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STABILITY IN A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAYS

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1. Introduction

Stability properties differential equations can be of great importance in applications. For linear delay equations stability of the trivial ($x(t) = 0$) solution is determined by the location of the zeros of its characteristic equation. Necessary and sufficient conditions for stability in terms of the parameters (coefficients, delays) of the equation are known only for the simplest equations, even in the case of linear constant delay equations. There are numerous sufficient conditions for guaranteeing stability for special equations (see e.g. [6]). One possible approach to find sufficient stability conditions is, analogously to the ODEs case, Liapunov's method. But, unfortunately, there is no general strategy to construct a Liapunov functional for a given equation, and if the equation is complicated (nonlinear, with several time- or state-dependent delays), obtaining a Liapunov functional can be very difficult if not impossible.

For nonlinear autonomous ODEs the linearization method is a very useful one, since we can deduce stability properties of the solution of the nonlinear equation from that of the corresponding linear equation, which is significantly easier to check. Recently, Cooke and Huang ([2]) extended this method for nonlinear delay equations with state-dependent delays of the form

$$\dot{x}(t) = g\left(x_t, \int_{-r_0}^{0} d\eta(s) g\Big(x(t+s-\tau(x_t))\Big)\right), \tag{1.1}$$

where $\tau : C \to [0, r_1]$, η is a matrix valued function of bounded variation, $r_0 > 0$, and r is such that $r \geq r_0 + r_1$.

The nonlinear delay system with state dependent delays

$$\dot{x}(t) = f\left(t, x(t), \int_{-r}^{0} d_s\mu(s,t,x_t)\, x(t+s)\right), \qquad t \geq 0, \tag{1.2}$$

was investigated in [7]. The term

$$\int_{-r}^{0} d_s\mu(s,t,x_t)x(t+s) \tag{1.3}$$

describing the delay dependence is a Stieltjes-integral of the solution segment $x(t+\cdot)$ with respect to $\mu(\cdot, t, x_t)$, which is a matrix valued function of bounded variations depending on time, t, and the state of the equation, x_t. Here $r > 0$ is fixed and $x_t : [-r, 0] \to \mathbb{R}^n$, $x_t(s) \equiv x(t+s)$.

To give some motivation and/or justification on the particular form selected by Eq. (1.3) for the delay terms, assume for example that the delayed term depends linearly on the state, i.e., has the form Lx_t, where L is a bounded linear operator on $C \equiv C([-r, 0], \mathbb{R}^n)$. In this case the Riesz Representation Theorem yields Eq. (1.3) with $\mu = \mu(s)$. If $L = L(t)$ depends on t, then by the same result we get that there exists $\mu = \mu(s, t)$ such that Eq. (1.3) holds. Therefore it seems like a natural extension of the above cases to assume the structure described by Eq. (1.3) for the state-dependent case. Moreover, representation Eq. (1.3) includes discrete and distributed constant and time-dependent delays, and the "usual" state-dependent delays, $x(t - \tau(t, x(t))$ or $x(t - \tau(t, x_t))$ as well. A nice feature of this form is that it also allows delayed terms of the form

$$\Lambda(t, x_t) = \sum_{i=1}^{\infty} A_i(t, x_t)x(t - \tau_i(t, x_t)) + \int_{-\tau_0}^{0} G(s, t, x_t)x(t+s)\, ds.$$

In this paper we shall obtain a linearization test similar to that of [2] for the autonomous version of Eq. (1.2). Note, that despite the significant technical differences between our presentation and that of [2] due to the different form of the two equations, the main ideas are of course the same, since both follow the steps of the proof of the ODEs case (see e.g. [10]), and the two results are equivalent in the sense that they both provide the same linear equation for nonlinear equations which can be rewritten in both forms. Example 4.4 will show an equation, which is not included in Eq. (1.1), but is covered by Eq. (1.2), and of course, examples can be constructed for the opposite direction as well.

We note, that the main difficulty to obtain linearization results for state-dependent delay equations is that it is difficult to differentiate the delayed term in the presence of state-dependent delays (see a detailed discussion of differentiability of solutions with respect to parameters for state-dependent delay equations in [7]). We shall define a bounded linear operator, $\mathcal{F} : C \to \mathbb{R}^n$ (see Eq. (3.5) below), as a candidate for the linearized equation about the trivial solution. This is not the "true" linearization at zero, since the delayed term is not necessarily differentiable at zero (in the space C), but using assumption (H2) (ii), we can get an estimate on the error replacing the right hand side of the equation by $\mathcal{F}x_t$ (see Lemma 3.2 below), which turns out to be sufficient to prove that the asymptotic stability of the corresponding linearized equation, Eq. (3.8), implies that of the nonlinear equation, Eq. (2.1).

Section 3 contains the main results, and in Section 4 we illustrate the method on several examples with constant, time- and state-dependent delays. In Section 6 we summarize the well-posedness results of [7] for Eq. (1.2).

2. Preliminaries

Consider the nonlinear state-dependent delay system (the autonomous version of Eq. (1.2))

$$\dot{x}(t) = f\left(x(t), \int_{-r}^{0} d_s\mu(s, x_t)\, x(t+s)\right), \qquad t \geq 0 \tag{2.1}$$

with initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{2.2}$$

(See also Chapter 5 in [7].) Introduce the simplifying notations:

$$\Lambda(\psi) \equiv \int_{-r}^{0} d_s\mu(s, \psi)\psi(s) \tag{2.3}$$

and

$$\lambda(\psi, \xi) \equiv \int_{-r}^{0} d_s\mu(s, \psi)\xi(s). \tag{2.4}$$

Then, of course, $\Lambda(\psi) = \lambda(\psi, \psi)$, and Eq. (2.1) can be written as

$$\dot{x}(t) = f(x(t), \Lambda(x_t)), \qquad t \geq 0.$$

We assume the following conditions throughout the paper:

(H1) (i) $f : \Omega_1 \times \Omega_2 \to \mathbb{R}^n$ is continuously differentiable, where Ω_1 and Ω_2 are open subsets of $\mathbb{R}^n$,

(ii) $0 \in \Omega_1 \cap \Omega_2$, and $f(0, 0) = 0$,

(H2) $\mu(\cdot, \psi)$ is a matrix valued function of bounded variation for every $\psi \in \Omega_3$, where $\Omega_3 \subset C$ open, such that

(i) $\sup\left\{\left|\int_{-r}^{0} d_s\mu(s, \psi)\xi(s)\right| \ : \ \psi \in \Omega_3, \ \xi \in C, \ |\xi|_C \leq 1\right\} < \infty,$

(ii) for every $\alpha > 0$ and $M > 0$ there exists a constant $L_2 = L_2(\alpha, M)$ such that for all $\xi \in W^{1,\infty}$, $t \in [0, \alpha]$ and $\psi, \bar{\psi} \in \Omega_3$, $|\psi|_C$, $|\bar{\psi}|_C \leq M$,

$$|\lambda(\psi, \xi) - \lambda(\bar{\psi}, \xi)| \leq L_2 |\xi|_{W^{1,\infty}} |\psi - \bar{\psi}|_C,$$

(H3) $\varphi \in W^{1,\infty}$, i.e., φ is Lipschitz-continuous.

