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On Numerical Approximations for a Class of Differential Equations with Time- and State-Dependent Delays

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Abstract — We establish limiting relations between solutions for a large class of functional differential equations with time- and state-dependent delays and solutions of appropriately selected sequences of approximating delay differential equations with piecewise constant arguments. The approximating equations, generated in the above process, lead naturally to discrete difference equations, well suited for computational purposes, and thus provide an approximation framework for simulation studies.

Keywords — Delay equations, State-dependent delays, Approximation, Euler's method, Equations with piecewise constant arguments.

1 Introduction and Problem Formulation

In this paper we consider an Euler-type approximation technique for functional differential equations (FDEs) with time- and state-dependent delays (see e.g. [4] and [9] for related developments). As the main result of this paper we present a "new" proof (in comparison to [4] and [9]) for convergence of the above method using appropriately selected sequences of approximating delay differential equations with piecewise constant arguments (EPCAs). (Note that in [5] a variety of EPCA based schemes were introduced and applied to linear equations with constant delays.) We refer the interested reader to [6] for extensive numerical studies of our method. In this direction, we mention [2], [8] and the references therein for different type of numerical approximation techniques for FDEs with time- and state-dependent delays.

In this paper, we consider the vector delay differential equation

$$\dot{x}(t) = f\Big(t, x(t), x(t-\tau(t,x(t)))\Big), \qquad t \geq 0 \tag{1.1}$$

with initial data

$$x(t) = \Phi(t), \qquad t \in [-\lambda, 0], \tag{1.2}$$

where $\lambda \equiv -\inf\{t-\tau(t,u) \ : \ t \geq 0,\ -\infty < u < \infty\}$.

Throughout this paper we shall use the notation $[t]_h \equiv [t/h]h$, where $h > 0$ and $[\cdot]$ is the greatest integer function. For fixed $h > 0$ we define the delay differential equation with piecewise constant arguments associated with (1.1) by

$$\dot{y}_h(t) = f\left([t]_h, y_h([t]_h), y_h\left([t]_h - \Big[\tau([t]_h, y_h([t]_h))\Big]_h\right)\right), \quad t \geq 0. \tag{1.3}$$

with initial condition corresponding to (1.2)

$$y_h(-kh) = \Phi(-kh), \qquad k = 0, 1, 2, \ldots, \quad -\lambda \leq -kh \leq 0. \tag{1.4}$$

By a solution of initial value problem (IVP) (1.3)-(1.4) we mean a function y_h defined on $\{-kh\ :\ k = 0, 1, \ldots,\quad -\lambda \le -kh \le 0\}$ by (1.4), which satisfies the following properties on R^+: (i) the function y_h is continuous on R^+, (ii) the derivative $\dot{y}_h(t)$ exists at each point $t \in R^+$ with the possible exception of the points kh ($k = 0, 1, 2, \ldots$) where finite one-sided derivatives exist, and (iii) the function y_h satisfies (1.3) on each interval $[kh, (k+1)h)$ for $k = 0, 1, 2, \ldots$.

In the next section we show that the solutions of IVP (1.3)-(1.4) approximate solutions of IVP (1.1)-(1.2) as $h \to 0^+$, uniformly on compact time intervals, and establish a rate of convergence estimate on approximate solutions as well.

We shall assume that the following conditions are satisfied:

(H1) $f \in C(R^+ \times R^{2N}, R^N)$, $\Phi : [-\lambda, 0] \to R^N$, is bounded, and $\tau \in C\left(R^+ \times R^N, R^+\right)$,

(H2') the function $f(t, u, v)$ is locally Lipschitz-continuous in u, v on $R^+ \times R^{2N}$, that is, for every $T > 0$ and $M > 0$ there exists a constant $L_1 = L_1(T, M)$ such that $\|f(t, u, v) - f(t, \bar{u}, \bar{v})\| \le L_1 \cdot (\|u - \bar{u}\| + \|v - \bar{v}\|)$, for $t \in [0, T]$, $u, v, \bar{u}, \bar{v} \in B(M)$,

(H2) the function $f(t, u, v)$ is locally Lipschitz-continuous on $R^+ \times R^{2N}$, that is, for every $T > 0$ and $M > 0$ there exists a constant $\tilde{L}_1 = \tilde{L}_1(T, M)$ such that $\|f(t, u, v) - f(\bar{t}, \bar{u}, \bar{v})\| \le \tilde{L}_1 \cdot (|t - \bar{t}| + \|u - \bar{u}\| + \|v - \bar{v}\|)$, for $t, \bar{t} \in [0, T]$, $u, v, \bar{u}, \bar{v} \in B(M)$,

