On Numerical Approximations for a Class of Differential Equations with Time- and State-Dependent Delays

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Abstract We establish limiting relations between solutions for a large class of functional differential equations with time- and state-dependent delays and solutions of appropriately selected sequences of approximating delay differential equations with piecewise constant arguments. The approximating equations, generated in the above process, lead naturally to discrete difference equations, well suited for computational purposes, and thus provide an approximation framework for simulation studies.

Keywords Delay equations, State-dependent delays, Approximation, Euler’s method, Equations with piecewise constant arguments.

1 Introduction and Problem Formulation

In this paper we consider an Euler-type approximation technique for functional differential equations (FDEs) with time- and state-dependent delays (see e.g. [4] and [9] for related developments). As the main result of this paper we present a “new” proof (in comparison to [4] and [9]) for convergence of the above method using appropriately selected sequences of approximating delay differential equations with piecewise constant arguments (EPCAs). (Note that in [5] a variety of EPCA based schemes were introduced and applied to linear equations with constant delays.) We refer the interested reader to [6] for extensive numerical studies of our method. In this direction, we mention [2], [8] and the references therein for different type of numerical approximation techniques for FDEs with time- and state-dependent delays.

In this paper, we consider the vector delay differential equation

\[ \dot{x}(t) = f \left( t, x(t), x(t - \tau(t, x(t))) \right), \quad t \geq 0 \tag{1.1} \]

with initial data

\[ x(t) = \Phi(t), \quad t \in [-\lambda, 0], \tag{1.2} \]

where \( \lambda \equiv \inf \{ t - \tau(t, u) : t \geq 0, -\infty < u < \infty \} \).

Throughout this paper we shall use the notation \( [t]_h \equiv [t/h]h \), where \( h > 0 \) and \([\cdot]\) is the greatest integer function. For fixed \( h > 0 \) we define the delay differential equation with piecewise constant arguments associated with (1.1) by

\[ \dot{y}_h(t) = f \left( [t]_h, y_h([t]_h), y_h \left( [t]_h - \left[ \tau([t]_h, y_h([t]_h)) \right] \right) \right), \quad t \geq 0 \tag{1.3} \]

with initial condition corresponding to (1.2)

\[ y_h(-kh) = \Phi(-kh), \quad k = 0, 1, 2, \ldots, -\lambda \leq -kh \leq 0 \tag{1.4} \]
By a solution of initial value problem (IVP) \((1.3)-(1.4)\) we mean a function \(y_k\) defined on \([-kh : k = 0, 1, \ldots, -\lambda \leq -kh \leq 0]\) by \((1.4)\), which satisfies the following properties on \(R^+\):

(i) the function \(y_k\) is continuous on \(R^+\), (ii) the derivative \(\dot{y}_k(t)\) exists at each point \(t \in R^+\) with the possible exception of the points \(kh (k = 0, 1, 2, \ldots)\) where finite one-sided derivatives exist, and

(iii) the function \(y_k\) satisfies \((1.3)\) on each interval \([kh, (k+1)h]\) for \(k = 0, 1, 2, \ldots\).

In the next section we show that the solutions of IVP (1.3)-(1.4) approximate solutions of IVP (1.1)-(1.2) as \(h \to 0^+\), uniformly on compact time intervals, and establish a rate of convergence estimate on approximate solutions as well.

We shall assume that the following conditions are satisfied:

(H1) \( f \in C(R^+ \times R_N, R_N), \Phi : [-\lambda, 0] \to R_N, \) is bounded, and \( \tau \in C(R^+ \times R_N, R^+) \).

(H2') the function \( f(t, u, v) \) is locally Lipschitz-continuous in \( u, v \) on \( R^+ \times \mathbb{R}^N \), that is, for every \( T > 0 \) and \( M > 0 \) there exists a constant \( L_1 = L_1(T, M) \) such that \( ||f(t, u, v) - f(t, \bar{u}, \bar{v})|| \leq L_1 \cdot (||u - \bar{u}|| + ||v - \bar{v}||) \), for \( t \in [0, T], u, v, \bar{u}, \bar{v} \in B(M) \).

