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Growth and Complex Oscillation of Differential Polynomials Generated by Solutions of Differential Equations

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Abstract. In this paper, we investigate the fixed points and the hyper order of the differential polynomial $g_f = d_2 f'' + d_1 f' + d_0 f$, where $d_0(z), d_1(z), d_2(z)$ are entire functions that are not all equal to zero with $\rho(d_j) < \infty$ $(j = 0, 1, 2)$ generated by solutions of the differential equation

$$f'' + A_1(z) f' + A_0(z) f = F,$$

where $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$, F are entire functions of finite order. Because of the control of differential equation, we can obtain some precise estimates of their hyper order and fixed points. We also investigate the relation between infinite order solutions of higher order linear differential equations with entire coefficients and finite order entire functions.

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1. Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory (see [7], [9], [11], [13]). In addition, we will use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote respectively the

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exponents of convergence of the zero-sequence and the sequence of distinct zeros of a meromorphic function f, $\rho(f)$ to denote the order of growth of f. A meromorphic function $\varphi(z)$ is called a small function with respect to $f(z)$ if $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$ except possibly a set of r of finite linear measure, where $T(r,f)$ is the Nevanlinna characteristic function of f. In order to express the rate of growth of meromorphic solutions of infinite order, we recall the following definition.

Definition 1.1 ([4], [13]) Let f be a meromorphic function. Then the hyper order $\rho_2(f)$ of $f(z)$ is defined by

$$\rho_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log T(r,f)}{\log r}. \tag{1.1}$$

Remark 1.1. If f is an entire function, then the hyper order $\rho_2(f)$ of $f(z)$ is defined by

$$\rho_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log T(r,f)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log\log\log M(r,\ f)}{\log r},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

To give the precise estimate of fixed points, we define:

Definition 1.2. ([4], [10], [12]) Let f be a meromorphic function and let $z_1, z_2, ...$ ($|z_j| = r_j$, $0 < r_1 \le r_2 \le ...$) be the sequence of the fixed points of f, each point being repeated only once. The exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\tau}(f) = \inf\left\{\tau > 0 : \sum_{j=1}^{+\infty} |z_j|^{-\tau} < +\infty\right\}.$$

Clearly,

$$\overline{\tau}(f) = \overline{\lim_{r\to+\infty}} \frac{\log \overline{N}\left(r, \frac{1}{f-z}\right)}{\log r}, \tag{1.2}$$

where $\overline{N}(r, \frac{1}{f-z})$ is the counting function of distinct fixed points of $f(z)$ in $\{z\colon |z| < r\}$.

Definition 1.3. ([4], [8], [10], [12]) Let f be a meromorphic function. Then $\overline{\lambda}_2(f)$, the hyper exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log \overline{N}\left(r, \frac{1}{f}\right)}{\log r}, \tag{1.3}$$

and $\overline{\tau}_2(f)$, the hyper exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\tau}_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log \overline{N}\left(r, \frac{1}{f-z}\right)}{\log r}. \qquad (1.4)$$

Consider the second order linear differential equation

$$f'' + A_1(z) f' + A_0(z) f = F, \qquad (1.5)$$

where $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$, F are transcendental entire functions with finite order. Many important results have been obtained on the fixed points of general transcendental meromorphic functions for almost four decades (see [14]). However, there are few studies on the fixed points of solutions of differential equations, even second order linear differential equations

$$f'' + A(z) f = 0, \qquad (1.6)$$

$$f'' + A(z) f = F, \qquad (1.7)$$

where $A(z)$ and $F(z) \not\equiv 0$ are entire functions. It was in the year 2000 that Z. X. Chen first pointed out the relation between the exponent of convergence of distinct fixed points and the rate of growth of solutions of equations (1.6), (1.7) and have obtained the following results.

