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Hopfield-type neural networks systems with piecewise constant argument

M. U. Akhmet[a,*], E. Yılmaz[a]

[a] *Department of Mathematics and Institute of Applied Mathematics, Middle East Technical University, 06531 Ankara, Turkey*

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Abstract. In this paper we consider Hopfield-type neural networks systems with piecewise constant argument of generalized type. Sufficient conditions for the existence of a unique equilibrium and a periodic solution are obtained. The stability of these solutions is investigated.

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1. Introduction and preliminaries

In recent years, dynamics of delayed neural networks have been studied and developed by many authors and many applications have been found in different areas such as associative memory, image processing, signal processing, pattern recognition and optimization (see [5, 7, 9, 10] and references cited therein). As is well known, such applications depend on the existence of an equilibrium point and its stability.

Differential equations with piecewise constant argument combine the properties of both the differential and difference equations. They play an important role in applications, see, for example, [11, 13]. Investigation of differential equations with piecewise constant arguments of delay and advanced type had been initiated in [6, 12], where the method of research was based on the reduction to discrete equations. Hence, qualitative properties of solutions which start at non-integer values can not be achieved. Particularly, one can not investigate the problem of stability completely, since only elements of a countable set are allowed to be discussed for initial moments. By introducing arbitrary piecewise constant functions as arguments, which can be interpreted

E-mail addresses: marat@metu.edu.tr (M. U. Akhmet), enes@metu.edu.tr (E. Yılmaz)
*Corresponding author. M. U. Akhmet is previously known as M. U. Akhmetov.

as piecewise constant deviated argument, the concept of differential equations with piecewise constant argument has been generalized in [1],[3], where an integral representation formula was proposed as another approach to meet the challenges discussed above.

One of the most crucial idea of the present paper is that we assume Hopfield-type neural networks may "memorize" values of the phase variable at certain moments of time to utilize the values during middle process till the next moment. Thus, we arrive to differential equations with piecewise constant delay. Obviously, the distances between the moments may be very variable. Consequently, the concept of generalized type of piecewise constant argument may be fruitful for the theory of neural networks.

Let us denote the set of all real numbers, natural numbers and integers by $\mathbb{R}, \mathbb{N}, \mathbb{Z}$, respectively, and a norm on $\mathbb{R}^m$ by $||\cdot||$ where $||u|| = \sum_{j=1}^{m} |u_i|$.

In the present paper we shall consider the following Hopfield-type neural networks system with piecewise constant argument

$$\begin{aligned} x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(t))) + d_i, \\ & \quad a_i > 0, \ i = 1, 2, \cdots, m. \end{aligned} \tag{1.1}$$

where $\beta(t) = \theta_k$ if $\theta_k \le t < \theta_{k+1}$, $k \in \mathbb{Z}$, $t \in \mathbb{R}$, is an identification function, $\theta_k, k \in \mathbb{Z}$, is a strictly increasing sequence of real numbers, $|\theta_k| \to \infty$ as $|k| \to \infty$, and there exists a positive real number $\bar{\theta}$ such that $\theta_{k+1} - \theta_k \le \bar{\theta}, k \in \mathbb{Z}$. Moreover, m denotes the number of neurons in the network, $x_i(t)$ corresponds to the state of the ith unit at time t, $f_j(x_j(t))$ and $g_j(x_j(\beta(t)))$ denote, respectively, the measures of activation to its incoming potentials of the unit j at time t and $\theta_k, k \in \mathbb{Z}$; b_{ij}, c_{ij}, d_i are real constants; b_{ij} denotes the synaptic connection weight of the unit j on the unit i at time t, c_{ij} denotes the synaptic connection weight of the unit j on the unit i at time θ_k, d_i is the input from outside the network to the unit i.