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(H3) the delay function \( \tau \) is locally Lipschitz-continuous in its second argument on \( R^+ \times R_N \), that is, for every \( T > 0 \) and \( M > 0 \) there exists a constant \( L_2 = L_2(T, M) \) such that \( ||\tau(t, u) - \tau(t, \bar{u})|| \leq L_2 \cdot ||u - \bar{u}|| \), for \( t \in [0, T], u, \bar{u} \in B(M) \).

(H4) the initial function \( \Phi \) is Lipschitz-continuous with Lipschitz constant \( L_3 \), that is \( ||\Phi(t) - \Phi(\bar{t})|| \leq L_3 ||t - \bar{t}|| \), for \( t, \bar{t} \in [-\lambda, 0] \).

Here and throughout \( ||\cdot|| \) denotes a norm on \( R_N \), \( R^+ \equiv [0, \infty) \), and \( B(M) \equiv \{x \in R_N : ||x|| \leq M\} \). If \( \lambda = \infty \) then the notations \([-\lambda, T]\) and \([-\lambda, \infty)\) should be interpreted as \((-\infty, T]\) and \((-\infty, \infty)\), respectively.

Note, that these conditions can be considered “standard” for the well-posedness of IVP (1.1)-(1.2), (see e.g. [3]).

2 Convergence Results

Using the notation \( a(k) \equiv y_k(kh) \), and applying the method of steps on the intervals \([kh, (k+1)h]\) one can easily see the following result.

Theorem 2.1 Assume (H1). Then IVP (1.3)-(1.4) has a unique solution in the form

\[
y_k(t) = a(k) + f\left(kh, a(k), a(k-d_k)\right) \cdot (t-kh),
\]

for \( t \in [kh, (k+1)h], \quad k = 0, 1, 2, \ldots \), where \( d_k \equiv \left[\tau(kh, a(k))/h\right] \), and the sequence \( a(k) \) satisfies the difference equation

\[
a(k+1) = a(k) + f\left(kh, a(k), a(k-d_k)\right) \cdot h, \quad k = 0, 1, 2, \ldots,
\]

\[
a(-k) = \Phi(-kh), \quad k = 0, 1, 2, \ldots \quad -\lambda \leq -kh \leq 0.
\]

Remark 2.2 The sequence \( a(k) \) is well-defined, because \( -\lambda \leq (k-d_k)h \leq kh \) for every \( k = 0, 1, 2, \ldots \).
We introduce the simplifying notations \( \sigma(t) \equiv t - \tau(t, x(t)) \) and \( \sigma_h(t) \equiv [t, \tau(t, x(t))]. \)

The following result shows that there exists \( \alpha > 0 \) such that for \( h > 0 \) the solutions of the corresponding initial value problems (1.3)- (1.4) on \( [0, \alpha] \) form a uniformly bounded family.

**Lemma 2.3** Assume (H1) and (H2'). Then for every \( M \geq 5 \sup_{-\lambda \leq s \leq 0} \| \Phi(s) \| \) there exist \( \alpha > 0 \) such that for every \( h > 0 \)

\[
\| y_h(t) \| \leq M, \quad t \in [0, \alpha].
\]

**Proof:** Fix an \( M \geq 5 \sup_{-\lambda \leq s \leq 0} \| \Phi(s) \| \). Integrating (1.3) and using elementary manipulations we have the following estimate

\[
\begin{align*}
\| y_h(t) - y_h(0) \| & \leq \int_0^t \| f\left([s]\sigma_h(s), y_h(s)\right) - f\left([s]\sigma_h(s), y_h(0)\right) \| \, ds \\
& \leq L_1 \int_0^t \left( \| y_h(s) - y_h(0) \| + \| y_h(s) - y_h(0) \| \right) \, ds + K_1 t \\
& \leq L_1 \int_0^t \left( \| y_h(s) - y_h(0) \| + \| y_h(s) - y_h(0) \| \right) \, ds + K_1 t,
\end{align*}
\]

where \( K_1 \equiv K_1(\alpha) \equiv \sup \{ \| f(t, \Phi(0), \Phi(s)) \| \} : t \in [0, \alpha], \ s \in [-\lambda, 0] \} \).

