b-Mt]\}, \ E_0 = [-b_0, 0] \times [-b,b], \tag{2.15}$$

where $a > 0$*,* $b_0 \in \mathbb{R}_+$*,* $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$*,* $b_i > 0$ *for* $i = 1, \ldots, n$ *and* $M = (M_1, \ldots, M_n) \in \mathbb{R}^n_+$*. We assume that* $b > Ma$*. Suppose that we have defined the sets* $J_+, J_- \subset \{1, \ldots, n\}$*. The cases* $J_+ = \emptyset$ *or* $J_- = \emptyset$ *are not excluded. Write*

$$\Delta_+ = \{(t,x) \in \partial E \ : \ \text{there is } i \in J_+ \text{ such that} x_i = b_i - M_i t\}, \tag{2.16}$$

$$\Delta_- = \{(t,x) \in \partial E \ : \ \text{there is } i \in J_- \text{ such that} x_i = -b_i + M_i t\} \tag{2.17}$$

and $\Delta = \Delta_+ \cup \Delta_-$*. Then (1.2), (1.3) reduces to the initial boundary value problem with solutions defined on the classical Haar pyramid. Write*

$$\alpha(t) = -b + Mt, \quad \beta(t) = b - Mt, \ t \in [0,a).$$

If

$$M_i \geq -\frac{1}{h}[f(t, x[i,\alpha], p, w, q) - f(t, x[i,\alpha], p, w, q - e_i h)], \quad i \in J_-, \tag{2.18}$$

$$M_i \leq -\frac{1}{h}[f(t, x[i,\beta], p, w, q) - f(t, x[i,\beta], p, w, q - e_i h)], \quad i \in J_+, \tag{2.19}$$

where $p \in \mathbb{R}$*,* $q \in \mathbb{R}^n$*,* $w \in C(B,\mathbb{R})$ *then conditions* 4), 5) *of Assumption* $H[\,f, \psi, E\,]$ *are satisfied. The theorems presented in Section 2 concern functional differential inequalities corresponding to initial boundary value problems with solutions defined on the Haar pyramid. These theorems are new.*

Suppose that $J_+ = J_- = \{1, \ldots, n\}$*. Then* $\Delta_0 = \emptyset$*,* $\Delta = \partial_0 E$ *and (1.2), (1.3) reduces to the Cauchy problem with solutions defined on the classical Haar pyramid. Theorems on functional differential inequalities generated by initial problems can be found in [5], Chapter 1.*

Remark 2.4. *Suppose that* E *is given by (1.1) and* $E_0 = [-b_0, 0] \times [b_*, b^*]$*. Assume that*
1) the partial derivatives $(\partial_{q_1} f, \ldots, \partial_{q_n} f) = \partial_q f$ *exist on* Ω *and* $\partial_q f \in C(\Omega, \mathbb{R}^n)$*;*
2) the differential inequalities

$$\alpha_i'(t) \geq -\partial_{q_i} f(t, x[i,\alpha], p, w, q), \ (t,x) \in \Delta, \ i \in I_-[t,x],$$

$$\beta_i'(t) \leq -\partial_{q_i} f(t, x[i,\beta], p, w, q), \ (t,x) \in \Delta, \ i \in I_+[t,x],$$

are satisfied. Then conditions 4), 5) *of Assumption* $H[\,f, \psi, E\,]$ *are satisfied.*

Remark 2.5. *Suppose that* $k \in \mathbb{Z}$*,* $0 \leq k \leq n$ *is fixed. For each* $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ *we write* $x = (x', x'')$ *where* $x' = (x_1, \ldots, x_k)$*,* $x'' = (x_{k+1}, \ldots, x_n)$*. We have* $x' = x$ *for* $k = n$ *and* $x'' = x$ *for* $k = 0$*. Write*

$$E = (0,a) \times [-b,b], \quad E_0 = [-b_0, 0] \times [-b,b], \tag{2.20}$$

where $a > 0$, $b_0 \in \mathbb{R}_+$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ *and* $b_i > 0$ *for* $1 \le i \le n$. *Put*

$$\Delta_0 = (0, a) \times ([-b', b'] \backslash (-b', b']) \times ([-b'', b''] \backslash [-b'', b'')), \quad \Delta = \partial_0 E \backslash \Delta_0. \tag{2.21}$$