Here $W^{1,\infty}$ is the Sobolev space of absolutely continuous functions $\psi : [-r, 0] \to \mathbb{R}^n$ with essentially bounded derivatives. The norm in this Banach-space is defined by $|\psi|_{W^{1,\infty}} \equiv \max\{\sup_{s\in[-r,0]} |\psi(s)|, \operatorname{ess\,sup}_{s\in[-r,0]} |\dot{\psi}(s)|\}$.

It is easy to see that in order have a well-posed problem, the initial function φ and the function μ have to satisfy that

$$\varphi(0) \in \Omega_1, \;\; \varphi \in \Omega_3, \quad \text{and} \quad \int_{-r}^{0} d_s\mu(s,\varphi)\,\varphi(s) \in \Omega_2. \tag{2.5}$$

We recall the following result from [7] concerning the well-posedness of IVP (2.1)-(2.2).

Theorem 2.1 *Assume that $(\bar{\varphi}, \mu, f)$ satisfy (H1)–(H3) and Eq. (2.5). Then there exist $\alpha > 0$ and $\delta > 0$ such that IVP (2.1)-(2.2) corresponding to (φ, μ, f) has unique solution on $[0, \alpha]$ for all $|\varphi - \bar{\varphi}|_C < \delta$.*

In the remaining part of this section we recall some results from [6] which we shall need in the sequel. Consider a linear delay equation with constant delays of the form:

$$\dot{x}(t) = \mathcal{L}x_t, \qquad t \geq 0, \tag{2.6}$$

where $\mathcal{L} \; : \; C \to \mathbb{R}^n$ is a bounded linear operator. It is well-known (e.g. [6]), that Eq. (2.6) has a unique solution, $x(t;\varphi)$, corresponding to any initial function $\varphi \in C$, defined on $t \in [-r, \infty)$. Moreover (see e.g. [6]), the family of linear operators, $\{S(t)\}_{t\geq 0}$, given by

$$S(t)\varphi \equiv x(\cdot;\varphi)_t, \qquad t \geq 0$$

defines a strongly continuous semigroup on C.

Let define

$$\omega_0 \equiv \sup\Big\{Re\,\lambda \;:\; det(\lambda I - \mathcal{L}e^{\lambda \cdot}) = 0\Big\},$$

i.e., ω_0 is the supremum of the real part of the characteristic roots of Eq. (2.6). We shall need the following lemma:

Lemma 2.2 (see e.g. in [6]) *If $\omega_0 < 0$, then for any $\omega_0 < \omega < 0$ there exists $M = M(\omega) \geq 1$ such that*

$$\|S(t)\| \leq Me^{\omega t}, \qquad t \geq 0.$$

Consider the perturbed equation

$$\dot{x}(t) = \mathcal{L}x_t + g(t), \qquad t \geq 0, \tag{2.7}$$

where $g \in L^1_{\text{loc}}([0,\infty), \mathbb{R}^n)$. Then Eq. (2.7) has a unique solution on $[0,\infty)$ for all initial function $\varphi \in C$, and the solution, $x(t)$ satisfies the following abstract variation of constant formula:

Lemma 2.3 (see e.g. [6]) *The solution, $x(t)$, of Eq. (2.7), corresponding to an initial function $\varphi \in C$ has the form:*

$$x_t = S(t)\varphi + \int_0^t S(t-s)X_0 g(s)\, ds,$$

where

$$X_0 \ : \ [-r,0] \to \mathbb{R}^{n\times n}, \qquad X_0(u) \equiv \begin{cases} 0, & u<0, \\ I, & u=0. \end{cases} \tag{2.8}$$

We shall need the following variation of Lemma 2.3.

Lemma 2.4 *The solution, $x(t)$, of Eq. (2.7) satisfies*

$$x_t = S(t-r)x_r + \int_0^{t-r} S(t-r-s)X_0 g(s+r)\, ds, \qquad t \geq r,$$

where X_0 is defined by Eq. (2.8).

Proof By applying Lemma 2.3, semigroup properties of $S(t)$, and change of variables we get

$$\begin{aligned} x_t &= S(t)\varphi + \int_0^t S(t-s)X_0 g(s)\, ds \\ &= S(t-r)S(r)\varphi + S(t-r)\int_0^r S(r-s)X_0 g(s)\, ds + \int_r^t S(t-s)X_0 g(s)\, ds \\ &= S(t-r)x_r + \int_0^{t-r} S(t-r-s)X_0 g(s+r)\, ds, \end{aligned}$$

which proves the lemma. □

3. Main results

First we introduce constants which we shall use throughout this section.

It follows from the assumption that Ω_1 and Ω_2 are open subsets of $\mathbb{R}^n$ and $0 \in \Omega_1 \cap \Omega_2$ that there exists a constant $\delta_1 > 0$ such that $\overline{\mathcal{G}}_{\mathbb{R}^n}(\delta_1) \subset \Omega_1 \cap \Omega_2$. Assumption (H1) implies that there exists a constant $L_1 = L_1(\delta_1)$ such that

$$|f(x,y) - f(\bar{x},\bar{y})| \leq L_1(|x-\bar{x}| + |y-\bar{y}|), \qquad \text{for } x,\bar{x},y,\bar{y} \in \overline{\mathcal{G}}_{\mathbb{R}^n}(\delta_1). \tag{3.1}$$

Assumption (H2) (i) and the linearity of $\lambda(\psi,\xi)$ in ξ yield that there exists a constant $L_3 > 0$ such that

$$|\lambda(\psi,\xi)| \leq L_3|\xi|_C, \qquad \psi \in \Omega_3. \tag{3.2}$$

Inequality (3.2) and $|x(t)| \leq |x_t|_C$ yield that

$$x(t) \in \overline{\mathcal{G}}_{\mathbb{R}^n}(\delta_1) \quad \text{and} \quad \Lambda(x_t) \in \overline{\mathcal{G}}_{\mathbb{R}^n}(\delta_1) \qquad \text{for } x_t \in \overline{\mathcal{G}}_C(\delta_2), \tag{3.3}$$

where $\delta_2 \equiv \delta_1 \min\{1, 1/L_3\}$.