Theorem A. [4] *For all non trivial solutions f of* (1.6), *the following hold:*
(i) If A is a polynomial with $\deg A = n \geq 1$, then we have $\overline{\tau}(f) = \rho(f) = \frac{n+2}{2}$.
(ii) If A is transcendental and $\rho(A) = \rho < +\infty$, then we have $\overline{\tau}(f) = \rho(f) = +\infty$ and $\overline{\tau}_2(f) = \rho_2(f) = \rho$.

Theorem B. [4] *Suppose that $F(z)$ and $A(z)$ have finite order and $F(z) \not\equiv zA(z)$. Then for all non trivial solutions f of* (1.7), *the conclusions of Theorem A hold except at most for one solution f_0.*

We know that a differential equation bears a relation to all derivatives of its solutions. Hence, linear differential polynomials generated by its solutions must have special nature because of the control of differential equations. The first main purpose of this paper is to study the growth, the oscillation and the relation between small functions and differential polynomials generated by solutions of the second order linear differential equation (1.5).

Before we state our results, we denote by

$$\alpha_1 = d_1 - d_2A_1,\ \beta_0 = d_2A_0A_1 - (d_2A_0)' - d_1A_0 + d_0', \qquad (1.8)$$

$$\alpha_0 = d_0 - d_2A_0,\ \ \beta_1 = d_2A_1^2 - (d_2A_1)' - d_1A_1 - d_2A_0 + d_0 + d_1', \qquad (1.9)$$

$$h = \alpha_1 \beta_0 - \alpha_0 \beta_1 \tag{1.10}$$

and

$$\psi(z) = \frac{\alpha_1 \left(\varphi' - (d_2 F)' - \alpha_1 F \right) - \beta_1 (\varphi - d_2 F)}{h}, \tag{1.11}$$

where $A_1(z)$, $A_0(z)$, F, d_j $(j = 0, 1, 2)$ and φ are entire functions with finite order.

Theorem 1.1. *Let $A_1(z)$, $A_0(z) \not\equiv 0$, F be entire functions of finite order. Let $d_0(z), d_1(z), d_2(z)$ be entire functions that are not all equal to zero with $\rho(d_j) < \infty$ $(j = 0, 1, 2)$ such that $h \not\equiv 0$. Let $\varphi(z)$ be an entire function with finite order such that $\psi(z)$ is not a solution of* (1.5)*. If f is an infinite order solution of* (1.5) *with $\rho_2(f) = \rho$, then the differential polynomial $g_f = d_2 f'' + d_1 f' + d_0 f$ satisfies*

$$\overline{\lambda}(g_f - \varphi) = \rho(g_f) = \rho(f) = \infty, \tag{1.12}$$

$$\overline{\lambda}_2(g_f - \varphi) = \rho_2(g_f) = \rho_2(f) = \rho. \tag{1.13}$$

Theorem 1.2. *Let $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$, $F \not\equiv 0$ be entire functions of finite order such that all solutions of equation* (1.5) *are of infinite order. Let $d_0(z), d_1(z), d_2(z)$ be entire functions that are not all equal to zero with $\rho(d_j) < \infty$ $(j = 0, 1, 2)$ such that $h \not\equiv 0$. Let φ be a finite order entire function. If f is a solution of equation* (1.5) *with $\rho_2(f) = \rho$, then the differential polynomial g_f satisfies* (1.12) *and* (1.13)*.*

Applying Theorem 1.2 for $\varphi(z) = z$, we obtain the following result.

Corollary 1.1. *Under the assumptions of Theorem 1.2, if f is a solution of equation* (1.5) *with $\rho_2(f) = \rho$, then the differential polynomial g_f has infinitely many fixed points and satisfies $\overline{\tau}(g_f) = \rho(g_f) = \rho(f) = \infty$, $\overline{\tau}_2(g_f) = \rho_2(g_f) = \rho_2(f) = \rho$.*

Theorem 1.3. *Let $A_1(z)$, $A_0(z), F, d_0(z), d_1(z), d_2(z), \varphi$ satisfy the hypotheses of Theorem 1.1, such that $A_1(z)$ or $A_0(z)$ is transcendental, then equation* (1.5) *has solution f that g_f satisfies* (1.12)*.*

In the following, we obtain a result which is an example of Theorem 1.2.