Then (1.2), (1.3) reduces to the mixed problem for the nonlinear functional differential equations. Suppose that f *satisfies the condition: if* $\tilde{q}, q \in \mathbb{R}^n$, $\tilde{q} = (\tilde{q}_1, \ldots, \tilde{q}_n)$, $q = (q_1, \ldots, q_n)$ *and* $\tilde{q}_i \le q_i$ *for* $i = 1, \ldots, k$, $\tilde{q}_i \ge q_i$ *for* $i = k+1, \ldots, n$, *then*

$$f(t, x, p, w, \tilde{q}) \le f(t, x, p, w, q),$$

where $(t, x, p, w) \in E \times \mathbb{R} \times C(B, \mathbb{R})$. *Then conditions* 4), 5) *of Assumption* $H[f, \psi, E]$ *are satisfied.*

Theorems presented in Section 2 concern, as particular cases, functional differential inequalities corresponding to mixed problems with solutions defined on rectangular domains.

Note that we do not assume that there exists $\partial_q f = (\partial_{q_1} f, \ldots, \partial_{q_n} f)$. It follows that our results are generalisations of theorems on functional differential inequalities presented in [5] (Chapter 5), see also [1].

Remark 2.6. *Suppose that* E *and* E_0 *are given by (2.20). Put* $\Delta_0 = \emptyset$, $\Delta = \partial_0 E$. *Then (1.2), (1.3) reduces to the Cauchy problem with solutions considered on rectangular domains.*
Suppose that
1) the partial derivatives $\partial_q f = (\partial_{q_1} f, \ldots, \partial_{q_1} f)$ *exist on* Ω *and* $\partial_q f \in C(\Omega, \mathbb{R}^n)$;
2) there is $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in (-b, b)$ *such that*

$$(x_i - \tilde{x}_i)\partial_{q_i} f(t, x, p, w, q) \le 0 \text{ on } \Omega \text{ for } 1 \le i \le n. \tag{2.22}$$

Then conditions 4), 5) *of Assumption* $H[f, \psi, E]$ *are satisfied. Section 2 contains new theorems on functional inequalities with solutions defined on rectangular domains.*

3. Comparison theorem

First order partial differential or functional differential equations have the following properties. Theorems on uniqueness of solutions to an initial or initial boundary value problem are consequences of suitable comparison theorems. They give estimates for functions of several variables by means of solutions of ordinary differential or functional differential equations.

The fundamental result, known as the Haar - Ważewski inequality, shows that a function of several variables which is of class C^1 on the Haar pyramid and satisfies a linear differential inequality can be estimated by solution of a linear ordinary differential equation ([7], [10]). There exist many generalizations of the above classical result. The differential inequality may be nonlinear with respect to the unknown function and assumptions on the regularity of the unknown function considered on the Haar pyramid may be weakened [9]. A functional differential version of the Haar - Ważewski inequality can be found in [5].

We extend the Haar - Ważewski inequality (1.2) on initial boundary value problems. We prove that a function satisfying a functional differential inequality and initial boundary conditions can be estimated by a solution of initial value problem for an ordinary functional differential equation.