The following assumptions will be needed throughout the paper:

(C1) The activation functions $f_j, g_j \in C(\mathbb{R}, \mathbb{R})$ with $f_j(0) = 0,\ g_j(0) = 0$ satisfy

$$|f_j(u) - f_j(v)| \le L_j |u - v|$$
$$|g_j(u) - g_j(v)| \le \bar{L}_j |u - v|$$

for all $u, v \in \mathbb{R}$, where $L_j, \bar{L}_j > 0$ are Lipschitz constants, for $j = 1, 2, \ldots, m$;

(C2) $\bar{\theta}\left[\alpha_3 + \alpha_2\right] < 1$;

(C3) $\bar{\theta}\Big[\alpha_2 + \alpha_3\left(1 + \bar{\theta}\alpha_2\right) e^{\bar{\theta}\alpha_3}\Big] < 1$,

where

$$\alpha_1 = \sum_{i=1}^{m}\sum_{j=1}^{m} |b_{ji}| L_i, \ \alpha_2 = \sum_{i=1}^{m}\sum_{j=1}^{m} |c_{ji}| \bar{L}_i, \ \alpha_3 = \sum_{i=1}^{m} a_i + \alpha_1.$$

Theorem 1.1. *Suppose (C1) holds. If the neural parameters* a_i, b_{ij}, c_{ij} *satisfy*

$$a_i > L_i \sum_{j=1}^{m} |b_{ji}| + \bar{L}_i \sum_{j=1}^{m} |c_{ji}|, \quad i = 1, \cdots, m.$$

Then, (1.1) has a unique equilibrium $x^* = (x_1^*, \cdots, x_m^*)^T$.

The proof of the theorem is almost identical to the verification in [10] with slight changes which are caused by the piecewise constant argument.

We understand a solution $x(t) = (x_1, \cdots, x_m)^T$ of (1.1) as a continuous function on $\mathbb{R}$ such that the derivative $x'(t)$ exists at each point $t \in \mathbb{R}$, with the possible exception of the points $\theta_k, k \in \mathbb{Z}$, where one-sided derivative exists and the differential equation (1.1) is satisfied by $x(t)$ on each interval (θ_k, θ_{k+1}) as well.

In the following theorem the conditions for the existence and uniqueness of solutions on $\mathbb{R}$ are established. The proof of the assertion is similar to that of Theorem 2.3 in [1].

Theorem 1.2. *Suppose that conditions (C1)-(C3) are fulfilled. Then, for every* $(t_0, x^0) \in \mathbb{R} \times \mathbb{R}^m$, *there exists a unique solution* $x(t) = x(t, t_0, x^0) = (x_1, \ldots, x_m)^T$, $t \in \mathbb{R}$, *of (1.1), such that* $x(t_0) = x^0$.

Now, let us give the following two equivalence lemmas of (1.1). The proofs are omitted here, since they are similar to that of Lemma 3.1 in [1].

Lemma 1.1. *A function* $x(t) = x(t, t_0, x^0) = (x_1, \ldots, x_m)^T$, *where* t_0 *is a fixed real number, is a solution of (1.1) on* $\mathbb{R}$ *if and only if it is a solution of the following integral equation on* $\mathbb{R}$*: For* $i = 1, \cdots, m$,

$$\begin{aligned} x_i(t) &= e^{-a_i(t-t_0)} x_i^0 + \int_{t_0}^{t} e^{-a_i(t-s)} \Big[\sum_{j=1}^{m} b_{ij} f_j(x_j(s)) \\ &\quad + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(s))) + d_i \Big] ds. \end{aligned} \tag{1.2}$$

Lemma 1.2. *A function* $x(t) = x(t, t_0, x^0) = (x_1, \ldots, x_m)^T$, *where* t_0 *is a fixed real number, is a solution of (1.1) on* $\mathbb{R}$ *if and only if it is a solution of the following integral equation on* $\mathbb{R}$*: For* $i = 1, \cdots, m$,

$$\begin{aligned} x_i(t) &= x_i^0 + \int_{t_0}^{t} \Big[- a_i x_i(s) + \sum_{j=1}^{m} b_{ij} f_j(x_j(s)) \\ &\quad + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(s))) + d_i \Big] ds. \end{aligned} \tag{1.3}$$