(H3') the delay function τ is locally Lipschitz-continuous in its second argument on $R^+ \times R^N$, that is, for every $T > 0$ and $M > 0$ there exists constant $L_2 = L_2(T, M)$ such that $|\tau(t, u) - \tau(t, \bar{u})| \le L_2 \|u - \bar{u}\|$, for $t \in [0, T]$, $u, \bar{u} \in B(M)$,

(H3) the delay function τ is locally Lipschitz-continuous on $R^+ \times R^N$, that is, for every $T > 0$ and $M > 0$ there exists a constant $\tilde{L}_2 = \tilde{L}_2(T, M)$ such that $|\tau(t, u) - \tau(\bar{t}, \bar{u})| \le \tilde{L}_2 \cdot (|t - \bar{t}| + \|u - \bar{u}\|)$, for $t, \bar{t} \in [0, T]$, $u\, \bar{u} \in B(M)$, and

(H4) the initial function Φ is Lipschitz-continuous with Lipschitz constant L_3, that is $\|\Phi(t) - \Phi(\bar{t})\| \le L_3 |t - \bar{t}|$, for $t, \bar{t} \in [-\lambda, 0]$.

Here and throughout $\|\cdot\|$ denotes a norm on R^N, $R^+ \equiv [0, \infty)$, and $B(M) \equiv \{x \in R^N\ :\ \|x\| \le M\}$. If $\lambda = \infty$ then the notations $[-\lambda, T]$ and $[-\lambda, \infty)$ should be interpreted as $(-\infty, T]$ and $(-\infty, \infty)$, respectively.

Note, that these conditions can be considered "standard" for the well-posedness of IVP (1.1)-(1.2), (see e.g. [3]).

2 Convergence Results

Using the notation $a(k) \equiv y_h(kh)$, and applying the method of steps on the intervals $[kh, (k+1)h)$ one can easily see the following result.

Theorem 2.1 *Assume (H1). Then IVP* (1.3)-(1.4) *has a unique solution in the form*

$$y_h(t) = a(k) + f\Big(kh, a(k), a(k - d_k)\Big) \cdot (t - kh), \tag{2.1}$$

for $t \in [kh, (k+1)h)$, $\quad k = 0, 1, 2, \ldots$, *where* $d_k \equiv \Big[\tau(kh, a(k))/h\Big]$, *and the sequence* $a(k)$ *satisfies the difference equation*

$$\begin{aligned} a(k+1) &= a(k) + f\Big(kh, a(k), a(k - d_k)\Big) \cdot h, \qquad k = 0, 1, 2, \ldots, \\ a(-k) &= \Phi(-kh), \qquad k = 0, 1, 2, \ldots, \quad -\lambda \le -kh \le 0. \end{aligned} \tag{2.2}$$

Remark 2.2 *The sequence* $a(k)$ *is well-defined, because* $-\lambda \le (k - d_k)h \le kh$ *for every* $k = 0, 1, 2, \ldots$

We introduce the simplifying notations $\sigma(t) \equiv t - \tau(t, x(t))$ and $\sigma_h(t) \equiv [t]_h - \Big[\tau([t]_h, y_h([t]_h))\Big]_h$.

The following result shows that there exists $\alpha > 0$ such that for $h > 0$ the solutions of the corresponding initial value problems (1.3)-(1.4) on $[0, \alpha]$ form a uniformly bounded family.

Lemma 2.3 *Assume (H1) and (H2'). Then for every $M \geq 5 \sup_{-\lambda \leq t \leq 0} \|\Phi(t)\|$ there exist $\alpha > 0$ such that for every $h > 0$*

$$\|y_h(t)\| \leq M, \qquad t \in [0, \alpha]. \tag{2.3}$$

PROOF: Fix an $M \geq 5 \sup_{-\lambda \leq t \leq 0} \|\Phi(t)\|$. Integrating (1.3) and using elementary manipulations we have the following estimate

$$\begin{aligned} \|y_h(t) - y_h(0)\| \;\; &\leq \;\; \int_0^t \Big\| f\Big([s]_h, y_h([s]_h), y_h(\sigma_h(s))\Big) - f\Big([s]_h, y_h(0), y_h(\sigma_h(0))\Big) \Big\| \, ds \\ &\quad + \int_0^t \Big\| f\Big([s]_h, y_h(0), y_h(\sigma_h(0))\Big) \Big\| \, ds. \end{aligned}$$