Corollary 1.2. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ and $Q(z) = \sum_{i=0}^{n} b_i z^i$ be nonconstant polynomials where a_i, b_i $(i = 0, 1, ..., n)$ are complex numbers, $a_n b_n \neq 0$ such that $\arg a_n \neq \arg b_n$ or $a_n = c b_n$ $(0 < c < 1)$ and $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1)$ be entire functions with $\rho(A_j) < n$ $(j = 0, 1)$. Let $d_0(z), d_1(z), d_2(z)$ be entire functions that are not all equal to zero* *with* $\rho(d_j) < n$ $(j = 0, 1, 2)$*, and let* $\varphi(z) \not\equiv 0$ *be an entire function with finite order. If* $f(z) \not\equiv 0$ *is a solution of the differential equation*

$$f'' + A_1(z) e^{P(z)} f' + A_0(z) e^{Q(z)} f = 0, \tag{1.14}$$

then the differential polynomial $g_f = d_2 f'' + d_1 f' + d_0 f$ *satisfies* $\overline{\lambda}(g_f - \varphi) = \rho(g_f) = \rho(f) = \infty$ *and* $\overline{\lambda}_2(g_f - \varphi) = \rho_2(g_f) = \rho_2(f) = n$.

The second main purpose of this paper is to investigate the relation between infinite order solutions of higher order linear differential equations with entire coefficients and finite order entire functions. We obtain the following result.

Theorem 1.4. *Let* $A_0, A_1, ..., A_{k-1}, F$ *be finite order entire functions, and let* φ *be a finite order entire function which is not a solution of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + ... + A_1 f' + A_0 f = F. \tag{1.15}$$

If f *is an infinite order solution of equation* (1.15) *with* $\rho_2(f) = \rho$*, then we have* $\overline{\lambda}(f - \varphi) = \rho(f) = \infty$ *and* $\overline{\lambda}_2(f - \varphi) = \rho_2(f) = \rho$.

Applying Theorem 1.4 for $\varphi(z) = z$, we obtain the following result.

Corollary 1.3. *Let* $A_0, A_1, ..., A_{k-1}, F$ *be finite order entire functions such that* $zA_0 + A_1 \not\equiv F$*. Then every infinite order solution* f *of equation* (1.15) *with* $\rho_2(f) = \rho$ *has infinitely many fixed points and satisfies* $\overline{\tau}(f) = \rho(f) = \infty$, $\overline{\tau}_2(f) = \rho_2(f) = \rho$.

In the following, we obtain a result which is an example of Theorem 1.4.

Corollary 1.4. *Let* $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, ..., k-1)$ *be nonconstant polynomials where* $a_{0,j}, ..., a_{n,j}$ $(j = 0, ..., k-1)$ *are complex numbers such that* $a_{n,j} a_{n,0} \neq 0$ $(j = 1, ..., k-1)$*, let* $A_j(z)$ $(\not\equiv 0)$ $(j = 0, ..., k-1)$ *be entire functions. Suppose that* $\arg a_{n,j} \neq \arg a_{n,0}$ *or* $a_{n,j} = c a_{n,0}$ $(0 < c < 1)$ $(j = 1, ..., k-1)$, $\rho(A_j) < n$ $(j = 0, ..., k-1)$*. Let* $\varphi \not\equiv 0$ *be a finite order entire function. Then every solution* $f(z) \not\equiv 0$ *of the equation*

$$f^{(k)} + A_{k-1}(z) e^{P_{k-1}(z)} f^{(k-1)} + ... + A_1(z) e^{P_1(z)} f' + A_0(z) e^{P_0(z)} f = 0, \tag{1.16}$$

where $k \geqslant 2$*, satisfies* $\overline{\lambda}(f - \varphi) = \rho(f) = \infty$ *and* $\overline{\lambda}_2(f - \varphi) = \rho_2(f) = n$*. In particularly every solution* $f(z) \not\equiv 0$ *of equation* (1.16) *has infinitely many fixed points and satisfies* $\overline{\tau}(f) = \rho(f) = \infty$, $\overline{\tau}_2(f) = \rho_2(f) = n$.