We shall need the following estimate.

Lemma 3.1 *Assume (H1)–(H3). Let x be the solution of IVP (2.1)-(2.2) corresponding to initial function φ satisfying $|\varphi|_C \leq \delta_2$. Assume that $\alpha > 0$ is such that $|x_t| \leq \delta_2$ for $0 \leq t \leq \alpha$. Then the solution x satisfies the inequality*

$$|x_t| \leq |\varphi|_C \exp\Big(L_1(1+L_3)t\Big), \qquad t \in [0, \alpha].$$

Proof Let $\alpha > 0$ satisfy the condition of the lemma, and let $t \in [0, \alpha]$. The integrated form of Eq. (2.1), and relations (3.1), (3.3) and (H1) (ii) yield the following estimates.

$$\begin{aligned} |x(t)| &\leq |\varphi(0)| + \int_0^t |f(x(u), \Lambda(x_u))|\, du \\ &\leq |\varphi|_C + L_1 \int_0^t |x(u)| + |\Lambda(x_u)|\, du \\ &\leq |\varphi|_C + L_1 \int_0^t |x(u)| + L_3|x_u|_C\, du. \end{aligned} \tag{3.4}$$

The assumption $|\varphi|_C \leq \delta_2$ and Eq. (3.4) imply that

$$\max_{-r \leq v \leq t} |x(v)| \leq |\varphi|_C + L_1(1+L_3) \int_0^t \max_{-r \leq v \leq u} |x(v)|\, du, \qquad t \in [0, \alpha],$$

which, using Gronwall-Bellman inequality, yields the statement of the lemma. □

Define the linear operator

$$\mathcal{F} \;:\; C \to \mathbb{R}^n, \qquad \mathcal{F}\psi \equiv \frac{\partial f}{\partial x}(0,0)\psi(0) + \frac{\partial f}{\partial y}(0,0)\lambda(0,\psi) \tag{3.5}$$

and the function

$$G \;:\; C \to \mathbb{R}^n, \qquad G(\psi) \equiv f(\psi(0), \Lambda(\psi)) - \mathcal{F}\psi. \tag{3.6}$$

Note, that $\mathcal{F}$ is a bounded operator, since by Eq. (3.2) it follows that

$$|\mathcal{F}\psi| \leq \left(\left\| \frac{\partial f}{\partial x}(0,0) \right\| + \left\| \frac{\partial f}{\partial y}(0,0) \right\| L_3 \right) |\psi|_C.$$

By this notation we can rewrite Eq. (2.1) as

$$\dot{x}(t) = \mathcal{F}x_t + G(x_t), \qquad t \geq 0, \tag{3.7}$$

and therefore we can consider it as a perturbation of the constant delay equation

$$\dot{x}(t) = \mathcal{F}x_t, \qquad t \geq 0 \tag{3.8}$$

by the function G.

We shall need the following estimate of G.

Lemma 3.2 *Assume (H1)–(H3). There exists a constant $N > 0$ such that for every $\eta > 0$ there exists a constant $\theta = \theta(\eta) > 0$ such that*

$$|G(\psi)| \leq N\Big(\eta + |\psi|_{W^{1,\infty}}\Big)|\psi|_C \tag{3.9}$$

for all $\psi \in W^{1,\infty}$ such that $|\psi|_C \leq \theta$.

Proof The definition of $\mathcal{F}$, (H1), and elementary estimates imply

$$\begin{aligned}
|G(\psi)| &\leq \left| f\Big(\psi(0), \Lambda(\psi)\Big) - \frac{\partial f}{\partial x}(0,0)\psi(0) - \frac{\partial f}{\partial y}(0,0)\lambda(0,\psi) \right| \\
&= \left| f\Big(\psi(0), \Lambda(\psi)\Big) - f(0,0) - \frac{\partial f}{\partial x}(0,0)\psi(0) - \frac{\partial f}{\partial y}(0,0)\lambda(0,\psi) \right| \\
&\leq \sup_{0\leq\nu\leq1} \left\| \frac{\partial f}{\partial x}(\nu\psi(0), \nu\Lambda(\psi)) - \frac{\partial f}{\partial x}(0,0) \right\| |\psi(0)| + \left\| \frac{\partial f}{\partial y}(0,0) \right\| |\Lambda(\psi) - \lambda(0,\psi)| \\
&\quad + \sup_{0\leq\nu\leq1} \left\| \frac{\partial f}{\partial y}(\nu\psi(0), \nu\Lambda(\psi)) - \frac{\partial f}{\partial x}(0,0) \right\| |\Lambda(\psi)|\,.
\end{aligned} \tag{3.10}$$

By the continuous differentiability of f guaranteed by (H1) (i), for every $\eta > 0$ there exists $0 < \theta_1(\eta) \leq \delta_1$ such that if $|x|, |y| < \theta_1(\eta)$ then

$$\left\| \frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(0,0) \right\| < \eta \quad \text{and} \quad \left\| \frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(0,0) \right\| < \eta$$

It follows from Eq. (3.2), $\theta_1(\eta) \leq \delta_1$ and the definition of δ_2 that the constant $\theta = \theta(\eta) \equiv \theta_1(\eta)\min\{1, 1/L_3\}$ satisfies $\theta \leq \delta_2$, and if $\psi \in \overline{\mathcal{G}}_C(\theta)$ then

$$\left\| \frac{\partial f}{\partial x}(\nu\psi(0), \nu\Lambda(\psi)) - \frac{\partial f}{\partial x}(0,0) \right\| < \eta \quad \text{and} \quad \left\| \frac{\partial f}{\partial y}(\nu\psi(0), \nu\Lambda(\psi)) - \frac{\partial f}{\partial y}(0,0) \right\| < \eta \tag{3.11}$$

for all $0 \leq \nu \leq 1$. It follows from assumption (H2) (ii) with $L_2 = L_2(\delta_1)$, $\theta \leq \delta_2$ and Eq. (3.3), that for $\psi \in \overline{\mathcal{G}}_C(\theta) \cap W^{1,\infty}$

$$\begin{aligned} |\Lambda(\psi) - \lambda(0,\psi)| &= |\lambda(\psi,\psi) - \lambda(0,\psi)| \\ &\leq L_2(\delta_1)|\psi|_{W^{1,\infty}}|\psi|_C. \end{aligned} \tag{3.12}$$