Assuming that $\alpha > 0$ is such, that (2.3) holds and using the Lipschitz-continuity of f on the set $[0, \alpha] \times B(M) \times B(M)$ with Lipschitz-constant, $L_1 = L_1(\alpha, M)$, we get

$$\begin{aligned} \|y_h(t) - y_h(0)\| \;\; &\leq \;\; L_1 \int_0^t \Big(\|y_h([s]_h) - y_h(0)\| + \|y_h(\sigma_h(s)) - y_h(\sigma_h(0))\| \Big) \, ds + K_1 t \\ &\leq \;\; L_1 \int_0^t \Big(\|y_h([s]_h) - y_h(0)\| + \|y_h(\sigma_h(s)) - y_h(0)\| \\ &\qquad + \|y_h(\sigma_h(0)) - y_h(0)\| \Big) \, ds + \; K_1 t, \qquad t \in [0, \alpha], \end{aligned} \tag{2.4}$$

where $K_1 \equiv K_1(\alpha) \equiv \sup\{\|f(t, \Phi(0), \Phi(s))\| \; : \; t \in [0, \alpha], \; s \in [-\lambda, 0]\}$.

We define the function $\psi_h(t) \equiv \max\Big\{ \sup_{-\lambda \leq s \leq 0} \|\Phi(s) - \Phi(0)\|, \; \max_{0 \leq s \leq t} \|y_h(s) - y_h(0)\| \Big\}$. Recalling that $\sigma_h(s) \leq s$ and $[s]_h \leq s$, and using the definition of ψ_h, inequality (2.4) implies

$$\|y_h(t) - y_h(0)\| \leq K_1 \alpha + L_1 \alpha \cdot \sup_{-\lambda \leq s \leq 0} \|\Phi(s) - \Phi(0)\| + \int_0^t 2 L_1 \psi_h(s) \, ds, \qquad t \in [0, \alpha].$$

Define the constant $K_2 \equiv K_2(\alpha) \equiv K_1 \alpha + \max\{L_1 \alpha, 1\} \cdot \sup_{-\lambda \leq s \leq 0} \|\Phi(s) - \Phi(0)\|$. It is easy to see that

$$\psi_h(t) \leq K_2 + \int_0^t 2 L_1 \psi_h(s) \, ds, \qquad t \in [0, \alpha]. \tag{2.5}$$

By applying the Gronwall-Bellman inequality for (2.5) we find that

$$\|y_h(t) - y_h(0)\| \leq \psi_h(t) \leq K_2 \cdot \exp(2 L_1 t), \qquad t \in [0, \alpha]. \tag{2.6}$$

It follows that if we can select $\alpha > 0$ such that

$$K_2 \cdot \exp(2 L_1 \alpha) + \|\Phi(0)\| \leq M \tag{2.7}$$

then (2.3) holds and our calculations are valid. Pick an apriori $T > 0$. Note that we can select Lipschitz-constant $L_1(\alpha, M)$ such that $L_1(\alpha, M) \leq L_1(T, M)$ for $\alpha \leq T$. Therefore we have that $K_2(\alpha) \leq K_1(T)\alpha + \max\{L_1(T, M)\alpha, 1\} \sup_{-\lambda \leq s \leq 0} \|\Phi(s) - \Phi(0)\| \leq 3 \sup_{-\lambda \leq s \leq 0} \|\Phi(s)\|$ for $\alpha > 0$ such that

$$L_1(T, M)\alpha \leq 1, \quad \text{and} \quad K_1(T)\alpha \leq \sup_{-\lambda \leq s \leq 0} \|\Phi(s)\|. \tag{2.8}$$

Now, assuming (2.8), we have that $K_2(\alpha) \cdot \exp\Big(2 L_1(\alpha, M)\alpha\Big) + \|\Phi(0)\| \leq 5 \sup_{-\lambda \leq t \leq 0} \|\Phi(t)\|$, provided that α satisfies

$$\exp\Big(2 L_1(T, M)\alpha\Big) \leq 4/3. \tag{2.9}$$

Select $\alpha \in (0,T]$ such that (2.8) and (2.9) hold, then (2.7) is satisfied, which proves the lemma.

