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Creating a chaos in a system with relay

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Abstract. We address a special initial value problem of a differential equation with relay function. The concept of Li-Yorke chaos [8] is considered.

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1. Introduction and Preliminaries

The method of construction of chaotic motions has been proposed in [1]-[3]. We consider a special initial value problem for relay systems and impulsive systems, whose initial moments of time are from a Cantor set. Using the map, which is topologically conjugate to symbolic dynamics, as the generator of moments of the relay switching in the multidimensional system, we observe in paper [1] Devaney's ingredients of chaos for a relay system with linear elements. Existence of a quasi-minimal set has been proved in [3]. The approach has been used, also, in [2] to construct the Li-Yorke chaos [8] for impulsive differential equations. In the present article we attempt to shape the chaos for the multidimensional non-linear relay system.

Let us recall the definition of chaos for maps. Consider an infinite nonvoid compact metric space (X, ρ) with metric ρ and a continuous map $T : X \to X$. A pair $(x, x') \in X \times X, x \neq x'$, is called a *Li-Yorke pair* [5] if it is *proximal* and *not asymptotic*, that is, $\liminf_{i\to\infty} \rho(T^i(x), T^i(\tilde{x})) = 0$ and $\limsup_{i\to\infty} \rho(T^i(x), T^i(\tilde{x})) > 0$, respectively.

The map $T : X \to X$ is Li-Yorke chaotic, if: it has points with all periods $p \in \mathbb{N}$; there exists an uncountable subset $X' \subset X$, the scrambled set, that does not contain periodic points and each pair $(x, x') \in X' \times X', x \neq x'$, is a Li-Yorke pair. Consider the sequence space [9]

$$\Sigma_2 = \{s = (s_0 s_1 s_2 \ldots) : s_j = 0 \text{ or } 1\}$$

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with the metric
$$d[s,t]=\sum_{i=0}^{\infty}\frac{|s_i-\tilde{s}_i|}{2^i},$$
where $\tilde{s}=(\tilde{s}_0\tilde{s}_1\ldots)\in\Sigma_2$, and the shift map $\sigma:\Sigma_2\to\Sigma_2$, such that $\sigma(s)=(s_1s_2\ldots)$. The pair (Σ_2,σ) is the symbolic dynamics. The map is continuous, $cardPer_n(\sigma)=2^n, Per(\sigma)$ is dense in Σ_2, and there exists a dense orbit in Σ_2. It is known that the dynamics (σ,Σ_2) is chaotic in the sense of Li-Yorke with a scrambled set Σ_2'.

Let $h:\Lambda\to\Lambda$, where Λ is a subset of the interval $[0,1]$, be a map topologically conjugate to σ, and Λ' is an image of Σ_2' by the congjugacy.

For every $t_0\in\Lambda$ one can construct a sequence $\kappa(t_0)$ of real numbers $\kappa_i, i\geq 0$, such that $\kappa_{i+1}=h(\kappa_i)$ and $\kappa_0=t_0$. Sequence $\zeta(t_0)=\{\zeta_i(t_0)\}$ in (2.1) is defined as $\zeta_i(t_0)=i+\kappa_i(t_0), i\geq 0$.

By applying the conjugacy of h and σ, one can verify that map h has the following useful chaotic properties.

Lemma 1.1. *If $t,t'\in\Lambda'$, then there exist sequences $k_i,l_i\to\infty$, as $i\to\infty$, such that $\max_{j=0,1,\ldots,l_i}|h^{k_i+j}(t)-h^{k_i+j}(t')|\to 0$ as $i\to\infty$.*

Lemma 1.2. *There exists a positive number η, such that for every pair $t,t'\in\Lambda', t\neq t'$, there exists a sequence $m_i\to\infty$, as $i\to\infty$, such that $|h^{m_i}(t)-h^{m_i}(t')|\geq\delta$.*

