International Journal of Qualitative Theory of Differential Equations and Applications
Vol. 1, No. 2 (2007), pp. 117–123

Existence of Unbounded Solutions in Rational Equations

E. Camouzis[a], E. A. Grove[b], Y. Kostrov[b], and G. Ladas[b,*]

[a]*Department of Mathematics and Natural Sciences, The American College of Greece, 6 Gravias Street, Aghia Paraskevi, 15342 Athens, Greece*
[b]*Department of Mathematics, University of Rhode Island, Kingston, Rhode Island, 02881-0816, USA*

Received April 12, 2007; accepted July 27, 2007

Abstract. We exhibit a range of parameters and a set of initial conditions where the rational difference equation

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^{2k} \beta_i x_{n-i}}{A + \sum_{j=0}^{k} B_{2j} x_{n-2j}}$$

has unbounded solutions.

AMS Subject Classifications: 39A10, 39A11

Keywords: Existence of unbounded solutions; Rational difference equations

1. Introduction

The existence of unbounded solutions in rational difference equations has recently been investigated in a series of papers. See [1]-[18].

We establish the existence of unbounded solutions in the rational difference equation

E-mail addresses: camouzis@acgmail.gr (E. Camouzis), grove@math.uri.edu (E. A. Grove), ekostrov@yahoo.com (Y. Kostrov), gladas@math.uri.edu (G. Ladas)
*Corresponding author

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^{2k} \beta_i x_{n-i}}{A + \sum_{j=0}^{k} B_{2j} x_{n-2j}} \quad , \qquad n = 0, 1, \ldots \tag{1.1}$$

where k is a positive integer, and where the parameters and initial conditions are non-negative real numbers chosen such that the the denominator is always positive. More precisely we exhibit a range of parameters and a set of initial conditions where Eq.(1.1) has unbounded solutions.

This result extends the known results of the special case #195 (see [11])

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + x_n} \quad , \qquad n = 0, 1, \ldots .$$

This result also includes the fourth order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2} + \varepsilon x_{n-3}}{A + B x_n + D x_{n-2}} \quad , \qquad n = 0, 1, \ldots$$

and shows that the 21 special cases #286, #344-351, #412-415, and #472-479 of Eq.(1.1) have unbounded solutions.

For the notation of the special cases, see [10] and [17].

2. Existence of Unbounded Solutions

In this section, we establish the existence of unbounded solutions of Eq.(1.1).

Theorem 2.1. *Assume that* $B_0 > 0$. *Set*

$$U = \frac{\beta_0 + \beta_2 + \cdots + \beta_{2k}}{B_0}$$

and assume that

$$\beta_1 > A + U(B_0 + B_2 + \cdots + B_{2k}).$$

Then Eq.(1.1) has unbounded solutions.

Proof. Let $\varepsilon > 0$ be chosen such that

$$\beta_1 > A + (U + \varepsilon)(B_0 + B_2 + \cdots + B_{2k}) \tag{2.1}$$

and let $\{x_n\}_{n=-2k}^{\infty}$ be a solution of Eq.(1.1) such that

$$0 < x_0, x_{-2}, \ldots, x_{-2k+2}, x_{-2k} < U + \varepsilon \tag{2.2}$$

and

$$x_{-1} > x_{-3} > \cdots > x_{-2k+1} > \frac{\alpha + \beta_1 x_0 + \beta_3 x_{-2} + \cdots + \beta_{2k-3} x_{-2k+4} + \beta_{2k-1} x_{-2k+2}}{\varepsilon B_0}. \tag{2.3}$$

We claim that for all $n \geq 0$, we have

$$x_{2n+1} > \frac{\alpha + \beta_0 x_n + \beta_2 x_{n-2} + \cdots + \beta_{2k-2} x_{n-(2k-2)} + \beta_{2k} x_{n-2k}}{A + B_0 x_n + B_2 x_{n-2} + \cdots + B_{2k-2} x_{n-(2k-2)} + B_{2k} x_{n-2k}} \tag{2.4}$$

and

$$0 < x_{2n+2} < U + \varepsilon. \tag{2.5}$$

Note that

$$\begin{aligned} x_1 &= \frac{\alpha + \beta_0 x_0 + \beta_1 x_{-1} + \beta_2 x_{-2} + \cdots + \beta_{2k-1} x_{-2k+1} + \beta_{2k} x_{-2k}}{A + B_0 x_0 + B_2 x_{-2} + \cdots + B_{2k-2} x_{-2k+2} + B_{2k} x_{-2k}} \\ &\geq \frac{\alpha + \beta_0 x_0 + \beta_2 x_{-2} + \cdots + \beta_{2k-2} x_{-2k+2} + \beta_{2k} x_{-2k}}{A + B_0 x_0 + B_2 x_{-2} + \cdots + B_{2k-2} x_{-2k+2} + B_{2k} x_{-2k}} \\ &\quad + \frac{\beta_1 x_1}{A + B_0 x_0 + B_2 x_{-2} + \cdots + B_{2k-2} x_{-2k+2} + B_{2k} x_{-2k}}. \end{aligned}$$