We comment that if the Lipschitz-constant of f is independent of M, (i.e., f is Lipschitz-continuous on $[0,T]\times R^{2n}$ for arbitrary $T>0$), or the function f is bounded on $[0,T]\times R^{2n}$, then (2.6) and the definition of K_2, or the latter case (1.3) yields that $\{y_h(t)\}$ is uniformly bounded on every compact time interval.

Lemma 2.3 allows us to obtain existence results for IVP (1.1)-(1.2) following the steps of the classical Cauchy-Peano Theorem (see e.g. [1] or [9]): the approximate solutions $\{y_h(t) \ : \ h>0\}$ form a uniformly bounded family of functions on some interval $[0,\alpha]$. It is easy to see using (1.3) that $y_h(t)$ for $h>0$ are also equicontinuous functions on $[0,\alpha]$, and therefore there exists a sequence $h_k \to 0$, such that the corresponding functions converge to a continuous function, i.e., $x(t) \equiv \lim_{k\to\infty} y_{h_k}(t)$ exists and continuous on $[-\lambda,\alpha]$. Then it is easy to show (using a continuity argument) that $x(t)$ satisfies (1.1), i.e., IVP (1.1)-(1.2) has a solution on $[0,\alpha]$. (We refer to [3] for related well-posedness result using fixed point arguments.)

In particular, we have the following theorem (see the detailed proof in [7]):

Theorem 2.4 *Assume that (H1) and (H2') hold and $\Phi \in C([-\lambda,0],R)$. Then there exists $T>0$ such that IVP (1.1)-(1.2) has a solution on $[-\lambda,T]$.*

Assuming that the initial function Φ is Lipschitz-continuous and the delay function τ is locally Lipschitz-continuous with respect to its second argument we can prove the following convergence result for our approximating scheme, which also implies the uniqueness of solutions of IVP (1.1)-(1.2).

Theorem 2.5 *Assume (H1), (H2'), (H3') and (H4). Then if IVP (1.1)-(1.2) has a solution $x(t)$ on $[0,T]$, then the solution is unique and $\lim_{h\to 0+}\max_{0\le t\le T}\|x(t)-y_h(t)\|=0$, where y_h is the solution of IVP (1.3)-(1.4). If in addition (H2) and (H3) hold, then there exist constants $M_3(T,\Phi)>0$ and $h_0>0$ such that $\|x(t)-y_h(t)\|\le M_3 h$, for $t\in[0,T]$ and $0<h\le h_0$.*

Proof: Let $T>0$ be such that IVP (1.1)-(1.2) has a solution, $x(t)$, on $[0,T]$. For the uniqueness of solution it is enough to prove that $\lim_{h\to 0+}\max_{0\le t\le T}\|x(t)-y_h(t)\|=0$. Let $M_1 \equiv \max\{\|x(t)\| \ : \ t\in[-\lambda,T]\}+1$. Suppose that there exists $h_0>0$ such that

$$\|y_h(t)\| \le M_1, \qquad t\in[0,T], \quad 0<h\le h_0. \tag{2.10}$$

Define the constant $M_2 \equiv \max\{\|f(t,u,v)\| \ : \ t\in[0,T],\ u,v\in B(M_1)\}$, then y_h satisfies

$$\|y_h(t_1)-y_h(t_2)\| \le M_2|t_1-t_2|, \qquad t_1,t_2\in[0,T], \quad 0<h\le h_0. \tag{2.11}$$

Equations (1.1), (1.3), the assumed relation (2.10) and the assumptions of the theorem yield the following inequalities

$$\begin{aligned}
\|x(t)-y_h(t)\| &\le \int_0^t \Big\| f\Big(s,x(s),x(\sigma(s))\Big) - f\Big([s]_h,x(s),x(\sigma(s))\Big)\Big\|\,ds \\
&\quad + \int_0^t \Big\| f\Big([s]_h,x(s),x(\sigma(s))\Big) - f\Big([s]_h,y_h([s]_h),y_h(\sigma_h(s))\Big)\Big\|\,ds \\
&\le \int_0^t \Big\| f\Big(s,x(s),x(\sigma(s))\Big) - f\Big([s]_h,x(s),x(\sigma(s))\Big)\Big\|\,ds \\
&\quad + L_1\int_0^t \Big(\|x(s)-y_h([s]_h)\| + \|x(\sigma(s))-y_h(\sigma_h(s))\|\Big)\,ds,
\end{aligned} \tag{2.12}$$