2. The Li-Yorke chaos

The main object of our investigation is the following special initial value problem
$$\begin{aligned}&z'(t)=Az(t)+f(z)+v(t,t_0),\\&z(t_0)=z_0,(t_0,z_0)\in\Lambda\times\mathbb{R}^n,\end{aligned}\tag{2.1}$$
where $z\in\mathbb{R}^n, t\in\mathbb{R}_+=[0,\infty), i\geq 0$. Cantor set $\Lambda\subset I=[0,1]$, and sequence of impulsive moments $\zeta_i(t_0)$ were described in the last section.
$$v(t,t_0)=\begin{cases}m_0 & \text{if} \quad \zeta_{2i}(t_0)<t\leq\zeta_{2i+1}(t_0),\ i\in\mathbb{Z},\\ m_1 & \text{if} \quad \zeta_{2i-1}(t_0)<t\leq\zeta_{2i}(t_0),\ i\in\mathbb{Z},\end{cases}$$
where $m_0,m_1\in\mathbb{R}^n$ are vectors. The function f satisfies the Lipshitz condition with a positive constant L, A is an $n\times n$ constant real valued matrix with real parts of eigenvalues all negative. Denote the maximal of them $\alpha<0$.

For a fixed $t_0\in\Lambda$, system (2.1) is a differential equation with discontinuous right hand side of a specific type when discontinuities happen on vertical planes in the $(t,z)-$space.

A function $z(t), z(t_0)=z_0$, is a solution of (2.1) on $[t_0,\infty)$ if: (i) $z(t)$ is continuous on $[t_0,\infty)$; (ii) the derivative $z'(t)$ exists at each point $t\in\mathbb{R}$ with the possible exception of the points $\zeta_i(t_0)$, where left-sided derivatives exist; (iii) equation (2.1) is satisfied on each interval $(\zeta_i(t_0), \zeta_{i+1}(t_0)], i \geq 0$.

It can be easily verified that problem (2.1) has a unique solution $z(t, t_0, z_0)$ for each $t_0 \in \Lambda, z_0 \in \mathbb{R}^n$.

There exists a positive number N such that $\|e^{At}\| \leq Ne^{\alpha t}, t \geq 0$.

The solution $z(t) = z(t, t_0, z_0), t_0 \in \Lambda, z_0 \in \mathbb{R}^n$, of (2.1) satisfies the following integral equation

$$z(t) = e^{A(t-t_0)}z_0 + \int_{t_0}^{t} e^{A(t-s)}[f(z(s)) + v(s, t_0)]\, ds. \tag{2.2}$$

In what follows we assume that $\sup_{\mathbb{R}^n} |f(z)| = M_0 < \infty, NL < \alpha$. Fix a sequence $\zeta(t_0), t_0 \in \Lambda$. Using the standard technique one can verify that all solutions eventually, as t increases, enter the tube with the radius $M = M_0[1 + \frac{N}{\alpha - NL}], t \in \mathbb{R}$. Moreover, if the sequence $\kappa(t_0)$ is periodic with a period $p \in \mathbb{N}$, then there is a solution of (2.1) with the same period, and its integral curve is placed in the tube. One can easily see that all these solutions are different for different p. Let us, introduce the following distance. If ϕ, ψ are continuous on $\mathbb{R}$ functions, then denote $\|\phi(t) - \psi(t)\|_J = \sup_J \|\phi(t) - \psi(t)\|$, where J is an interval of $\mathbb{R}$. We use the following definitions. They are taken from [5, 8, 9] and adapted for (2.1).

Definition 2.1. *A pair of solutions of (2.1)* $z(t) = z(t, t_0, z_0)$, $z_1(t) = z(t, t_1, z_1)$, $t_0, t_1 \in \Lambda$, *is proximal if for each* $\epsilon > 0, E > 0$ *there exists an interval* $J \subset [t_0, \infty)$ *with length not less than* E *such that* $\|z_1(t) - z(t)\|_E < \epsilon$.

Definition 2.2. *The solutions of (2.1)* $z(t) = z(t, t_0, z_0)$, $z_1(t) = z(t, t_1, z_1)$, $t_0, t_1 \in \Lambda$, *are not asymptotic if there exist positive numbers* ϵ_0 *and a sequence* ξ_i, $i \geq 0$, $\xi_i \to \infty$, *as* $i \to \infty$, *such that* $||z_1(\xi_i) - z(\xi_i)|| > \epsilon_0$.