For a function $z \in C(E_0 \cup E, \mathbb{R})$, we put

$$E_+[z] = \{(t,x) \in E \, : \, |z(t,x)| \geq |z(t,y)| \quad \text{for } y \in [\alpha(t), \beta(t)]\}.$$

Assumption $H[\sigma]$. The function $\sigma\colon [0,a) \times \mathbb{R}_+ \times C(\Gamma, \mathbb{R}_+) \to \mathbb{R}_+$ satisfies condition (V_0) and

(i) σ is continuous and satisfies monotonicity condition: if $\xi, \tilde{\xi} \in C(\Gamma, \mathbb{R}_+)$ and $\xi(\tau) \leq \tilde{\xi}(\tau)$ for $\tau \in \Gamma$ then $\sigma(t,p,\xi) \leq \sigma(t,p,\tilde{\xi})$ where $(t,p) \in [0,a) \times \mathbb{R}_+$;

(ii) the function $\kappa\colon [0,a) \to \mathbb{R}$ is continuous and $0 \leq \kappa(t) \leq t$ for $t \in [0,a)$;

(iii) for every $\eta \in C([-b_0, 0], \mathbb{R}_+)$ the maximal solution of the Cauchy problem

$$\omega'(t) = \sigma(t, \omega(t), \omega_{\kappa(t)}), \quad \omega(t) = \eta(t), \quad t \in [-b_0, 0], \tag{3.1}$$

is defined on $[-b_0, a)$.

We will need the following lemma on functional differential inequalities.

Lemma 3.1. *Suppose that Assumption $H[\sigma]$ is satisfied and*
1) $\xi, \tilde{\xi} \in C([-b_0, a), \mathbb{R}_+)$ and $\xi(0) < \tilde{\xi}(0)$;
2) denoted by

$$T_+ = \{t \in (0,a) : \xi(\tau) < \tilde{\xi}(\tau) \quad \text{for } \tau \in [0,t), \ \xi(t) = \tilde{\xi}(t)\},$$

and assume that

$$D_-\xi(t) - \sigma(t, \xi(t), \xi_{\kappa(t)}) < D_-\tilde{\xi}(t) - \sigma(t, \tilde{\xi}(t), \tilde{\xi}_{\kappa(t)}), \quad t \in T_+$$

where D_- is the left hand side Dini derivative.
Then $\xi(t) < \tilde{\xi}(t)$ for $t \in [0,a)$.

The proof of the above lemma can be found in [5].
For $w \in C(B, \mathbb{R})$, we write

$$\|w\|_{D[t,x]} = \max\{|w(\tau,y)| \, : \, (\tau,y) \in D[t,x]\}.$$

Now we will prove the main theorem of this Section.

Theorem 3.1. *Suppose that the Assumption $H[\sigma]$ is satisfied and*
1) function $u\colon E_0 \cup E \to \mathbb{R}$ is the class of C^* and

$$|u(t,x)| \leq \tilde{\eta}(t) \quad \text{on } E_0, \tag{3.2}$$

where $\tilde{\eta} \in C([-b_0, 0], \mathbb{R}_+)$*;*
2) $\omega(\,\cdot\,, \tilde{\eta})\colon [-b_0, a) \to \mathbb{R}_+$ *is the maximal solution of the problem*

$$\omega'(t) = \sigma(t, \omega(t), \omega_{\kappa(t)}), \quad \omega(t) = \tilde{\eta}(t) \quad \text{for } t \in [-b_0, 0], \tag{3.3}$$

and the following boundary inequality holds:

$$|u(t,x)| \le \omega(t, \tilde{\eta}) \quad \text{on } \Delta_0; \tag{3.4}$$

3) the function u *satisfies the functional differential inequality*

$$\begin{aligned} |\partial_t u(t,x)| \quad \le \quad & \sigma(t, |u(t,x)|, \|u_{\psi(t,x)}\|_{D[\psi(t,x)]}) + \sum_{i \in I_-[t,x]} \alpha_i'(t)\, |\partial_{x_i} u(t,x)| \\ & - \sum_{i \in I_+[t,x]} \beta_i'(t)\, |\partial_{x_i} u(t,x)|, \end{aligned} \tag{3.5}$$

where $(t,x) \in E_+[u] \backslash \Delta_0$*.*
Then

$$|u(t,x)| \le \omega(t, \tilde{\eta}) \quad \text{on } E. \tag{3.6}$$

Proof. Defining

$$\xi(t) = \max\{|u(\tau, y)| : (\tau, y) \in E_0 \cup E], \tau \le t\}, \quad t \in [-b_0, a),$$