where $L_1 = L_1(T,M_1)$. Using the notation $\eta_h(s) \equiv [s]_h - \Big[\tau([s]_h,x([s]_h))\Big]_h$, and elementary manipulations on the last term of the right hand side of (2.12) we have

$$\|x(t)-y_h(t)\| \le \int_0^t \Big\| f\Big(s,x(s),x(\sigma(s))\Big) - f\Big([s]_h,x(s),x(\sigma(s))\Big)\Big\|\,ds$$

$$+ L_1 \int_0^t \Big(\|x(s) - y_h(s)\| + \|y_h(s) - y_h([s]_h)\| + \|x(\sigma(s)) - x(\eta_h(s))\| + \|x(\eta_h(s)) - y_h(\eta_h(s))\| + \|y_h(\eta_h(s)) - y_h(\sigma_h(s))\| \Big)\, ds. \tag{2.13}$$

Now we shall estimate the term $\|y_h(\eta_h(s)) - y_h(\sigma_h(s))\|$ in (2.13).
Let $A \equiv \min\{\eta_h(s), \sigma_h(s)\}$ and $B \equiv \max\{\eta_h(s), \sigma_h(s)\}$. We need to study three cases: First we assume that $0 \le A \le B$. Then (2.11) yields $\|y_h(\eta_h(s)) - y_h(\sigma_h(s))\| \le M_2|A - B|$. If $A \le B \le 0$ then the assumed Lipschitz-continuity of Φ implies $\|y_h(\eta_h(s)) - y_h(\sigma_h(s))\| \le L_3|A - B|$. Finally, if $A \le 0 \le B$ then inequality (2.11) and the Lipschitz-continuity of Φ imply

$$\begin{aligned} \|y_h(\eta_h(s)) - y_h(\sigma_h(s))\| &\le \|y_h(A) - y_h(0)\| + \|y_h(0) - y_h(B)\| \\ &\le \max\{L_3, M_2\} \cdot |A - B|. \end{aligned}$$

Thus in all the above cases using the Lipschitz-continuity of τ we have

$$\begin{aligned} \|y_h(\eta_h(s)) - y_h(\sigma_h(s))\| &\le K_3|\tau([s]_h, x([s]_h)) - \tau([s]_h, y_h([s]_h))| + 2K_3h \\ &\le K_3L_2\|x([s]_h) - y_h([s]_h)\| + 2K_3h, \end{aligned} \tag{2.14}$$

where $K_3 \equiv \max\{L_3, M_2\}$ and $L_2 = L_2(T, M_1)$.

Let $\varepsilon_h(t) \equiv \|x(t) - y_h(t)\|$. Combining (2.13) and (2.14) and using the definition of ε_h we get

$$\varepsilon_h(t) \le L_1 \int_0^t \Big(\varepsilon_h(s) + \varepsilon_h(\eta_h(s)) + K_3L_2\varepsilon_h([s]_h) \Big)\, ds + g_h(t), \tag{2.15}$$

where

$$\begin{aligned} g_h(t) \equiv & \int_0^t \bigg(\Big\| f\Big(s, x(s), x(\sigma(s))\Big) - f\Big([s]_h, x(s), x(\sigma(s))\Big) \Big\| + \\ & + L_1 \Big(\|y_h(s) - y_h([s]_h)\| + \|x(\sigma(s)) - x(\eta_h(s))\| + 2K_3h \Big) \bigg)\, ds. \end{aligned} \tag{2.16}$$

Let $\psi_h(t) \equiv \max_{0 \le s \le t} \varepsilon_h(s)$. Now, $-\lambda \le \eta_h(s) \le s$, for every $s \ge 0$ and if $\eta_h(s) \le 0$ then $\varepsilon_h(\eta_h(s)) = 0$ therefore it follows from (2.15) that $\psi_h(t)$ satisfies

$$\psi_h(t) \le \int_0^t L_1(2 + K_3L_2)\psi_h(s)\, ds + g_h(t).$$

Since $g_h(t)$ is a monotone nondecreasing function, an application of the Gronwall-Bellmann inequality yields

$$\|x(t) - y_h(t)\| \le \psi_h(t) \le \exp\Big(L_1(2 + K_3L_2)T\Big) \cdot g_h(T), \qquad t \in [0, T]. \tag{2.17}$$