Now

$$\frac{\beta_1}{A + B_0 x_0 + B_2 x_{-2} + \cdots + B_{2k-2} x_{-2k+2} + B_{2k} x_{-2k}} > 1$$

if and only if

$$\beta_1 > A + B_0 x_0 + B_2 x_{-2} + \cdots + B_{2k-2} x_{-2k+2} + B_{2k} x_{-2k}.$$

We know by (2.1) that

$$\begin{aligned} \beta_1 &> A + (U+\varepsilon)(B_0 + B_2 + \cdots + B_{2k}) \\ &= A + B_0(U+\varepsilon) + B_2(U+\varepsilon) + \cdots + B_{2k}(U+\varepsilon) \\ &> A + B_0 x_0 + B_2 x_{-2} + \cdots + B_{2k} x_{-2k}. \end{aligned}$$

It follows that

$$x_1 > \frac{\alpha + \beta_0 x_0 + \beta_2 x_{-2} + \cdots + \beta_{-2k+2} + \beta_{2k} x_{-2k}}{A + B_0 x_0 + B_2 x_{-2} + \cdots + B_{2k-2} x_{-2k+2} + B_{2k} x_{-2k}} + x_{-1}.$$

Now $x_2 > 0$ since $\beta_1 > 0$.

Also,

$$\begin{aligned} x_2 &= \frac{\alpha+\beta_0 x_1+\beta_1 x_0+\beta_2 x_{-1}+\cdots+\beta_{2k-1}x_{-2k+2}+\beta_{2k}x_{-2k+1}}{A+B_0x_1+B_2x_{-1}+\cdots+B_{2k-2}x_{-2k+3}+B_{2k}x_{-2k+1}} \\ &\leq \frac{\beta_0 x_1}{B_0 x_1}+\frac{(\beta_2+\beta_4+\cdots+\beta_{2k})x_1}{B_0x_1} \\ &\quad +\frac{\alpha+\beta_1x_0+\beta_3x_{-2}+\cdots+\beta_{2k-1}x_{-2k+2}}{A+B_0x_1+B_2x_{-1}+B_4x_{-3}+\cdots+B_{2k}x_{-2k+1}}. \end{aligned}$$

Now

$$\begin{aligned} &\frac{\alpha+\beta_1x_0+\beta_3x_{-2}+\cdots+\beta_{2k-1}x_{-2k+2}}{A+B_0x_1+B_2x_{-1}+B_4x_{-3}+\cdots+B_{2k}x_{-2k+1}} \\ &\qquad\leq \frac{\alpha+\beta_1x_0+\beta_3x_{-2}+\cdots+\beta_{2k-1}x_{-2k+2}}{B_0x_1+B_2x_{-1}+B_4x_{-3}+\cdots+B_{2k}x_{-2k+1}} \\ &\qquad\leq \frac{\alpha+\beta_1x_0+\beta_3x_{-2}+\cdots+\beta_{2k-1}x_{-2k+2}}{(B_0+B_2+B_4+\cdots+B_{2k})x_{-2k+1}}. \end{aligned}$$

Recall that

$$\begin{aligned} &\frac{\alpha+\beta_1x_0+\beta_3x_{-2}+\cdots+\beta_{2k-1}x_{-2k+2}}{B_0+B_2+B_4+\cdots+B_{2k}}\cdot\frac{1}{x_{-2k+1}} \\ &\qquad< \frac{\alpha+\beta_1x_0+\beta_3x_{-2}+\cdots+\beta_{2k-1}x_{-2k+2}}{B_0+B_2+B_4+\cdots+B_{2k}} \\ &\qquad\quad\times\frac{\varepsilon B_0}{\alpha+\beta_1x_0+\beta_3x_{-2}+\cdots+\beta_{2k-1}x_{-2k+1}} \\ &\qquad= \frac{\varepsilon B_0}{B_0+B_2+B_4+\cdots B_{2k}} \\ &\qquad\leq \varepsilon. \end{aligned}$$

Thus

$$x_2 < \frac{\beta_0}{B_0}+\frac{\beta_2+\beta_4+\cdots+\beta_{2k}}{B_0}+\varepsilon = U+\varepsilon.$$

The proof of the claim follows by induction.