We define the function \( \psi(t) \equiv \max \left\{ \sup_{-\lambda \leq s \leq 0} \| \Phi(s) - \Phi(0) \|, \ \sup_{0 \leq s \leq t} \| \Phi(s) - \Phi(0) \| \right\} \). Recalling that \( \sigma_h(s) \leq s \) and \( [s] \sigma \leq s \), and using the definition of \( \psi_h \), inequality (2.4) implies

\[
\| y_h(t) - y_h(0) \| \leq K_1 \alpha + L_1 \alpha \cdot \sup_{-\lambda \leq s \leq 0} \| \Phi(s) - \Phi(0) \| + \int_0^t 2L_1 \psi_h(s) \, ds, \quad t \in [0, \alpha].
\]

Define the constant \( K_2 \equiv K_2(\alpha) \equiv K_1(\alpha) + \max \{ L_1(\alpha, 1) \cdot \sup_{-\lambda \leq s \leq 0} \| \Phi(s) - \Phi(0) \| \} \). It is easy to see that

\[
\psi_h(t) \leq K_2 + \int_0^t 2L_1 \psi_h(s) \, ds, \quad t \in [0, \alpha].
\]

By applying the Gronwall-Bellman inequality for (2.5) we find that

\[
\| y_h(t) - y_h(0) \| \leq \psi_h(t) \leq K_2 \cdot \exp(2L_1 t), \quad t \in [0, \alpha].
\]

It follows that if we can select \( \alpha > 0 \) such that

\[
K_2 \cdot \exp(2L_1 \alpha) + \| \Phi(0) \| \leq M
\]

then (2.3) holds and our calculations are valid. Pick an apriori \( T > 0 \). Note that we can select Lipschitz-constant \( L_1(\alpha, M) \) such that \( L_1(\alpha, M) \leq L_1(T, M) \) for \( \alpha \leq T \). Therefore we have that \( K_2(\alpha) \leq K(T) \alpha + \max \{ L_1(T, M, \alpha, 1) \} \sup_{-\lambda \leq s \leq 0} \| \Phi(s) - \Phi(0) \| \leq 3 \sup_{-\lambda \leq s \leq 0} \| \Phi(s) \| \) for \( \alpha > 0 \) such that

\[
L_1(T, M) \alpha \leq 1, \quad \text{and} \quad K(T) \alpha \leq \sup_{-\lambda \leq s \leq 0} \| \Phi(s) \|.
\]

Now, assuming (2.8), we have that

\[
K_2(\alpha) \cdot \exp \left( 2L_1(\alpha, M) \alpha \right) + \| \Phi(0) \| \leq 5 \sup_{-\lambda \leq t \leq 0} \| \Phi(t) \|,
\]

provided that \( \alpha \) satisfies

\[
\exp \left( 2L_1(T, M) \alpha \right) \leq 4/3.
\]
Select $\alpha \in (0, T]$ such that (2.8) and (2.9) hold, then (2.7) is satisfied, which proves the lemma.

We comment that if the Lipschitz-constant of $f$ is independent of $M$, (i.e., $f$ is Lipschitz-continuous on $[0, T] \times \mathbb{R}^n$ for arbitrary $T > 0$), or the function $f$ is bounded on $[0, T] \times \mathbb{R}^n$, then (2.6) and the definition of $K_2$, or the latter case (1.3) yields that $\{y_h(t)\}$ is uniformly bounded on every compact time interval.