Remark 1.2. In Theorem 1.1 and Theorem 1.2, if we don't have the condition $h \not\equiv 0$, then the differential polynomial can be of finite order. For example if $d_2(z) \not\equiv 0$ is a finite order entire function and $d_0(z) = A_0 d_2(z)$, $d_1(z) = A_1 d_2(z)$, then we have $h \equiv 0$ and $g_f = d_2(z) F$ is of finite order.

2. Auxiliary Lemmas

Lemma 2.1. (see [6, p. 412]). *Let the differential equation*

$$f^{(k)} + a_{k-1}f^{(k-1)} + ... + a_0 f = 0 \tag{2.1}$$

be satisfied in the complex plane by the linearly independent meromorphic functions $f_1, f_2, ..., f_k$. *Then the coefficients* $a_{k-1}, ..., a_0$ *are meromorphic in the plane with the following properties:*

$$m(r, a_j) = O\{\log[\max(T(r, f_s) : s = 1, ..., k)]\} \quad (j = 0, ..., k-1). \tag{2.2}$$

Lemma 2.2. *Suppose that* $A_0, A_1, ..., A_{k-1}, F \not\equiv 0$ *are meromorphic functions with at least one* A_s $(0 \leq s \leq k-1)$ *being transcendental. If all solutions of*

$$f^{(k)} + A_{k-1}f^{(k-1)} + ... + A_1 f' + A_0 f = F \tag{2.3}$$

are meromorphic, then (2.3) *has an infinite order solution.*

Proof. Since all the solutions of (2.3) are meromorphic, all the solutions of the corresponding homogeneous differential equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + ... + A_1 f' + A_0 f = 0 \tag{2.4}$$

of (2.3) are meromorphic. Now assume that $\{f_1, ..., f_k\}$ is a fundamental solution set of (2.4). Then by Lemma 2.1, we have for $j = 0, ..., k-1$

$$m(r, A_j) = O\{\log[\max(T(r, f_s) : s = 1, ..., k)]\}. \tag{2.5}$$

Since A_s is transcendental, at least one of $f_1, ..., f_k$ is of infinite order of growth. Suppose f_1 satisfies $\rho(f_1) = \infty$.
If f_0 is a solution of (2.3), then every solution f of (2.3) can be written in the form

$$f = C_1 f_1 + C_2 f_2 + ... + C_k f_k + f_0, \tag{2.6}$$

where $C_1, C_2, ..., C_k$ are arbitrary constants. If $\rho(f_0) = \infty$, then Lemma 2.2 holds. If $\rho(f_0) < \infty$, then $f = f_1 + f_0$ is a meromorphic solution of (2.3) and $\rho(f) = \infty$.

Lemma 2.3. [5] *Let* $A_0, A_1, ..., A_{k-1}, F (\not\equiv 0)$ *be finite order meromorphic functions. If* f *is a meromorphic solution with* $\rho(f) = +\infty$ *of equation* (2.3), *then* $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$.

Lemma 2.4. [2] *Let* $A_0, A_1, ..., A_{k-1}, F (\not\equiv 0)$ *be finite order meromorphic functions. If* f *is a meromorphic solution of equation* (2.3) *with* $\rho(f) = +\infty$ *and* $\rho_2(f) = \rho$, *then* f *satisfies* $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho$.