By combining Eq. (3.10), Eq. (3.11) and Eq. (3.12) we get for $\psi \in \overline{\mathcal{G}}_C(\theta) \cap W^{1,\infty}$ that

$$\begin{aligned} |G(\psi)| &\leq \eta|\psi|_C + \eta L_3|\psi|_C + \left\| \frac{\partial f}{\partial y}(0,0) \right\| L_2(\delta_1)|\psi|_{W^{1,\infty}}|\psi|_C \\ &\leq N(\eta + |\psi|_{W^{1,\infty}})|\varphi|_C, \end{aligned}$$

where $N \equiv \max\left\{1 + L_3,\ \left\| \frac{\partial f}{\partial y}(0,0) \right\| L_2(\delta_1)\right\}$. □

Let $S(t)$ be the semigroup generated by the linear constant-delay Eq. (3.8), and ω_0 be the supremum of the real part of the characteristic roots of equation Eq. (3.8). (See Section 2 for the definition of $S(t)$ and ω_0.) We show that the stability properties of the trivial solution of the nonlinear state-dependent autonomous equation Eq. (2.1) can be obtained by that of the linear constant-delay Eq. (3.8).

Theorem 3.3 *Assume (H1)–(H3), and that the semigroup $S(t)$ is asymptotically stable, i.e., $\omega_0 < 0$. Then for every $\omega > \omega_0$ there exist $K = K(\omega) > 0$ and $\delta = \delta(\omega) > 0$ such that for all $\varphi \in G_C(\delta)$ the corresponding solution, $x(t)$, of IVP (2.1)-(2.2) is defined for $t \in [0,\infty)$, and satisfies*

$$|x(t)| \leq K e^{\omega t}|\varphi|_C, \qquad t \geq 0.$$

Proof Fix an arbitrary $\omega_0 < \omega < 0$ and fix ω^* such that $\omega_0 < \omega^* < \omega$. Then by Lemma 2.2, there exists a constant $M = M(\omega^*) \geq 1$ such that

$$|S(t)\varphi|_C \leq M e^{\omega^* t}|\varphi|_C, \qquad t \geq 0, \quad \varphi \in C. \tag{3.13}$$

Let $x(t)$ be the solution of Eq. (3.7) (or equivalently Eq. (2.1)) corresponding to an initial function $\varphi \in C$. By Lemma 2.4 we get that

$$x_t = S(t-r)x_r + \int_0^{t-r} S(t-r-s)X_0 G(x_{s+r})\, ds, \qquad t \geq r, \tag{3.14}$$

where X_0 is defined by Eq. (2.8).

Let $N > 0$ be the constant given by Lemma 3.2, define

$$\eta \equiv \frac{\omega - \omega^*}{4MN},$$

and let $\theta(\eta)$ be the constant corresponding to this η from Lemma 3.2. Finally, define two more constants

$$\delta_3 \equiv \min\left\{\delta_2,\ \frac{\omega-\omega^*}{4MN},\ \frac{\omega-\omega^*}{4MNL_1(1+L_3)\delta_2},\ \theta(\eta)\right\},$$

and

$$\delta \equiv \delta_3 \exp\Big(-L_1(1+L_3)r\Big)\frac{1}{M}e^{\omega^* r}.$$

We comment, that $\frac{1}{M}e^{\omega^* r} \le 1$ since $M \ge 1$ and $\omega^* < 0$, and hence $\delta \le \delta_3 \le \delta_2$.

Let $|\varphi|_C < \delta$. Then by Eq. (3.3) and $\delta \le \delta_2$ it follows that $\varphi(0) \in \Omega_1$ and $\Lambda(\varphi) \in \Omega_2$, and therefore Theorem 2.1 implies that there exists a solution if IVP (2.1)-(2.2) $x(t)$ corresponding to φ on an interval $[0,\alpha]$. Since, by Eq. (3.3) and Theorem 2.1, the solution is continuable till $x_t \in \mathcal{G}_C(\delta_2)$, and since Lemma 3.1 and the definition of δ imply the relation $|x_r|_C < \delta_3 \le \delta_2$, it follows that there exists $r < t_1 \le \alpha$ such that $|x_t|_C < \delta_3$ on $t \in [0,t_1)$. Suppose that there exists t_2 such that $r < t_2 \le \alpha$ and the solution satisfies

$$|x_t|_C < \delta_3 \qquad \text{for } t \in [0,t_2), \quad \text{and} \quad |x_{t_2}|_C = \delta_3. \tag{3.15}$$

For $t \in [r,t_2)$ and $|\varphi|_C \le \delta$, estimate Eq. (3.1), Eq. (3.2), Eq. (3.15), $\delta_3 \le \delta_2$ and the definition of δ_3 imply that

$$\begin{aligned} |\dot{x}(t)| &= |f(x(t),\Lambda(x_t))| \\ &\le L_1(|x(t)| + |\Lambda(x_t)|) \\ &\le L_1(1+L_3)|x_t| \\ &\le L_1(1+L_3)\delta_3 \\ &\le \frac{\omega-\omega^*}{4MN}. \end{aligned} \tag{3.16}$$

Then Eq. (3.16) yields that

$$\sup_{t-r\le s\le t} |\dot{x}(s)| \le \frac{\omega-\omega^*}{4MN},$$

and hence, by using Eq. (3.15), we also have

$$|x_t|_{W^{1,\infty}} \le \frac{\omega-\omega^*}{4MN}, \qquad \text{for } t \in [r,t_2), \quad |\varphi|_C \le \delta. \tag{3.17}$$