2. Stability of equilibrium

In this section, we will give sufficient conditions for the global asymptotic stability of the equilibrium x^*. The system (1.1) can be reduced as follows. Let $y_i = x_i - x_i^*$, for each $i = 1, \cdots, m$. Then,

$$\begin{aligned} y_i'(t) &= -a_i y_i(t) + \sum_{j=1}^{m} b_{ij}\phi_j(y_j(t)) + \sum_{j=1}^{m} c_{ij}\psi_j(y_j(\beta(t))), \\ & \quad i = 1, 2, \cdots, m, \end{aligned} \tag{2.1}$$

where $\phi_i(y_i) = f_i(y_i + x_i^*) - f_i(x_i^*)$ and $\psi_i(y_i) = g_i(y_i + x_i^*) - g_i(x_i^*)$. For each $j = 1, \cdots, m$, $\phi_j(\cdot)$, $\psi_j(\cdot)$, are Lipschitzian since $f_j(\cdot)$, $g_j(\cdot)$ are Lipschitzian with $L_j, \bar{L}_j$ respectively, and $\phi_j(0) = 0$, $\psi_j(0) = 0$.

For simplicity of notation in the sequel, let us denote

$$\zeta = \left\{1 - \bar{\theta}\left[\alpha_2 + \alpha_3\left(1 + \bar{\theta}\alpha_2\right)e^{\bar{\theta}\alpha_3}\right]\right\}^{-1}.$$

The following lemma, which plays an important role in the proofs of further theorems has been considered in [4]. But, for convenience of the reader we place the full proof of the assertion.

Lemma 2.1. *Let $y(t) = (y_1(t), \cdots, y_m(t))^T$ be a solution of (2.1) and (C1)-(C3) be satisfied. Then, the following inequality*

$$||y(\beta(t))|| \leq \zeta ||y(t)|| \tag{2.2}$$

holds for all $t \in \mathbb{R}$.

Proof. For a fixed $t \in \mathbb{R}$, there exists $k \in \mathbb{Z}$ such that $t \in [\theta_k, \theta_{k+1})$. Then, from Lemma 1.2, we have

$$\begin{aligned} ||y(t)|| &= \sum_{i=1}^{m} |y_i(t)| \\ &\leq ||y(\theta_k)|| + \sum_{i=1}^{m}\int_{\theta_k}^{t}\left[a_i|y_i(s)| + \sum_{j=1}^{m}|b_{ji}|L_i|y_i(s)| + \sum_{j=1}^{m}|c_{ji}|\bar{L}_i|y_i(\theta_k)|\right]ds \\ &\leq (1 + \bar{\theta}\alpha_2)||y(\theta_k)|| + \int_{\theta_k}^{t}\alpha_3||y(s)||ds. \end{aligned}$$

The Gronwall-Bellman Lemma yields that

$$||y(t)|| \leq \left(1 + \bar{\theta}\alpha_2\right)e^{\bar{\theta}\alpha_3}||y(\theta_k)||. \tag{2.3}$$

Furhermore, for $t \in [\theta_k, \theta_{k+1})$ we have

$$\begin{aligned} ||y(\theta_k)|| &\leq ||y(t)|| + \sum_{i=1}^{m}\int_{\theta_k}^{t}\left[a_i|y_i(s)| + \sum_{j=1}^{m}|b_{ji}|L_i|y_i(s)| + \sum_{j=1}^{m}|c_{ji}|\bar{L}_i|y_i(\theta_k)|\right]ds \\ &\leq ||y(t)|| + \bar{\theta}\alpha_2||y(\theta_k)|| + \int_{\theta_k}^{t}\alpha_3||y(s)||ds. \end{aligned}$$

The last inequality and (2.3) imply that

$$||y(\theta_k))|| \quad \leq \quad ||y(t)|| + \bar{\theta}\alpha_2||y(\theta_k)|| + \bar{\theta}\alpha_3\left(1+\bar{\theta}\alpha_2\right)e^{\bar{\theta}\alpha_3}||y(\theta_k)||.$$

Thus, it follows from condition (C3) that

$$||y(\theta_k)|| \leq \zeta||y(t)||, \quad t \in [\theta_k, \theta_{k+1}).$$

Accordingly, (2.2) holds for all $t \in \mathbb{R}$, which is the desired conclusion. □

From now on we need the following assumption:

(C4) $\gamma - \alpha_1 - \zeta\alpha_2 > 0$, where $\gamma = \min\limits_{1\leq i\leq m} a_i$ is positive.