Lemma 2.5. *Let $A_1(z)$, $A_0(z)$ $(\not\equiv 0)$, F be entire functions of finite order. Let $d_0(z), d_1(z), d_2(z)$ be entire functions that are not all equal to zero with $\rho(d_j) < \infty$ $(j = 0, 1, 2)$ such that $h \not\equiv 0$, where h is defined in (1.10). If f is an infinite order solution of (1.5) with $\rho_2(f) = \rho$, then the differential polynomial $g_f = d_2 f'' + d_1 f' + d_0 f$ satisfies*

$$\rho(g_f) = \rho(f) = \infty, \ \rho_2(g_f) = \rho_2(f) = \rho. \tag{2.7}$$

Proof. Suppose that f is a solution of equation (1.5) with $\rho(f) = \infty$ and $\rho_2(f) = \rho$. Substituting $f'' = F - A_1 f' - A_0 f$ into g_f, we get

$$g_f - d_2 F = (d_1 - d_2 A_1) f' + (d_0 - d_2 A_0) f. \tag{2.8}$$

Differentiating both sides of equation (2.8) and replacing f'' with $f'' = F - A_1 f' - A_0 f$, we obtain

$$g_f' - (d_2 F)' - (d_1 - d_2 A_1) F = \left[d_2 A_1^2 - (d_2 A_1)' - d_1 A_1 - d_2 A_0 + d_0 + d_1'\right] f'$$

$$+ \left[d_2 A_0 A_1 - (d_2 A_0)' - d_1 A_0 + d_0'\right] f. \tag{2.9}$$

Then by (1.8), (1.9), (2.8) and (2.9), we have

$$\alpha_1 f' + \alpha_0 f = g_f - d_2 F, \tag{2.10}$$

$$\beta_1 f' + \beta_0 f = g_f' - (d_2 F)' - (d_1 - d_2 A_1) F. \tag{2.11}$$

Set

$$h = \alpha_1 \beta_0 - \alpha_0 \beta_1 = (d_1 - d_2 A_1)\left(d_2 A_0 A_1 - (d_2 A_0)' - d_1 A_0 + d_0'\right)$$

$$- (d_0 - d_2 A_0)\left(d_2 A_1^2 - (d_2 A_1)' - d_1 A_1 - d_2 A_0 + d_0 + d_1'\right). \tag{2.12}$$

By $h \not\equiv 0$ and $(2.10) - (2.12)$, we obtain

$$f = \frac{\alpha_1 \left(g_f' - (d_2 F)' - \alpha_1 F\right) - \beta_1 (g_f - d_2 F)}{h}. \tag{2.13}$$

If $\rho(g_f) < \infty$, then by (2.13) we get $\rho(f) < \infty$ and this is a contradiction. Hence $\rho(g_f) = \infty$.

Now, we prove that $\rho_2(f) = \rho$. By (2.8) we have $\rho_2(g_f) \leqslant \rho_2(f)$ and by (2.13) we get $\rho_2(f) \leqslant \rho_2(g_f)$. This yields $\rho_2(g_f) = \rho_2(f) = \rho$.

Lemma 2.6. [1] *Let $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, \ldots, k-1)$ be nonconstant polynomials where $a_{0,j}, \ldots, a_{n,j}$ $(j = 0, \ldots, k-1)$ are complex numbers such that $a_{n,j} a_{n,0} \neq 0$*

$(j=1,...,k-1)$, let $A_j(z)$ $(\not\equiv 0)$ $(j=0,...,k-1)$ be entire functions. Suppose that $\arg a_{n,j} \neq \arg a_{n,0}$ or $a_{n,j} = ca_{n,0}$ $(0<c<1)$ $(j=1,...,k-1)$, $\rho(A_j) < n$ $(j=0,...,k-1)$. Then every solution $f(z) \not\equiv 0$ of the equation

$$f^{(k)} + A_{k-1}(z)\, e^{P_{k-1}(z)} f^{(k-1)} + ... + A_1(z)\, e^{P_1(z)} f^{'} + A_0(z)\, e^{P_0(z)} f = 0, \qquad (2.14)$$

where $k \geq 2$, is of infinite order and $\rho_2(f) = n$.