we have that $\xi \in C([-b_0, a), \mathbb{R}_+)$ and the statement (3.6) is equivalent to the inequality $\xi(t) \le \omega(t, \tilde{\eta})$ for $t \in [0, a)$. Let $0 < \tilde{a} < a$ be fixed. We denote by $\omega(\,\cdot\,, \tilde{\eta}, \epsilon)$ the maximal solution of the Cauchy problem

$$\omega'(t) = \sigma(t, \omega(t), \omega_{\kappa(t)}) + \epsilon, \quad \omega(t) = \tilde{\eta}(t) + \epsilon, \quad t \in [-b_0, 0]. \tag{3.7}$$

There exist $\tilde{\epsilon} > 0$ such that for every $0 < \epsilon < \tilde{\epsilon}$ the solution $\omega(\,\cdot\,, \tilde{\eta}, \epsilon)$ is defined on $[-b_0, \tilde{a})$ and

$$\lim_{\epsilon \to 0} \omega(t, \tilde{\eta}, \epsilon) = \omega(t, \tilde{\eta}) \quad \text{uniformly on } \Gamma \cup [0, \tilde{a}).$$

We will show that

$$\xi(t) < \omega(t, \tilde{\eta}, \epsilon) \quad \text{for } t \in [0, \tilde{a}). \tag{3.8}$$

We will prove the inequality by using Lemma 3.1. It follows that the initial estimate $\xi(t) \le \eta(t) + \epsilon$, $t \in [-b_0, 0]$, is satisfied. Set $\tilde{T}_+$

$$\tilde{T}_+ = \{t \in (0, a) : \xi(\tau) < \omega(\tau, \tilde{\eta}, \epsilon), \quad \tau \in [0, t), \quad \xi(t) = \omega(t, \tilde{\eta}, \epsilon)\}. \tag{3.9}$$

We prove

$$D_-\xi(t) < \sigma(t, \xi(t), \xi_{\kappa(t)}) + \epsilon \quad \text{for } t \in \tilde{T}_+. \tag{3.10}$$

Let $t \in \tilde{T}_+$ be fixed. Then $\xi(t) = \omega(t, \tilde{\eta}, \epsilon)$ and $\xi(t) = |u(t,x)|$ for some $x \in [\alpha(t), \beta(t)]$. It is clear that $(t,x) \in E_+[u]$. It follows from (3.4) that $(t,x) \notin \Delta_0$. Then we have that (i) $\xi(t) = u(t,x)$ or (ii) $\xi(t) = -u(t,x)$. We will consider the case (i). Let $\gamma = (\gamma_1, \ldots, \gamma_n)\colon [0,t] \to \mathbb{R}^n$ be defined by

$$\gamma_i(\tau) = x_i \quad \text{for } i \in I_0[t,x],$$
$$\gamma_i(\tau) = \beta_i(\tau) \quad \text{for } i \in I_+[t,x], \quad \gamma_i(\tau) = \alpha_i(\tau) \quad \text{for } i \in I_-[t,x],$$

and $\tilde{\xi}(\tau) = u(\tau, \gamma(\tau))$, $\tau \in [0,t]$. Then, we have $\tilde{\xi}(\tau) \le \xi(\tau)$ for $\tau \in [0,t)$ and $\tilde{\xi}(t) = \xi(t)$. We deduce that

$$\partial_{x_i} u(t,x) \ge 0 \quad \text{for } i \in I_+[t,x], \quad \partial_{x_i} u(t,x) \le 0 \quad \text{for } i \in I_-[t,x],$$
$$\partial_{x_i} u(t,x) = 0 \quad \text{for } i \in I_0[t,x].$$