Lemma 2.3 allows us to obtain existence results for IVP (1.1)-(1.2) following the steps of the classical Cauchy-Peano Theorem (see e.g. [1] or [9]): the approximate solutions $\{y_h(t) : h > 0\}$ form a uniformly bounded family of functions on some interval $[0, \alpha]$. It is easy to see using (1.3) that $y_h(t)$ for $h > 0$ are also equicontinuous functions on $[0, \alpha]$, and therefore there exists a sequence $h_k \to 0$, such that the corresponding functions converge to a continuous function, i.e., $x(t) \equiv \lim_{h \to 0} y_h(t)$ exists and continuous on $[-\lambda, \alpha]$. Then it is easy to show (using a continuity argument) that $x(t)$ satisfies (1.1), i.e., IVP (1.1)-(1.2) has a solution on $[0, \alpha]$. (We refer to [3] for related well-posedness result using fixed point arguments.)

In particular, we have the following theorem (see the detailed proof in [7]):

**Theorem 2.4** Assume that (H1) and (H2') hold and $\Phi \in C([-\lambda, 0], R)$. Then there exists $T > 0$ such that IVP (1.1)-(1.2) has a solution on $[-\lambda, T]$.

Assuming that the initial function $\Phi$ is Lipschitz-continuous and the delay function $\tau$ is locally Lipschitz-continuous with respect to its second argument we can prove the following convergence result for our approximating scheme, which also implies the uniqueness of solutions of IVP (1.1)-(1.2).

**Theorem 2.5** Assume (H1), (H2'), (H3') and (H4). Then if IVP (1.1)-(1.2) has a solution $x(t)$ on $[0, T]$, then the solution is unique and $\lim_{h \to 0^+} \max_{0 \leq t \leq T} \|x(t) - y_h(t)\| = 0$, where $y_h$ is the solution of IVP (1.3)-(1.4). If in addition (H2) and (H3) hold, then there exist constants $M_3(T, \Phi) > 0$ and $h_0 > 0$ such that $\|x(t) - y_h(t)\| \leq M_3 h$, for $t \in [0, T]$ and $0 < h \leq h_0$.

**Proof:** Let $T > 0$ be such that IVP (1.1)-(1.2) has a solution, $x(t)$, on $[0, T]$. For the uniqueness of solution it is enough to prove that $\lim_{h \to 0^+} \max_{0 \leq t \leq T} \|x(t) - y_h(t)\| = 0$. Let $M_1 \equiv \max \{|x(t)| : t \in [-\lambda, T]\} + 1$. Suppose that there exists $h_0 > 0$ such that

$$
\|y_h(t)\| \leq M_1, \quad t \in [0, T], \quad 0 < h \leq h_0. \tag{2.10}
$$

Define the constant $M_2 \equiv \max \{|f(t, u, v)| : t \in [0, T], \ u, v \in B(M_1)\}$, then $y_h$ satisfies

$$
\|y_h(t_1) - y_h(t_2)\| \leq M_2 |t_1 - t_2|, \quad t_1, t_2 \in [0, T], \quad 0 < h \leq h_0. \tag{2.11}
$$

Equations (1.1), (1.3), the assumed relation (2.10) and the assumptions of the theorem yield the following inequalities

$$
\|x(t) - y_h(t)\| \leq \int_0^t \|f\left(s, x(s), x(\sigma(s))\right) - f\left([s]_h, x(s), x(\sigma(s))\right)\|\,ds
+ \int_0^t \|f\left([s]_h, x(s), x(\sigma(s))\right) - f\left([s]_h, y_h([s]_h), x(\sigma(s))\right)\|\,ds
\leq \int_0^t \|f\left(s, x(s), x(\sigma(s))\right) - f\left([s]_h, x(s), x(\sigma(s))\right)\|\,ds
+ L_1 \int_0^t \left(\|x(s) - y_h([s]_h)\| + \|x(\sigma(s)) - y_h(\sigma([s]_h))\|\right)\,ds, \tag{2.12}
$$

where $L_1 = L_1(T, M_1)$. Using the notation $\eta_h(s) \equiv [s]_h - [\tau([s]_h, x([s]_h))], \ h$, and elementary manipulations on the last term of the right hand side of (2.12) we have

$$
\|x(t) - y_h(t)\| \leq \int_0^t \|f\left(s, x(s), x(\sigma(s))\right) - f\left([s]_h, x(s), x(\sigma(s))\right)\|\,ds
$$