Lemma 2.7. [3] *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ and $Q(z) = \sum_{i=0}^{n} b_i z^i$ be nonconstant polynomials where a_i, b_i $(i=0,1,...,n)$ are complex numbers, $a_n b_n \neq 0$ such that $\arg a_n \neq \arg b_n$ or $a_n = cb_n$ $(0<c<1)$. We denote index set by*

$$\Lambda = \{0, P, Q, 2P, P+Q\}. \qquad (2.15)$$

If H_j $(j \in \Lambda)$ and $H_{2Q} \not\equiv 0$ are all meromorphic functions of orders that are less than n, setting $\Psi_2(z) = \sum_{j\in\Lambda} H_j(z)\, e^j$, then $\Psi_2(z) + H_{2Q} e^{2Q} \not\equiv 0$.

3. Proof of Theorem 1.1

Suppose that f is a solution of equation (1.5) with $\rho(f) = \infty$ and $\rho_2(f) = \rho$. Set $w(z) = d_2 f^{''} + d_1 f^{'} + d_0 f - \varphi$. Since $\rho(\varphi) < \infty$, then by Lemma 2.5, we have $\rho(w) = \rho(g_f) = \rho(f) = \infty$ and $\rho_2(w) = \rho_2(g_f) = \rho_2(f) = \rho$. In order to prove $\overline{\lambda}(g_f - \varphi) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \rho$, we need to prove only $\overline{\lambda}(w) = \infty$ and $\overline{\lambda}_2(w) = \rho$. Substituting $g_f = w + \varphi$ into (2.13), we get

$$f = \frac{\alpha_1 w^{'} - \beta_1 w}{h} + \psi, \qquad (3.1)$$

where α_1, β_1, h, ψ are defined in (1.8) − (1.11). Substituting (3.1) into equation (1.5), we obtain

$$\frac{\alpha_1}{h} w^{'''} + \phi_2 w^{''} + \phi_1 w^{'} + \phi_0 w$$
$$= F - \left(\psi^{''} + A_1(z)\,\psi^{'} + A_0(z)\,\psi\right) = A, \qquad (3.2)$$

where ϕ_j $(j=0,1,2)$ are meromorphic functions with $\rho(\phi_j) < \infty$ $(j=0,1,2)$. Since $\psi(z)$ is not a solution of (1.5), it follows that $A \not\equiv 0$. Then, by Lemma 2.3 and Lemma 2.4, we obtain $\overline{\lambda}(w) = \lambda(w) = \rho(w) = \infty$, $\overline{\lambda}_2(w) = \lambda_2(w) = \rho_2(w) = \rho$, i.e., $\overline{\lambda}(g_f - \varphi) = \rho(g_f) = \rho(f) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \rho_2(g_f) = \rho_2(f) = \rho$.

Remark 3.1. The condition "$\psi(z)$ is not a solution of the equation (1.5)" in Theorem 1.1, is necessary because if $\psi(z)$ is a solution of the equation (1.5), then we have

$$F - \left(\psi'' + A_1(z)\psi' + A_0(z)\psi\right) \equiv 0.$$

4. Proof of Theorem 1.2

By the hypotheses of Theorem 1.2 all solutions of equation (1.5) are of infinite order. From (1.11), we see that $\psi(z)$ is a meromorphic function of finite order, hence $\psi(z)$ is not a solution of (1.5). By Theorem 1.1, we obtain Theorem 1.2.

5. Proof of Theorem 1.3

By Lemma 2.2, we know that equation (1.5) has an infinite order solution f. Then, by Theorem 1.1, g_f satisfies (1.12).