Then

$$\begin{aligned} D_-\xi(t) \le D_-\tilde{\xi}(t) &= \tilde{\xi}'(t) \\ &= \partial_t u(t,x) + \sum_{i=1}^{n} \gamma_i'(t)\,(\partial_{x_i} u(t,x)) \\ &= \partial_t u(t,x) + \sum_{i \in I_+[t,x]}^{n} \beta_i'(t)\,|\partial_{x_i} u(t,x)| - \sum_{i \in I_-[t,x]}^{n} \alpha_i'(t)\,|\partial_{x_i} u(t,x)| \\ &\le \sigma(t, \xi(t), \xi_{\kappa(t)}) \\ &< \sigma(t, \xi(t), \xi_{\kappa(t)}) + \epsilon, \end{aligned} \tag{3.11}$$

which proves (3.10). In a similar way, we can prove (3.10) in the case (ii).

We conclude from Lemma 3.1 that inequality (3.8) is satisfied and consequently, $|u(t,x)| < \omega(t, \tilde{\eta}, \epsilon)$ for $(t,x) \in E \cap ([0,\tilde{a}) \times \mathbb{R}^n)$. From the above inequality, we obtain in the limit, letting ϵ tend to 0, inequality (3.6) on $E \cap ([0,\tilde{a}) \times \mathbb{R}^n)$. By the arbitrariness of $0 < \tilde{a} < a$ the assertion follows. □

In the following, we prove that the difference between two solutions of the functional differential equation (1.2) can be estimated by a solution of a suitable ordinary differential equation.

Assumption $H_0[\,f, \psi, \Delta\,]$ The function $f\colon \Omega \to \mathbb{R}$ satisfies the condition (V) and

1) $\alpha, \beta\colon [0,a) \to \mathbb{R}^n$ are of class C^1 and $\alpha(t) < \beta(t)$ for $t \in [0,a)$;
2) for each $(t,x) \in \Delta$, $i \in I_-[t,x]$, $h \neq 0$, we have

$$\alpha_i'(t) \ge -\frac{1}{h}[f(t, x[i,\alpha], p, w, q + e_i h) - f(t, x[i,\alpha], p, w, q)]$$

where $p \in \mathbb{R}$, $w \in C(B,\mathbb{R})$, $q \in \mathbb{R}^n$;
3) for each $(t,x) \in \Delta$, $i \in I_+[t,x]$, $h \neq 0$, we have

$$\beta_i'(t) \le -\frac{1}{h}[f(t, x[i,\beta], p, w, q + e_i h) - f(t, x[i,\beta], p, w, q)]$$

where $p \in \mathbb{R}$, $w \in C(B,\mathbb{R})$, $q \in \mathbb{R}^n$;
4) $\kappa \in C([0,a), \mathbb{R}_+)$, $\psi_* \in C(E, \mathbb{R}^n)$ and $\kappa(t) \le t$ for $t \in [0,a)$ and $\alpha(t) \le \psi_*(t,x) \le \beta(t)$ for $(t,x) \in E$.

Theorem 3.2. *Suppose that Assumptions $H[\sigma]$ and $H_0[f,\psi,\Delta]$ are satisfied and*
1) the estimate

$$|f(t,x,p,w,q)-f(t,x,\tilde{p},\tilde{w},q)|\le\sigma(t,|p-\tilde{p}|,\|w-\tilde{w}\|_B) \tag{3.12}$$

holds on Ω;
2) the functions $u,v\colon E_0\cup E\to\mathbb{R}$ are solutions of (1.2) and they are of class C^*;
3) $\tilde{\eta}\in C([-b_0,0],\mathbb{R}_+)$ and $|u(t,x)-v(t,x)|\le\tilde{\eta}(t)$ on E_0 and boundary estimate

$$|u(t,x)-v(t,x)|\le\omega(t,\eta),\quad (t,x)\in\Delta_0 \tag{3.13}$$

is satisfied, where $\omega(\cdot,\eta)$ is the maximal solution of (3.3).
Under these assumptions, we have