...
\[ + L_1 \int_0^t \left( \|x(s) - y_h(s)\| + \|y_h(s) - y_h([s]_h)\| + \|x(\sigma(s)) - y(\eta_h(s))\| \\
+ \|y(\eta_h(s)) - y_h(\eta_h(s))\| + \|y_h(\eta_h(s)) - y_h(\sigma_h(s))\| \right) \, ds. \]  

(2.13)

Now we shall estimate the term \(\|y_h(\eta_h(s)) - y_h(\sigma_h(s))\|\) in (2.13).

Let \(A \equiv \max \{\eta_h(s), \sigma_h(s)\}\) and \(B \equiv \max \{y_h(s), \sigma_h(s)\}\). We need to study three cases: First we assume that \(0 \leq A \leq B\). Then (2.11) yields \(\|y_h(\eta_h(s)) - y_h(\sigma_h(s))\| \leq M_2 |A - B|\). If \(A \leq B \leq 0\) then the assumed Lipschitz-continuity of \(\Phi\) implies \(\|y_h(\eta_h(s)) - y_h(\sigma_h(s))\| \leq L_3 |A - B|\). Finally, if \(A \leq 0 \leq B\) then inequality (2.11) and the Lipschitz-continuity of \(\Phi\) imply

\[ \|y_h(\eta_h(s)) - y_h(\sigma_h(s))\| \leq \|y_h(A) - y_h(0)\| + \|y_h(0) - y_h(B)\| \leq \max\{L_3, M_2\} \cdot |A - B|. \]

(2.14)

Thus in all the above cases using the Lipschitz-continuity of \(\tau\) we have

\[ \|y_h(\eta_h(s)) - y_h(\sigma_h(s))\| \leq K_3 \tau([s]_h, x([s]_h)) + 2K_3 h \]

(2.15)

where \(K_3 \equiv \max \{L_3, M_2\}\) and \(L_2 = L_2(T, M_1)\).

Let \(\varepsilon_h(t) \equiv \|x(t) - y_h(t)\|\). Combining (2.13) and (2.14) and using the definition of \(\varepsilon_h\) we get

\[ \varepsilon_h(t) \leq L_1 \int_0^t \left( \|f(s, x(s)) - x(\sigma(s))\| + L_1 \left( \|y_h(s) - y_h([s]_h)\| + \|x(\sigma(s)) - y(\eta_h(s))\| + 2K_3 h \right) \right) \, ds + g_h(t), \]  

(2.16)

where

\[ g_h(t) \equiv \int_0^t \left( \|f(s, x(s)) - x(\sigma(s))\| + L_1 \left( \|y_h(s) - y_h([s]_h)\| + \|x(\sigma(s)) - y(\eta_h(s))\| + 2K_3 h \right) \right) \, ds. \]

(2.17)

Let \(\psi_h(t) \equiv \max_{0 \leq s \leq t} \varepsilon_h(s)\). Now, \(-\lambda \leq \eta_h(s) \leq s\) for every \(s \geq 0\) and if \(\eta_h(s) \leq 0\) then \(\varepsilon_h(\eta_h(s)) = 0\). Therefore it follows from (2.15) that \(\psi_h(t)\) satisfies

\[ \psi_h(t) \leq \int_0^t L_1 (2 + K_3 L_2) \psi_h(s) \, ds + g_h(t). \]

(2.18)

Since \(g_h(t)\) is a monotone nondecreasing function, an application of the Gronwall-Bellman inequality yields

\[ \|x(t) - y_h(t)\| \leq \psi_h(t) \leq \exp \left( L_1 (2 + K_3 L_2) T \right) \cdot g_h(T), \quad t \in [0, T]. \]

(2.19)