Since for $t \in [r,t_2)$, $|\varphi|_C < \delta_3$ and $0 \le s \le t$ relation Eq. (3.15) yields that $|x_{s+r}|_C \le \delta_3 \le \theta(\eta)$, then Lemma 3.2, Eq. (3.13), Eq. (3.14), Eq. (3.17) and the relation $|X_0 z|_C = |z|$ (for $z \in \mathbb{R}^n$) imply that

$$|x_t|_C \ \le\ \|S(t-r)\||x_r|_C + \int_0^{t-r} \|S(t-r-s)\||G(x_{s+r})|\,ds$$

$$
\begin{aligned}
&\le Me^{\omega^*(t-r)}|x_r|_C + \int_0^{t-r} MNe^{\omega^*(t-r-s)}\Big(\eta + |x_{s+r}|_{W^{1,\infty}}\Big)|x_{s+r}|_C\,ds \\
&\le Me^{\omega^*(t-r)}|x_r|_C + \int_r^{t} MNe^{\omega^*(t-s)}\Big(\eta + \frac{\omega-\omega^*}{4MN}\Big)|x_s|_C\,ds.
\end{aligned}
$$

Multiplying both sides by $e^{-\omega^* t}$ and changing a variable in the integral we get

$$
|x_t|_C e^{-\omega^* t} \le Me^{-\omega^* r}|x_r|_C + \int_r^t MNe^{-\omega^* s}\Big(\eta + \frac{\omega-\omega^*}{4MN}\Big)|x_s|_C\,ds.
$$

Applying Gronwall-Bellman inequality for the function $|x_t|_C e^{-\omega^* t}$ we get

$$
|x_t|_C e^{-\omega^* t} \le Me^{-\omega^* r}|x_r|_C \exp\left(MN\left(\eta + \frac{\omega-\omega^*}{4MN}\right)t\right), \qquad r \le t \le t_2,
$$

or equivalently, for $r \le t \le t_2$

$$
|x_t|_C \le Me^{-\omega^* r}|x_r|_C \exp\left(\left(MN\left(\eta + \frac{\omega-\omega^*}{4MN}\right) + \omega^*\right)t\right).
$$

From the definition of η it follows that

$$
\begin{aligned}
|x_t|_C &\le Me^{-\omega^* r}|x_r|_C \exp\left(\left(\frac{\omega-\omega^*}{2} + \omega^*\right)t\right) \\
&< Me^{-\omega^* r}|x_r|_C e^{\omega t}, \qquad r \le t \le t_2.
\end{aligned} \tag{3.18}
$$

Then this estimate, Lemma 3.1 and the definition of δ imply for $|\varphi|_C < \delta$ that

$$
\begin{aligned}
|x_t|_C &< Me^{-\omega^* r}|\varphi|_C e^{L_1(1+L_3)r} e^{\omega t} \\
&< \delta_3, \qquad r \le t \le t_2,
\end{aligned}
$$

which contradicts to the definition of t_2. Therefore $|x_t| < \delta_3$ for $r \le t \le \alpha$, but this implies that $\alpha = \infty$, and Eq. (3.18) holds for all $t \ge r$, therefore, by Eq. (3.15) and Eq. (3.18), the statement of the theorem is proved with $K \equiv Me^{\omega^* r}\delta_3$. □

Remark 3.4 *We note, that if $\omega_0 > 0$, i.e., the trivial solution of the linear equation is unstable, then so is the trivial solution of the nonlinear equation. Since instability results are of less interest in applications, and the detailed proof is rather lengthy, technical, and also similar to the state-independent case, we omit it. (See Section 10.1 in [6] for the state-independent case.)*

4. Applications

In this section we show examples, when by the linearization technique of the previous section, we can find conditions implying asymptotic stability of a nonlinear delay equation. The applicability of this linearization method depends on whether we are able to check the asymptotic stability of the linearized equation, which is a difficult problem in general, but in the examples we present in this section we can refer to existing conditions from the literature.

Example 4.1 Consider the scalar constant delay equation

$$\dot{x}(t) = -ax(t-1)(1+x(t)), \qquad t \geq 0, \quad (a>0). \tag{4.1}$$

This equation arises as we transform the delayed logistic equation

$$\dot{x}(t) = \tau x(t)(1 - x(t-\tau)/K)$$

by the new variable $y(t) = -1 + x(t)/K$, and change the time scale. (See e.g. [9].) It is known (e.g. [9]), that the trivial solution of Eq. (4.1) is asymptotically stable for $a < \pi/2$, and unstable for $a > \pi/2$. We can obtain this result by using Theorem 3.3. Equation Eq. (4.1) has the form Eq. (2.1) with $r = 1$, $f(x,y) = -ay(1+x)$ and $\lambda(\psi,\xi) = \xi(-1)$. Since $\frac{\partial f}{\partial x}(0,0) = 0$, $\frac{\partial f}{\partial y}(0,0) = -a$, the linearized equation Eq. (3.8) for this equation is

$$\dot{x}(t) = -ax(t-1), \qquad t \geq 0. \tag{4.2}$$

Since the trivial solution of Eq. (4.2) is asymptotically stable for $a < \pi/2$, and unstable for $a > \pi/2$ (see e.g. [6]), the same result holds for the trivial solution of Eq. (4.1) by Theorem 3.3 and Remark 3.4.

Example 4.2 Consider the scalar delay equation

$$\dot{x}(t) = x(t)\Big(a + bx(t-\tau) - cx^2(t-\tau)\Big), \qquad t \geq 0,$$

where $a > 0$ and $c > 0$. This is a delayed Lotka-Volterra type population model introduced by Gopalsamy and Ladas (see e.g. in [9]). The equation has a unique positive equilibrium point, $\bar{x} = (b + \sqrt{b^2 + 4ac})/(2c)$. By the new variable $y(t) = x(t) - \bar{x}$ we can transform the equilibrium point to zero, and get the equation

$$\dot{y}(t) = -(y(t) + \bar{x})\Big((2c\bar{x} - b)y(t-\tau) + cy^2(t-\tau)\Big), \qquad t \geq 0. \tag{4.3}$$

We can rewrite Eq. (4.3) in the form Eq. (2.1) with $f(u,v) = -(u+\bar{x})\Big((2c\bar{x} - b)v + cv^2\Big)$ and $\lambda(\psi,\xi) = \xi(-\tau)$. Since $\frac{\partial f}{\partial u}(0,0) = 0$ and $\frac{\partial f}{\partial v}(0,0) = -\bar{x}(2c\bar{x} - b)$, the linearized form of Eq. (4.3) is

$$\dot{x}(t) = -\bar{x}(2c\bar{x} - b)x(t-\tau), \qquad t \geq 0,$$

which is asymptotically stable if $0 < \bar{x}(2c\bar{x} - b)\tau < \pi/2$, or equivalently,

$$\frac{b\sqrt{b^2 + 4ac} + b^2 + 4ac}{2c}\tau < \frac{\pi}{2},$$

and therefore under this assumption the trivial solution of Eq. (4.3) is asymptotically stable as well.