Theorem 2.1. *Assume that (C1)-(C4) are fulfilled.Then, the zero solution of (2.1) is globally asymptotically stable.*

Proof. Let $y(t) = (y_1(t), \cdots, y_m(t))^T$ be an arbitrary solution of (2.1). From Lemma 1.1, we have

$$\begin{aligned} ||y(t)|| &\leq e^{-\gamma(t-t_0)}||y_0|| + \sum_{i=1}^{m}\int_{t_0}^{t} e^{-\gamma(t-s)}\left[\sum_{j=1}^{m}|b_{ji}|L_i|y_i(s)| + \sum_{j=1}^{m}|c_{ji}|\bar{L}_i|y_i(\beta(s))|\right]ds \\ &\leq e^{-\gamma(t-t_0)}||y_0|| + (\alpha_1+\zeta\alpha_2)\int_{t_0}^{t} e^{-\gamma(t-s)}||y(s)||ds. \end{aligned}$$

It follows that

$$e^{\gamma(t-t_0)}||y(t)|| \quad \leq \quad ||y_0|| + (\alpha_1+\zeta\alpha_2)\int_{t_0}^{t} e^{\gamma(s-t_0)}||y(s)||ds.$$

By virtue of Gronwall-Bellman inequality, we obtain that

$$||y(t)|| \leq e^{-(\gamma-\alpha_1-\zeta\alpha_2)(t-t_0)}||y_0||.$$

The last inequality, in conjunction with $(C4)$, deduces that the zero solution of system (2.1) is globally asymptotically stable. □

3. Existence and stability of periodic solutions

In this part, we study the existence and global asymptotic stability of the periodic solution of (1.1). The following conditions are to be assumed:

(C5) there exists a positive integer p such that $\theta_{k+p} = \theta_k + \omega,\ k \in \mathbb{Z}$ with a fixed positive real period ω;

(C6) $\kappa\left[\omega\left(\alpha_1+\zeta\alpha_2\right)\right]<1, \quad$ where $\kappa=\frac{1}{1-e^{-\gamma\omega}}$.

Theorem 3.1. *Assume that conditions (C1)-(C3) and (C5)-(C6) are valid. Then, the system (1.1) has a unique ω–periodic solution.*

We omit the proof of this assertion, since it can be proved in the same way as existence of the periodic solution for the quasilinear system of ordinary differential equations in noncritical case [8].

Theorem 3.2. *Assume that conditions (C1)-(C6) are valid. Then, the periodic solution of (1.1) is globally asymptotically stable.*

Proof. By Theorem 3.1, we know that (1.1) has an ω–periodic solution $x^*(t) = (x_1^*,\cdots,x_m^*)^T$. Suppose that $x(t)=(x_1,\cdots,x_m)^T$ is an arbitrary solution of (1.1) and let $z(t)=x(t)-x^*(t)=(x_1-x_1^*,\cdots,x_m-x_m^*)^T$. Then, from Lemma 1.1, we have

$$\begin{aligned}||z(t)|| &\leq e^{-\gamma(t-t_0)}||z_0||+\sum_{i=1}^{m}\int_{t_0}^{t}e^{-\gamma(t-s)}\left[\sum_{j=1}^{m}|b_{ji}|L_i|z_i(s)|+\sum_{j=1}^{m}|c_{ji}|\bar{L}_i|z_i(\beta(s))|\right]ds\\ &\leq e^{-\gamma(t-t_0)}||z_0||+(\alpha_1+\zeta\alpha_2)\int_{t_0}^{t}e^{-\gamma(t-s)}||z(s)||ds.\end{aligned}$$

Also, the previous inequality can be written as,

$$e^{\gamma(t-t_0)}||z(t)|| \quad\leq\quad ||z_0||+(\alpha_1+\zeta\alpha_2)\int_{t_0}^{t}e^{\gamma(s-t_0)}||z(s)||ds.$$

By applying Gronwall-Bellman inequality, we obtain that

$$||z(t)||\leq e^{-(\gamma-\alpha_1-\zeta\alpha_2)(t-t_0)}||z_0||.$$

Thus, using (C4), the periodic solution of system (1.1) is globally asymptotically stable. □

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