Define the functions \(\omega_v(h) \equiv \max \{\|x(t) - x(\bar{t})\| : t, \bar{t} \in [-\lambda, T], |t - \bar{t}| \leq h\} \), \(\omega_v(h) \equiv \max \{|\tau(t, u) - \tau(\bar{t}, u)| : t, \bar{t} \in [0, T], |t - \bar{t}| \leq h, u \in B(M_1)\}\), and \(\omega_v(h) \equiv \max \{|f(t, u, v) - f(\bar{t}, u, v)| : t, \bar{t} \in [0, T], |t - \bar{t}| \leq h, u, v \in B(M_1)\}\). Then from (2.16) and from the inequality

\[ |\sigma(s) - \eta_h(s)| \leq |\tau(s, x(s)) - \tau([s]_h, x(s))| + |\tau([s]_h, x(s)) - \tau([s]_h, x([s]_h))| + h \]

\[ \leq |\tau(s, x(s)) - \tau([s]_h, x(s))| + L_2 \|x(s) - x([s]_h)\| + h \]

\[ \leq \omega_v(h) + L_2 \omega_v(h) + h \]

it follows that

\[ g_h(T) \leq \left( \omega_v(h) + L_1 M_2 h + L_1 \omega_v(h) + L_2 \omega_v(h) + h \right) T + 2L_1 K_3 h T. \]

(2.20)

The function \(x(t)\) is uniformly continuous on \([-\lambda, T]\) (we have that the initial function \(\Phi\) is Lipschitz-continuous on \([-\lambda, 0]\), and hence \(\omega_v(h) \to 0\) as \(h \to 0^+\), and similarly by; the uniform
continuity of \( f \) and \( \tau \) on \( [0, T] \times B(M_1) \times B(M_1) \) and \( [0, T] \times B(M_1) \), respectively, we also have that \( \omega_f(h) \) and \( \omega_\tau(h) \) go to zero as \( h \to 0^+ \). Therefore (2.18) yields that
\[
g_h(T) \to 0, \quad \text{as } h \to 0^+.
\]
(2.19)

Select \( h_0 > 0 \) such that
\[
\exp \left( L_1(2 + K_3 L_2) T \right) \left( \omega_f(h_0) + L_1 M_2 h_0 + L_1 \omega_\tau(h_0) + 2 L_1 K_3 h_0 \right) T \leq 1,
\]
then from inequalities (2.17), (2.18) and definition of \( M_1 \) it follows that \( h_0 \) satisfies (2.10). Hence our calculation is valid, and (2.17) with (2.19) imply the first statement of the theorem.

To show the second statement of the theorem, observe that the definition of \( M_1 \) and equation (1.1) yield that \( \|x(t)\| \leq \max \{ \|f(t, u, v)\| \} \quad t \in [0, T], \ u, v \in B(M_1) \} \) for \( t \in [0, T] \), hence \( \omega_\tau(h) \leq K_4 h \), where \( K_4 \equiv \max \{ L_4, \max \{ \|f(t, u, v)\| \} \} \) \( t \in [0, T], \ u, v \in B(M_1) \} \). Assumption (H2) and (H3) imply \( \omega_f(h) \leq \tilde{L}_2 h \) and \( \omega_\tau(h) \leq \tilde{L}_2 h \), respectively, where \( \tilde{L}_2 = \tilde{L}_2(T, M_1) \) and \( \tilde{L}_1 = \tilde{L}_1(T, M_1) \). Inequalities (2.17) and (2.18) yield the second statement of the theorem using these estimates of \( \omega_\tau(h) \), \( \omega_f(h) \) and \( \omega_\tau(h) \), with constant
\[
M_3 \equiv \exp \left( L_1(2 + K_3 L_2) T \right) \left( \tilde{L}_1 + L_1 M_2 + L_1 K_4 (\tilde{L}_2 + L_2 K_4 + 1) + 2 L_1 K_3 \right) T.
\]
The proof of the theorem is complete.

We close the paper by noting that our results can be extended in a rather straightforward fashion to the case when the equation has multiple delays.

References