In particular, $\{x_{2n+1}\}_{n=-k}^{\infty}$ is a strictly monotonically increasing sequence of positive real numbers.

We claim that

$$\lim_{n\to\infty} x_{2n+1} = \infty.$$

For the sake of contradiction, suppose that there exists a positive real number $S > 0$ such that

$$\lim_{n\to\infty} x_{2n+1} = S.$$

Similarly to the method of Full Limiting Sequences (see [15] and [16]). there exist non-negative real numbers $L_1, L_0, L_{-1}, \ldots, L_{-2k+1}, L_{-2k}$ and a sub-sequence $\{x_{n_i}\}_{i=0}^{\infty}$ of $\{x_n\}_{n=-k}^{\infty}$ such that the following statements are true:

1. $L_{-2j+1} = \lim_{i\to\infty} x_{n_i-(2j-1)} = S$ for all $0 \le j \le k$.
2. $L_{-2j} = \lim_{i\to\infty} x_{n_i-2j} \le U + \varepsilon$ for all $0 \le j \le k$.

Hence

$$\begin{aligned} S &= L_1 = \lim_{i\to\infty} x_{n_i+1} \\ &= \lim_{i\to\infty} \frac{\alpha + \beta_0 x_{n_i} + \beta_1 x_{n_i-1} + \beta_2 x_{n_i-2} + \cdots + \beta_{2k} x_{n_i-2k}}{A + B_0 x_{n_i} + B_2 x_{n_i-2} + \cdots + B_{2k} x_{n_i-2k}} \\ &\ge \lim_{i\to\infty} \frac{\beta_1 x_{n_i-1}}{A + B_0 x_{n_i} + B_2 x_{n_i-2} + \cdots + B_{2k} x_{n_i-2k}} \\ &= \frac{\beta_1 L_{-1}}{A + B_0 L_0 + B_2 L_{-2} + \cdots + B_{2k} L_{-2k}} \\ &= \frac{\beta_1 S}{A + B_0 L_0 + B_2 L_{-2} + \cdots + B_{2k} L_{-2k}}. \end{aligned}$$

So as $S > 0$ and $\beta_1 S > 0$, it follows that

$$A + B_0 L_0 + B_2 L_{-2} + \cdots + B_{-2k} L_{-2k} > 0.$$

Thus

$$S = L_1 = \frac{\alpha + \beta_0 L_0 + \beta_1 L_{-1} + \beta_2 L_{-2} + \cdots + \beta_{2k} L_{-2k}}{A + B_0 L_0 + B_2 L_{-2} + \cdots + B_{2k} L_{-2k}}$$

and so

$$\begin{aligned} & S(A + B_0 L_0 + B_2 L_{-2} + \cdots + B_{2k} L_{-2k}) \\ &= \alpha + \beta_0 L_0 + \beta_1 L_{-1} + \beta_{-2} L_{-2} + \cdots + \beta_{2k} L_{-2k} \\ &= \alpha + (\beta_0 L_0 + \beta_2 L_{-2} + \cdots + \beta_{2k} L_{-2k}) + S(\beta_1 + \beta_3 + \cdots + \beta_{2k-1}). \end{aligned}$$

It follows that

$$\begin{aligned}
S[A+(B_0+B_2+\cdots+B_{2k})(U+\varepsilon)] \;&\geq\; S[A+B_0L_0+B_2L_{-2}+\cdots+B_{2k}L_{-2k}] \\
&= \alpha+(\beta_0L_0+\beta_2L_{-2}+\cdots+\beta_{2k}L_{-2k})+S(\beta_1+\beta_3+\cdots+\beta_{2k-1}) \\
&= \alpha+(\beta_0L_0+\beta_2L_{-2}+\cdots+\beta_{2k}L_{-2k})+S\beta_1+S(\beta_3+\cdots+\beta_{2k-1}) \\
&> \alpha+(\beta_0L_0+\beta_2L_{-2}+\cdots+\beta_{2k}L_{-2k}) \\
&\qquad +S[A+(U+\varepsilon)(B_0+B_2+\cdots+B_{2k})]+S(\beta_3+\cdots+\beta_{2k-1}) \\
&\geq S[A+(B_0+B_2+\cdots+B_{2k})(U+\varepsilon)],
\end{aligned}$$

which is a contradiction, and the proof is complete. □

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