Example 4.3 Consider the scalar delay equation with state-dependent delay

$$\dot{x}(t) = x(t)\Big(a - bx(t) - \sum_{i=1}^{m} b_i x(t - \tau_i) - cx(t - \tau(x_t))\Big), \qquad t \geq 0,$$

where

$$a > 0, \quad \text{and} \quad b > \sum_{i=1}^{m} |b_i| + |c|. \tag{4.4}$$

This population model with state-dependent delay term was studied in [1], where it was shown that Eq. (4.4) yields that the unique positive equilibrium, $\bar{x} = a/(b + \sum_{i=1}^{m} b_i + c)$, of the equation is globally asymptotically stable (for initial functions $\varphi(s) > M$ with some $M > 0$). We can show this result (for local asymptotic stability) by using linearization technique. By the new variable $y(t) = x(t) - \bar{x}$ we transform the equilibrium point to the origin, and the corresponding equation is

$$\dot{y}(t) = -(y(t) + \bar{x})\Big(by(t) + \sum_{i=1}^{m} b_i y(t - \tau_i) + cy(t - \tau(y_t + \bar{x}))\Big), \tag{4.5}$$

which has the form Eq. (2.1) with $f(u, v) = -(u+\bar{x})(bu+v)$, $\lambda(\psi, \xi) = \sum_{i=1}^{m} b_i \xi(-\tau_i) + c\xi(-\tau(\psi + \bar{x}))$. (Here and later, $\bar{x}$ in the argument of τ denotes a constant function with value equal to $\bar{x}$.) We have that $\frac{\partial f}{\partial u}(0, 0) = -b\bar{x}$, $\frac{\partial f}{\partial v}(0, 0) = -\bar{x}$, and $\lambda(0, \xi) = \sum_{i=1}^{m} b_i \xi(-\tau_i) + c\xi(-\tau(\bar{x}))$. Therefore the linearized equation of Eq. (4.5) is

$$\dot{x}(t) = -b\bar{x}x(t) - \bar{x}\left(\sum_{i=1}^{m} b_i x(t - \tau_i) + cx(t - \tau(\bar{x}))\right). \tag{4.6}$$

By a result from [6] (page 154) it follows that Eq. (4.4) yields the asymptotic stability of the trivial solution of Eq. (4.6), for arbitrary delay function $\tau(\cdot)$, which, by Theorem 3.3, implies that the trivial solution of Eq. (4.5) is asymptotically stable as well.

Example 4.4 Consider the scalar constant delay equation

$$\dot{x}(t) = \gamma x(t)\left(1 - \sum_{i=1}^{m} \frac{a_i x(t - \tau_i)}{1 + c_i x(t - \tau_i)}\right). \tag{4.7}$$

This is the so-called Michaelis-Menton single species growth equation (see e.g. in [9]). We assume that

$$\gamma > 0,\ a_i > 0,\ c_i > 0,\ \tau_i > 0, \quad \text{and} \quad \sum_{i=1}^{m} \frac{a_i}{1+c_i} = 1.$$

The last assumption yields that $\bar{x} = 1$ is a positive equilibrium point of Eq. (4.7). It was shown in [9] that $\gamma r \le 1$ implies the global asymptotic stability of $\bar{x}$, where $r = \max_{i=1,\dots,m} \tau_i$.

By letting $y(t) = x(t) - 1$, we get

$$\dot{y}(t) = -\gamma(y(t)+1) \sum_{i=1}^{m} \frac{a_i y(t-\tau_i)}{(1+c_i)(1+c_i+c_i y(t-\tau_i))}. \tag{4.8}$$

We can rewrite Eq. (4.8) in the form of Eq. (2.1), by selecting $f(u,v) = -\gamma(u+1)v$, and

$$\lambda(\psi,\xi) = \sum_{i=1}^{m} \frac{a_i}{(1+c_i)(1+c_i+c_i\psi(-\tau_i))}\xi(-\tau_i).$$

We have that $\frac{\partial f}{\partial u}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = -\gamma$, therefore the corresponding linearized equation is

$$\dot{x}(t) = -\gamma \sum_{i=1}^{m} \frac{a_i}{(1+c_i)^2} x(t-\tau_i). \tag{4.9}$$

By a condition from e.g. [5] or [8], it follows that the trivial solution of Eq. (4.8) is asymptotically stable if

$$\gamma \sum_{i=1}^{m} \frac{a_i}{(1+c_i)^2}\tau_i < 1.$$

It follows from the assumptions $\sum_{i=1}^{m} \frac{a_i}{1+c_i} = 1$, $c_i > 0$ and $r = \max_{i=1,\dots,m} \tau_i$ that

$$\gamma \sum_{i=1}^{m} \frac{a_i}{(1+c_i)^2}\tau_i < \gamma r \sum_{i=1}^{m} \frac{a_i}{1+c_i} = \gamma r,$$

therefore the condition $\gamma r \le 1$ implies that trivial solution of Eq. (4.9), and hence that of Eq. (4.8) is asymptotically stable.

Note, that the delayed term of Eq. (4.8) can not be written in the form given by the Stieltjes-integral in Eq. (1.1), and hence this equation is not included in Eq. (1.1) (without multiple delay terms).

Example 4.5 In [9] the scalar equation

$$\dot{x}(t) = f\left(\int_{-r}^{-\sigma} x(t+s)\,d\mu(s)\right) - g(x(t))$$

has been studied, where $r > \sigma > 0$, and

(i) $\mu(s)$ is nondecreasing and $\mu(-\sigma) - \mu(-r) = 1$,

(ii) $f(x)$ is strictly decreasing, $f(0) > 0$, $\lim_{x\to\infty} f(x) = 0$,

(iii) $g(x)$ is strictly increasing, $g(0) = 0$, $\lim_{x\to\infty} g(x) = \infty$,

and a condition was derived for the global asymptotic stability of the unique positive equilibrium.

We study the local asymptotic stability of the state-dependent version of this equation, i.e., consider

$$\dot{x}(t) = f\left(\int_{-r}^{-\sigma} x(t+s)\, d\mu(s, x_t)\right) - g(x(t)), \tag{4.10}$$

where we assume $r > \sigma > 0$, (ii), (iii) above and modify (i) as

(i') for all $\psi \in C$, the function $\mu(\cdot, \psi)$ is nondecreasing and $\mu(-\sigma, \psi) - \mu(-r, \psi) = 1$.