Definition 2.3. *A couple* $z(t) = z(t, t_0, z_0), z_1(t) = z(t, t_1, z_1)$, $t_0, t_1 \in \Lambda$, *of solutions of (2.1) is a Li-Yorke pair if they are proximal and not asymptotic.*

Definition 2.4. *Problem (2.1) is Li-Yorke chaotic on* Λ' *if:*

1. *there exist solutions* $\phi(t, t_0)$ *with all periods* $p \in \mathbb{N}$;
2. *each couple of solutions* $z(t) = z(t, t_0, z_0)$, $z_1(t) = z(t, t_1, z_1)$, *with* $t_0, t_1 \in \Lambda', t_0 \neq t_1$, *is Li-Yorke pair;*

Theorem 2.1. *Problem (2.1) is Li-Yorke chaotic on* Λ'.

Proof. Let us show that each pair of solutions is proximal. Fix numbers $t_0, t_1 \in \Lambda', t_0 \neq t_1$, solutions $z(t) = z(t, t_0, z_0)$, $z_1(t) = z(t, t_1, z_1)$, $z_0, z_1 \in \mathbb{R}^n$, of (2.1), and $E, \epsilon > 0$. There exists a number $\bar{T}$ such that both solutions z, z_1 are in the tube with the radius M if $t \geq \bar{T}$. By the proximal property of map h, Lemma 1.1, and its uniform continuity, there exist arbitrarily large numbers $\tilde{T} > \bar{T}, E_1 > 0$, such that

$\|\zeta_i(t_1)-\zeta_i(t_0)\| < \delta$, where $\zeta_i(t_1), \zeta_i(t_0) \in (\tilde{T}, \tilde{T}+E_1+E)$. We shall find a sufficiently large E_1 so that $\|z(t)-z_1(t)\|_J < \epsilon$ if $J = (\tilde{T}+E_1, \tilde{T}+E_1+E)$. We have that

$$\begin{aligned} z(t) &= e^{At}z(\tilde{T}) + \int_{\tilde{T}}^{t} e^{A(t-s)} f(z(s))ds + \int_{\tilde{T}}^{t} e^{A(t-s)} v(s,t_0)ds, \\ z_1(t) &= e^{At}z(\tilde{T}) + \int_{\tilde{T}}^{t} e^{A(t-s)} f(z_1(s))ds + \int_{\tilde{T}}^{t} e^{A(t-s)} v(s,t_1)ds. \end{aligned}$$

Consequently,

$$\begin{aligned} \|z(t)-z_1(t)\| &\le Ne^{\alpha(t-\tilde{T})}\|z(\tilde{T})-z_1(\tilde{T})\| + \int_{\tilde{T}}^{t} Ne^{\alpha(t-s)}L\|z(s)-z_1(s)\|ds \\ &\quad + \int_{\tilde{T}}^{t} Ne^{\alpha(t-s)}\|v(s,t_0)-v(s,t_1)\|ds \\ &\le Ne^{\alpha(t-\tilde{T})}\|z(\tilde{T})-z_1(\tilde{T})\| + \int_{\tilde{T}}^{t} Ne^{\alpha(t-s)}L\|z(s)-z_1(s)\|ds \\ &\quad + \int_{\tilde{T}}^{t} Ne^{\alpha(t-s)}\delta\|m_0-m_1\|ds. \end{aligned}$$

Next, we denote $u(t) = \|z(t)-z_1(t)\|e^{-\alpha t}$, and apply Lemma 2.2 [6], to obtain that

$$\|z(t)-z_1(t)\| \le \frac{N\delta\|m_0-m_1\|}{\alpha+NL}[e^{(\alpha+NL)(t-\tilde{T})}-1] + Ne^{(\alpha+NL)(t-\tilde{T})}\|z(\tilde{T})-z_1(\tilde{T})\|.$$

On the basis of the last inequality one can easily see that $\|z(t)-z_1(t)\| < \epsilon$ if $t \in J$, where E_1 is sufficiently large, and δ is a sufficiently small positive number.