$$|u(t,x)-v(t,x)|\le\omega(t,\tilde{\eta})\quad \text{on } E. \tag{3.14}$$

Proof. We will show that function $u-v$ satisfies assumptions of Theorem 3.1. The initial boundary inequalities follows from assumption 3). Let $(t,x)\in E_+[u-v]$. We prove that

$$\begin{aligned}|\partial_t(u-v)(t,x)| \;&\le\; \sigma(t,|(u-v)(t,x)|,\|(u-v)_{\psi(t,x)}\|_{D[t,x]})\\ &\quad+\sum_{i\in I_-[t,x]}\alpha_i'(t)\,|\partial_{x_i}(u-v)(t,x)|\\ &\quad+\sum_{i\in I_+[t,x]}\beta_i'(t)\,|\partial_{x_i}(u-v)(t,x)|.\end{aligned} \tag{3.15}$$

We have that (i) $(u-v)(t,x)=|(u-v)(t,x)|$ or (ii) $(u-v)(t,x)=-|(u-v)(t,x)|$. We consider the case (i). It follows that

$$\begin{aligned}\partial_{x_i}(u-v)(t,x) &= 0 \quad \text{for } i\in I_0[t,x],\\ \partial_{x_i}(u-v)(t,x) &\ge 0 \quad \text{for } i\in I_+[t,x],\quad \partial_{x_i}(u-v)(t,x)\le 0 \;\text{ for } i\in I_-[t,x].\end{aligned}$$

Then we have

$$\begin{aligned}&\partial_t(u-v)(t,x)\\ &\quad= f(t,x,u(t,x),u_{\psi(t,x)},\partial_x u(t,x))-f(t,x,v(t,x),v_{\psi(t,x)},\partial_x v(t,x))\\ &\quad\le \sigma(t,|(u-v)(t,x)|,\|(u-v)_{\psi(t,x)}\|_{D[\psi(t,x)]})\\ &\qquad+f(t,x,v(t,x),v_{\psi(t,x)},\partial_x u(t,x))-f(t,x,v(t,x),v_{\psi(t,x)},\partial_x v(t,x))\\ &\quad\le \sigma(t,|(u-v)(t,x)|,\|(u-v)_{\psi(t,x)}\|_{D[\psi(t,x)]})\\ &\qquad+\sum_{i\in I_-[t,x]}\alpha_i'(t)(-\partial_{x_i}(u-v)(t,x))-\sum_{i\in I_+[t,x]}\beta_i'(t)\partial_{x_i}(u-v)(t,x)\\ &\quad= \sigma(t,|(u-v)(t,x)|,\|(u-v)_{\psi(t,x)}\|_{D[\psi(t,x)]})\\ &\qquad+\sum_{i\in I_-[t,x]}\alpha_i'(t)|\partial_{x_i}(u-v)(t,x)|-\sum_{i\in I_+[t,x]}\beta_i'(t)|\partial_{x_i}(u-v)(t,x)|.\end{aligned}$$

In the similar way, we show that

$$\begin{aligned}\partial_t(u-v)(t,x) \quad \geq \quad & -\sigma(t,|(u-v)(t,x)|,\|(u-v)_{\psi(t,x)}\|_{D[\psi(t,x)]}) \\ & -\sum_{i\in I_-[t,x]} \alpha_i'(t)|\partial_{x_i}(u-v)(t,x)| + \sum_{i\in I_+[t,x]} \beta_i'(t)|\partial_{x_i}(u-v)(t,x)|.\end{aligned}$$

That shows (3.15) in case (i). In the similar way, we prove (3.15) in the case (ii). That shows that all assumptions of Theorem 3.1 are satisfied and the assertion follows. □

The following result is a consequence of Theorem 3.1.