6. Proof of Corollary 1.2

Suppose that $f(z) \not\equiv 0$ is a solution of equation (1.14). Then by Lemma 2.6 we have $\rho(f) = \infty$ and $\rho_2(f) = n$. First we suppose that $d_2 \not\equiv 0$. Set

$$\alpha_1 = d_1 - d_2 A_1 e^P, \quad \alpha_0 = d_0 - d_2 A_0 e^Q, \tag{4.1}$$

$$\beta_1 = d_2 A_1^2 e^{2P} - \left((d_2 A_1)' + P' d_2 A_1 + d_1 A_1\right) e^P - d_2 A_0 e^Q + d_0 + d_1', \tag{4.2}$$

$$\beta_0 = d_2 A_0 A_1 e^{P+Q} - \left((d_2 A_0)' + Q' d_2 A_0 + d_1 A_0\right) e^Q + d_0'. \tag{4.3}$$

$$\begin{aligned} h = \alpha_1\beta_0 - \alpha_0\beta_1 = \left(d_1 - d_2 A_1 e^P\right)\Big[d_2 A_0 A_1 e^{P+Q} - ((d_2 A_0)' \\ + Q' d_2 A_0 + d_1 A_0) e^Q + d_0'\Big] - \left(d_0 - d_2 A_0 e^Q\right)\Big[d_2 A_1^2 e^{2P} - ((d_2 A_1)' \\ + P' d_2 A_1 + d_1 A_1) e^P - d_2 A_0 e^Q + d_0 + d_1'\Big]. \end{aligned} \tag{4.4}$$

Now check all the terms of h. Since the term $d_2^2 A_1^2 A_0 e^{2P+Q}$ is eliminated, by (4.4) we can write $h = \Psi_2(z) - d_2^2 A_0^2 e^{2Q}$, where $\Psi_2(z)$ is defined as in Lemma 2.7. By $d_2 \not\equiv 0$, $A_0 \not\equiv 0$ and Lemma 2.7 we see that $h \not\equiv 0$. By Theorem 1.2 the differential polynomial $g_f = d_2 f'' + d_1 f' + d_0 f$ satisfies $\overline{\lambda}(g_f - \varphi) = \rho(g_f) = \rho(f) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \rho_2(g_f) = \rho_2(f) = n$.

Now suppose $d_2 \equiv 0$, $d_1 \not\equiv 0$ or $d_2 \equiv 0$, $d_1 \equiv 0$ and $d_0 \not\equiv 0$. Using a similar reasoning to that above we get $\overline{\lambda}(g_f - \varphi) = \rho(g_f) = \rho(f) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \rho_2(g_f) = \rho_2(f) = n$.

7. Proof of Theorem 1.4

Suppose that f is a solution of equation (1.15) with $\rho(f) = \infty$ and $\rho_2(f) = \rho$. Set $w = f - \varphi$. Then by $\rho(\varphi) < \infty$, we have $\rho(w) = \rho(f - \varphi) = \rho(f) = \infty$ and $\rho_2(w) = \rho_2(f - \varphi) = \rho_2(f) = \rho$. Substituting $f = w + \varphi$ into equation (1.15), we obtain

$$w^{(k)} + A_{k-1}w^{(k-1)} + ... + A_1 w' + A_0 w$$
$$= F - \left(\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + ... + A_1\varphi' + A_0\varphi\right) = W. \quad (5.1)$$

Since φ is not a solution of equation (1.15), we have $W \not\equiv 0$. By Lemma 2.3 and Lemma 2.4, we get $\overline{\lambda}(w) = \overline{\lambda}(f - \varphi) = \rho(w) = \rho(f - \varphi) = \infty$ and $\overline{\lambda}_2(w) = \overline{\lambda}_2(f - \varphi) = \rho_2(w) = \rho_2(f - \varphi) = \rho$, i. e., $\overline{\lambda}(f - \varphi) = \rho(f) = \infty$ and $\overline{\lambda}_2(f - \varphi) = \rho_2(f) = \rho$.

8. Proof of Corollary 1.4

Suppose that $f(z) \not\equiv 0$ is a solution of equation (1.16). Then by Lemma 2.6, we have $\rho(f) = \infty$ and $\rho_2(f) = n$. By using Theorem 1.4, we obtain Corollary 1.4.

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