Under this assumptions, Eq. (4.10) has a unique positive equilibrium point, $\bar{x}$, since the function

$$\begin{aligned} f\left(\int_{-r}^{-\sigma} \bar{x}\, d\mu(s, \bar{x})\right) - g(\bar{x}) &= f\Big(\bar{x}(\mu(-\sigma, \bar{x}) - \mu(-r, \bar{x}))\Big) - g(\bar{x}) \\ &= f(\bar{x}) - g(\bar{x}) \end{aligned}$$

has a unique positive zero. (Here and later, $\bar{x}$ in the second argument of μ denotes a constant function with value $\bar{x}$.) Using $y(t) = x(t) - \bar{x}$ and an argument similar to the one above, we get

$$\dot{y}(t) = f\left(\int_{-r}^{-\sigma} y(t+s)\, d\mu(s, y_t + \bar{x}) + \bar{x}\right) - g\Big(y(t) + \bar{x}\Big). \tag{4.11}$$

We can rewrite Eq. (4.11) in the form Eq. (2.1) with $F(u, v) = f(v + \bar{x}) - g(u + \bar{x})$, and $\lambda(\psi, \xi) = \int_{-r}^{-\sigma} \xi(s)\, d\mu(s, \psi + \bar{x})$. We have that $\frac{\partial F}{\partial u}(0, 0) = -g'(\bar{x})$ and $\frac{\partial F}{\partial v}(0, 0) = f'(\bar{x})$. Therefore the linearized version of Eq. (4.11) is

$$\dot{x}(t) = -g'(\bar{x})x(t) + f'(\bar{x}) \int_{-r}^{-\sigma} x(t+s)\, d\mu(s, \bar{x}). \tag{4.12}$$

Note that $g'(\bar{x}) > 0$ and $f'(\bar{x}) < 0$ by the assumptions. Theorem 1.1 of [8] yields that the trivial solution of Eq. (4.12) is asymptotically stable if

$$-f'(\bar{x}) \int_{-r}^{-\sigma} s\, d\mu(s, \bar{x}) < \frac{3}{2},$$

and therefore by our theorem, if this condition is satisfied, then the trivial solution of Eq. (4.11) is asymptotically stable as well.

5. References

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6. Appendix

In this section, for the convenience of the reader, we summarize the well-posedness results of [7] for IVP

$$\dot{x}(t) = f\left(t, x(t), \int_{-r}^{0} d_s\mu(s,t,x_t)\, x(t+s)\right), \qquad t \in [0,T], \tag{6.1}$$

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{6.2}$$

In this section we use the notations

$$\lambda(t,\psi,\xi) \equiv \int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s),$$

and $\Lambda(t,\psi) \equiv \lambda(t,\psi,\psi)$. We assume the following hypotheses:

(A1) $f : [0,T] \times \Omega_1 \times \Omega_2 \to \mathbb{R}^n$ is continuous, where Ω_1 and Ω_2 are open subsets of $\mathbb{R}^n$,

(A2) $\mu(\cdot, t, \psi)$ is a matrix valued function of bounded variation for every $t \in [0,T]$, $\psi \in \Omega_3$, where $\Omega_3 \subset C$ open, such that

(i) $\sup\left\{ \left| \int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s) \right| \ : \ t \in [0,T], \ \psi \in \Omega_3, \ \xi \in C, \ |\xi|_C \leq 1 \right\} < \infty$,

(ii) for each $\xi \in C$ the function $[0,T] \times \Omega_3 \to \mathbb{R}^n$, $(t,\psi) \mapsto \int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s)$ is continuous,

(A3) $\varphi \in C$,

(A4) for every $\alpha > 0$, $M > 0$ there exists a constant $L_1 = L_1(\alpha, M)$ such that for all $t \in [0,\alpha]$, $x, \bar{x} \in \Omega_1$, $y, \bar{y} \in \Omega_2$, $|x|$, $|\bar{x}|$, $|y|$, $|\bar{y}| \leq M$

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \leq L_1\Big(|x-\bar{x}| + |y-\bar{y}|\Big),$$

(A5) for every $\alpha > 0$ and $M > 0$ there exists a constant $L_2 = L_2(\alpha, M)$ such that for all $\xi \in W^{1,\infty}$, $t \in [0,\alpha]$ and $\psi, \bar{\psi} \in \Omega_3$, $|\psi|_C$, $|\bar{\psi}|_C \leq M$

$$|\lambda(t,\psi,\xi) - \lambda(t,\bar{\psi},\xi)| \leq L_2|\xi|_{W^{1,\infty}}|\psi - \bar{\psi}|_C,$$

(A6) $\varphi \in W^{1,\infty}$, i.e., φ is Lipschitz-continuous.

Introduce the Banach space $BC([0,T] \times \Omega_1 \times \Omega_2; \mathbb{R}^n)$ as the space of bounded continuous functions $f : [0,T] \times \Omega_1 \times \Omega_2 \to \mathbb{R}^n$ with norm $\|f\| \equiv \sup\{|f(t,x,y)| \ : \ t \in [0,T], \ x \in \Omega_1, \ y \in \Omega_2\}$. Introduce $\Theta_C(T, \Omega_3)$ as the Banach space of functions $\mu : [0,T] \times \Omega_3 \to \mathrm{NBV}([-r,0];\mathbb{R}^n)$ which satisfy (A2) (i) and (ii), where $\mu(\cdot,t,\psi)$ is the image function corresponding to $t \in [0,T]$ and $\psi \in C$. The norm in $\Theta_C(T,\Omega_3)$ is defined by $\|\mu\| \equiv \sup\left\{ \left| \int_{-r}^{0} d_s\mu(s,t,\psi)\xi(s) \right| < \infty \ : \ t \in [0,T], \ \psi \in \Omega_3, \ \xi \in C, \ |\xi|_C \leq 1 \right\}$.