Define the functions $\omega_x(h) \equiv \max\{\, \|x(t) - x(\bar{t})\| \;:\; t, \bar{t} \in [-\lambda, T],\, |t - \bar{t}| \le h\}$, $\omega_\tau(h) \equiv \max\{|\tau(t, u) - \tau(\bar{t}, u)| \;:\; t, \bar{t} \in [0, T],\; |t - \bar{t}| \le h,\; u \in B(M_1)\}$, and $\omega_f(h) \equiv \max\{\, \|f(t, u, v) - f(\bar{t}, u, v)\| \;:\; t, \bar{t} \in [0, T],\; |t - \bar{t}| \le h,\; u, v \in B(M_1)\}$. Then from (2.16) and from the inequality

$$\begin{aligned} |\sigma(s) - \eta_h(s)| &\le |\tau(s, x(s)) - \tau([s]_h, x(s))| + |\tau([s]_h, x(s)) - \tau([s]_h, x([s]_h)| + h \\ &\le |\tau(s, x(s)) - \tau([s]_h, x(s))| + L_2\|x(s) - x([s]_h)\| + h \\ &\le \omega_\tau(h) + L_2\omega_x(h) + h \end{aligned}$$

it follows that

$$g_h(T) \le \Big(\omega_f(h) + L_1M_2h + L_1\omega_x\Big(\omega_\tau(h) + L_2\omega_x(h) + h\Big) + 2L_1K_3h \Big)T. \tag{2.18}$$

The function $x(t)$ is uniformly continuous on $[-\lambda, T]$ (we have that the initial function Φ is Lipschitz-continuous on $[-\lambda, 0]$), and hence $\omega_x(h) \to 0$, as $h \to 0^+$, and similarly by, the uniform continuity of f and τ on $[0,T]\times B(M_1)\times B(M_1)$ and $[0,T]\times B(M_1)$, respectively, we also have that $\omega_f(h)$ and $\omega_\tau(h)$ go to zero as $h\to 0^+$. Therefore (2.18) yields that

$$g_h(T)\to 0, \qquad \text{as } h\to 0+. \tag{2.19}$$

Select $h_0>0$ such that

$$\exp\Big(L_1(2+K_3L_2)T\Big)\cdot\Big(\omega_f(h_0)+L_1M_2h_0+L_1\omega_x\Big(\omega_\tau(h_0)+L_2\omega_x(h_0)+h_0\Big)+2L_1K_3h_0\Big)T\le 1,$$

then from inequalities (2.17), (2.18) and definition of M_1 it follows that h_0 satisfies (2.10). Hence our calculation is valid, and (2.17) with (2.19) imply the first statement of the theorem.

To show the second statement of the theorem, observe that the definition of M_1 and equation (1.1) yield that $\|\dot{x}(t)\|\le\max\{\|f(t,u,v)\| \;:\; t\in[0,T],\ u,v\in B(M_1)\}$ for $t\in[0,T]$, hence $\omega_x(h)\le K_4h$, where $K_4\equiv\max\Big\{L_3,\max\{\|f(t,u,v)\| \;:\; t\in[0,T],\ u,v\in B(M_1)\}\Big\}$. Assumption (H2) and (H3) imply $\omega_f(h)\le\tilde{L}_1h$ and $\omega_\tau(h)\le\tilde{L}_2h$, respectively, where $\tilde{L}_2=\tilde{L}_2(T,M_1)$ and $\tilde{L}_1=\tilde{L}_1(T,M_1)$. Inequalities (2.17) and (2.18) yield the second statement of the theorem using these estimates of $\omega_x(h)$, $\omega_f(h)$ and $\omega_\tau(h)$, with constant

$$M_3\equiv\exp\Big(L_1(2+K_3L_2)T\Big)\cdot\Big(\tilde{L}_1+L_1M_2+L_1K_4(\tilde{L}_2+L_2K_4+1)+2L_1K_3\Big)\,T.$$

The proof of the theorem is complete.

We close the paper by noting that our results can be extended in a rather straightforward fashion to the case when the equation has multiple delays.

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