Consider a pair of solutions $z(t) = z(t,t_0,z_0), z_1(t) = z(t,t_1,z_1)$, with $t_0, t_1 \in \Lambda'$, $t_0 \ne t_1$. By Lemma 1.2 there exists a sequence i_k, $i_k \to \infty$ as $k \to \infty$, such that $|\kappa_{i_k}(t_0) - \kappa_{i_k}(t_1)| > \eta$.

Fix i_k, and assume that $\kappa_{i_k}(t_0) < \kappa_{i_k}(t_1)$. The case $\kappa_{i_k}(t_0) > \kappa_{i_k}(t_1)$ can be analyzed similarly. There exists a positive number ν, sufficiently small so that

$$\nu_1 = -\frac{N\|m_0-m_1\|}{\alpha}[1-e^{\alpha\eta}] - N\nu e^{\alpha\eta} + \frac{NL\nu}{\alpha}[1-e^{\alpha\eta}] > 0.$$

Denote $\epsilon_0 = \min\{\nu, \nu_1\}$. We shall show that there is a number ξ_k between $\zeta_{i_k}(t_0)$ and $\zeta_{i_k}(t_1)$ that satisfies $\|z(\xi_k)-z_1(\xi_k)\| \ge \epsilon_0$. Assume on the contrary that $\|z(t)-z_1(t)\| <$

ϵ_0, $t \in [\zeta_{i_k}(t_0), \zeta_{i_k}(t_1)]$. Then,

$$\begin{aligned}
&\|z(\zeta_{i_k}(t_1)+\eta) - z_1(\zeta_{i_k}(t_1)+\eta)\| \\
&\qquad \geq \quad \|e^{A\eta}\|\|z(\zeta_{i_k}(t_1)) - z_1(\zeta_{i_k}(t_1)) \\
&\qquad\qquad - \int_{\zeta_{i_k}(t_1)}^{\zeta_{i_k}(t_1)+\eta} \|e^{A(\zeta_{i_k}(t_1)+\eta-s)}\|\|z(s) - z_1(s)\|ds \\
&\qquad\qquad - \int_{\zeta_{i_k}(t_1)}^{\zeta_{i_k}(t_1)+\eta} \|e^{A(\zeta_{i_k}(t_1)+\eta-s)}\|\|v(s,t_0) - v(s,t_1)\|ds \\
&\qquad \geq \quad -\frac{N\|m_0 - m_1\|}{\alpha}[1 - e^{\alpha\eta}] - N\nu e^{\alpha\eta} + \frac{NL\nu}{\alpha}[1 - e^{\alpha\eta}] \geq \epsilon_0.
\end{aligned}$$

We get a contradiction, which proves the assertion. Evaluations made do not depend on the choice of k. Existence of periodic solutions is obvious. The theorem is proved. □

Remark 2.1. *The constant ϵ_0 is common for all chaotic solutions in the last proof. In paper [4] we have weakened the condition by discussing a map, which is conjugate to a Li-Yorke chaotic map, which is not necessarily the symbolic dynamics.*

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References

[1] M. U. Akhmet, Devaney chaos of a relay system, Communications in Nonlinear Science and Numerical Simulation, 14 (2009) 1486-1493.

[2] M. U. Akhmet, Li-Yorke chaos in the system with impacts, J. Math. Anal. Appl. 351 (2009) 804-810.

[3] M. U. Akhmet, Dynamical synthesis of quasi-minimal sets, International Journal of Bifurcation and Chaos, Int. J. Bifurcation and Chaos, 19, no. 7, (2009) 1-5.

[4] M. U. Akhmet, Shadowing property of the hybrid system (submitted).

[5] E. Akin, S. Kolyada, Li-Yorke sensitivity, Nonlinearity 16 (2003) 1421–1433.

[6] E. A. Barbashin, Introduction to the theory of stability, Wolters-Noordhoff Publishing, Groningen, 1970.

[7] R. Devaney, An introduction to chaotic dynamical systems, Addison-Wesley, Menlo Park California, 1990.

[8] T. Y. Li, J. Yorke, Period three implies chaos, Amer. Math. Monthly 87 (1975) 985-992.

[9] C. Robinson, Dynamical Systems: Stability, Symbolic dynamics, and Chaos, CRC Press, Boca Raton, Ann Arbor, London, Tokyo, 1995.