Define two versions of parameter spaces $\Gamma_0(T,\Omega_1,\Omega_2,\Omega_3) \equiv C \times \Theta_C(T,\Omega_3) \times BC\Big([0,T] \times \Omega_1 \times \Omega_2; \ \mathbb{R}^n\Big)$ and $\Gamma_1(T,\Omega_1,\Omega_2,\Omega_3) \equiv W^{1,\infty} \times \Theta_C(T,\Omega_3) \times BC\Big([0,T] \times \Omega_1 \times \Omega_2; \ \mathbb{R}^n\Big)$ with norms $\|\gamma\|_{\Gamma_0} \equiv |\varphi|_C + \|\mu\| + \|f\|$ and $\|\gamma\|_{\Gamma_1} \equiv |\varphi|_{W^{1,\infty}} + \|\mu\| + \|f\|$, respectively, and two versions of sets of feasible parameters $\Pi_0(T,\Omega_1,\Omega_2,\Omega_3) \equiv \Big\{(\varphi,\mu,f) \in \Gamma_0(T,\Omega_1,\Omega_2,\Omega_3) \ : \ \varphi(0) \in \Omega_1, \ \varphi \in \Omega_3, \text{ and } \int_{-r}^{0} d_s\mu(s,0,\varphi)\,\varphi(s) \in \Omega_2\Big\}$, and $\Pi_1(T,\Omega_1,\Omega_2,\Omega_3) \equiv \Pi_0(T,\Omega_1,\Omega_2,\Omega_3) \cap \Gamma_1(T,\Omega_1,\Omega_2,\Omega_3)$, respectively.

We have the following results on the local existence of solutions of IVP (6.1)-(6.2).

Theorem 6.1 *Assume (A1)–(A3). Given $\bar{\gamma} \equiv (\bar{\varphi}, \bar{\mu}, \bar{f}) \in \Pi_0(T,\Omega_1,\Omega_2,\Omega_3)$ then there exist positive constants $\alpha = \alpha(\bar{\gamma})$ and $\delta = \delta(\bar{\gamma})$ such that if $\gamma \equiv (\varphi,\mu,f) \in \Gamma_0(T,\Omega_1,\Omega_2,\Omega_3)$ and $\|\gamma - \bar{\gamma}\|_{\Gamma_0} < \delta$ then $\gamma \in \Pi_0(T,\Omega_1,\Omega_2,\Omega_3)$, and IVP (6.1)-(6.2) corresponding to γ has a solution, $x(t;\gamma)$, on $[-r,\alpha]$.*

The next theorem shows that (A1)–(A6) guarantee the existence of unique solution of IVP (6.1)-(6.2).

Theorem 6.2 *Let* $\gamma \in \Pi_0(T, \Omega_1, \Omega_2, \Omega_3)$ *and assume that (A1)–(A6) are satisfied. Then there exists* $\alpha > 0$ *such that IVP (6.1)-(6.2) has a unique solution on* $[0, \alpha]$.

The following examples show that if we violate assumptions (A4), (A5) and (A6), then we may also loose uniqueness of the solution.

Example 6.3 Consider the scalar IVP

$$\dot{x}(t) = 4\sqrt{x(t-\tau(t))}, \qquad t \geq 0, \tag{6.3}$$

$$x(t) = 0, \qquad -1 \leq t \leq 0, \tag{6.4}$$

where $\tau(t) \equiv \min\{t/2, 1\}$. It is easy to see that IVP (6.3)-(6.4) has two solutions on $[0, 2]$: $x_1(t) = 0$ and $x_2(t) = t^2$.

Example 6.4 Consider the scalar IVP with state-dependent delay

$$\dot{x}(t) = x\Big(t - \tau(x(t))\Big), \qquad t \geq 0, \tag{6.5}$$

$$x(t) = -2t, \qquad -2 \leq t \leq 0, \tag{6.6}$$

where $\tau(x) \equiv 2\min\Big\{\sqrt{|x|}, 1\Big\}$. It is easy to check that this IVP has two solutions: $x_1(t) = 0$, $t \geq 0$ and $x_2(t) = t^2$ for $t \in [0, 1]$. We can rewrite IVP (6.5)-(6.6) in the form

$$\dot{x}(t) = \int_{-2}^{0} d_s\mu(s, x_t)x(t+s), \qquad t \geq 0, \tag{6.7}$$

$$x(t) = -2t, \qquad -2 \leq t \leq 0, \tag{6.8}$$

by defining

$$\mu(s, \psi) \equiv \chi_{[-\tau(\psi(0)),0]}(s), \qquad s \in [-2, 0].$$

We have that if $|\psi(0)| \leq 1$ then

$$\lambda(\psi, \xi) = \int_{-r}^{0} d_s\mu(s, \psi)\xi(s) = \xi(-\tau(\psi(0))) = \xi\left(-2\sqrt{|\psi(0)|}\right),$$

which does not satisfy (A5). (It is enough to consider $\xi(s) = s$, and constant functions for ψ.)

Example 6.5 Consider the scalar IVP with state-dependent delay

$$\dot{x}(t) = x\Big(t - \tau(x(t))\Big), \qquad t \geq 0$$

$$x(t) = \begin{cases} 1, & -2 \leq t \leq -1 \\ 1 - 2\sqrt{1+t}, & -1 \leq t \leq -\frac{3}{4} \\ \frac{4}{3}t + 1, & -\frac{3}{4} \leq t \leq 0, \end{cases}$$

where $\tau(x) = \min\{|x|, 2\}$. The initial function is not Lipschitz-continuous (hence (A6) is not satisfied), therefore the uniqueness is not guaranteed by Theorem 6.2. In fact, the IVP has two solutions: $t+1$ is solution for $t \in [0,1]$ and the analytic expression on $[0, 0.5]$ for the other solution is $t + 1 - t^2$.

It is easy to see that the solution of IVP (6.1)-(6.2) is a $W^{1,\infty}$ function assuming (A1)–(A6). The next theorem shows that in the norm of Γ_1, the solution of IVP (6.1)-(6.2) is Lipschitz-continuous with respect to the parameters.

Theorem 6.6 *Assume that $\bar{\gamma} = (\bar{\varphi}, \bar{\mu}, \bar{f}) \in \Pi_1(T, \Omega_1, \Omega_2, \Omega_3)$ satisfies (A1)–(A6). Then there exist constants $\alpha > 0$, $\delta > 0$ and $L_3 = L_3(\alpha, \bar{\gamma}, \delta)$, such that IVP (6.1)-(6.2) has a unique solution on $[0, \alpha]$ for all $\gamma \in \mathcal{G}_{\Gamma_1(T,\Omega_1,\Omega_2,\Omega_3)}(\bar{\gamma}; \delta)$, and*

$$|x(\cdot;\gamma)_t - x(\cdot;\bar{\gamma})_t|_{W^{1,\infty}} \leq L_3 \|\gamma - \bar{\gamma}\|_{\Gamma_1}, \quad t \in [0, \alpha].$$

For the proofs and more details we refer the interested reader to [7].