Theorem 3.3. *Suppose that Assumption* $H[\sigma]$ *and* $H_0[f,\psi,\Delta]$ *are satisfied*
1) the function $\tilde{\omega}(t)=0$, $t\in[-b_0,a)$, *is the maximal solution of (3.3) with* $\eta(t)=0$ *for* $t\in[-b_0,0]$;
2) the estimate (3.12) holds.
Under these assumptions, the mixed problem (1.2), (1.3) admits at most one solution $u\colon E_0\cup E\to\mathbb{R}$ *of class* C^*.

It is clear that conditions 2), 3) of Assumption $H_0[f,\psi,\Delta]$ are important in our considerations. To show this, we give examples of the sets E and we formulate corresponding assumptions on Δ.

Remark 3.1. *Let* E, E_0 *and* Δ *be the sets defined in Remark 2.3. Suppose that conditions (2.18), (2.19) are satisfied for* $p\in\mathbb{R}$, $q\in\mathbb{R}^n$, $w\in C(B,\mathbb{R})$. *Then conditions* 2), 3) *of Assumption* $H_0[f,\psi,E]$ *are satisfied. Theorems 3.2 and 3.3 concern initial boundary value problems for (1.2) with solutions defined on the Haar pyramid. These theorems are new.*

Suppose that $J_+=J_-=\{1,\ldots,n\}$. *Then* $\Delta_0=\emptyset$, $\Delta=\partial_0 E$ *and (1.2), (1.3) reduces to the Cauchy problem with solutions defined on the classical Haar pyramid.*

Remark 3.2. *Let* E, E_0 *and* Δ *be the sets defined in Remark 2.5. Then (1.2), (1.3) reduces to a mixed problem for nonlinear functional differential equations. Suppose that*

$$-\frac{1}{h}[f(t,x[i,\alpha],p,w,q+e_ih)-f(t,x[i,\alpha],p,w,q)]\leq 0,\quad i=1,\ldots,k,$$

and

$$-\frac{1}{h}[f(t,x[i,\beta],p,w,q+e_ih)-f(t,x[i,\beta],p,w,q)]\geq 0,\quad i=k+1,\ldots,n.$$

The conditions 2), 3) *of Assumption* $H_0[f,\psi,\Delta]$ *are satisfied.*

Theorems 3.2 and 3.3 concern mixed problems with solutions defined on rectangular domains.

Note that we do not assume that there exist the derivatives $(\partial_{q_1}f,\ldots,\partial_{q_n}f)$. *It follows that our results are generalizations of theorems on mixed problems presented in [1], (see also [5], Chapter 5).*

Remark 3.3. *Let E, E_0 and Δ be the sets defined in Remark 2.6. Suppose that there exist the derivatives* $(\partial_{q_1} f, \ldots, \partial_{q_n} f)$, $\partial_q f \in C(\Omega, \mathbb{R})$ *and there is* $\tilde{x} \in (-b, b)$ *such that condition (2.22) is satisfied. Then conditions* 2), 3) *of Assumption* $H_0[\, f, \psi, \Delta\,]$ *are satisfied. Section 3 contains new theorems on initial problems for functional differential equations with solutions defined on a rectangular domain.*

Remark 3.4. *Note that the connection with a functional differential comparison problem is essential for the uniqueness criterion stated in Theorem 3.3. The following example points out this property. If* $\beta \geq \alpha > 1$ *and* $A, B \in \mathbb{R}_+$, *then the maximal solution of the Cauchy problem*

$$\omega'(t) = A \sqrt[\alpha]{\omega(t^\beta)} + B\omega(t), \ \omega(0) = 0 \tag{3.16}$$

is $\bar{\omega}(t) = 0$ *for* $t \in [0, a]$ *where* $a \leq 1$. *Note that maximal solution of (3.16) with* $\alpha > 1$ *and* $\beta = 1$ *is positive on* $(0, a]$.

Remark 3.5. *Results presented in the paper can be extended on functional differential systems*

$$\partial_t z_i(t, x) = f_i(t, x, z(t, x), z_{\psi(t,x)}, \partial_x z_i(t, x)), \ i = 1, \ldots, m,$$

where $z = (z_1, \ldots, z_m)$.

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