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BOUNDEDNESS OF POSITIVE SOLUTIONS OF A SYSTEM OF NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this manuscript the system of nonlinear delay differential equations $\dot{x}_i(t) = \sum_{j=1}^{n}\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)h_{ij}(x_j(t-\tau_{ij\ell}(t))) - \beta_i(t)f_i(x_i(t)) + \rho_i(t)$, $t \geq 0$, $1 \leq i \leq n$ is considered. Sufficient conditions are established for the uniform permanence of the positive solutions of the system. In several particular cases, explicit formulas are given for the estimates of the upper and lower limit of the solutions. In a special case, the asymptotic equivalence of the solutions is investigated.

1. **Introduction.** Nonlinear differential equations with delays frequently appear as model equations in physics, engineering, economics and biology. Next we recall some typical applications.

Compartmental systems are used to model many processes in pharmacokinetics, metabolism, epidemiology and ecology (see [19, 20, 24]). The nonlinear donor-controlled compartmental system

$$\dot{q}_i(t) = -k_{ii}f_i(q_i(t)) + \sum_{\substack{j=1\\ j\neq i}}^{n} k_{ij}f_j(q_j(t-\tau_{ij})) + I_i, \qquad i = 1,\dots,n$$

was studied in [8, 9]. Here $q_i(t)$ is the mass of the ith compartment at time t, $k_{ij} > 0$ represent the transfer or rate coefficients, $I_i \geq 0$ is the inflow to the ith compartment, and in this model it is assumed that the intercompartmental flows are functions of the state of the donor compartments only in the form $k_{ij}f_j(q_j)$ with some positive nonlinear function f_j.

The classical Hopfield neural networks [23] have been successfully used in many different fields of engineering applications, e.g., in signal processing, image processing, pattern recognition. In hardware implementations time delays appear naturally due to finite switching speed of the amplifiers, see, e.g., [3, 13, 17]. We recall the system

$$C_i\dot{u}_i(t) = -\frac{1}{R_i}u_i(t) + \sum_{j=1}^{n} T_{ij}g_j(u_j(t-\tau_{ij})) + I_i, \qquad i = 1,\dots,n,$$

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which was studied in [17]. Here $C_i > 0$, $R_i > 0$ and I_i are capacity, resistance, bias, respectively, T_{ij} is the interconnection weight, and g_i is a strictly monotone increasing nonlinear function with $g_i(0) = 0$. In [11] the existence, uniqueness and global stability of asymptotically periodic solutions of the bidirectional associative memory (BAM) network

$$\begin{aligned}\dot{x}_i(t) &= -a_i(t)x_i(t) + \sum_{j=1}^{k} p_{ji}(t)f_j(y_j(t-\tau_{ji})) + I_i(t), \quad i = 1,\ldots,n,\\ \dot{y}_j(t) &= -b_j(t)y_j(t) + \sum_{i=1}^{n} q_{ij}(t)f_i(x_j(t-\sigma_{ij})) + J_i(t), \quad j = 1,\ldots,k\end{aligned}$$

was examined.

In [12] the delay model

$$\begin{aligned}\dot{R}(t) &= f(T(t-\tau_3)) - d_1R(t)\\ \dot{L}(t) &= r_1R(t-\tau_1) - d_2L(t)\\ \dot{T}(t) &= r_2L(t-\tau_3) - d_3T(t)\end{aligned}$$

was considered for the control of the secretion of the hormone testosterone. Here $R(t)$, $L(t)$ and $T(t)$ are the concentrations of the gonadotropinreleasing hormone, luteinizing hormone and testosterone, respectively, r_1, r_2, d_1, d_2, d_3 are positive constants. Global stability of a positive equilibrium and oscillations of the solutions were investigated depending on the values of a parameter in the formula of the positive nonlinear function f.

In [5] the two-dimensional system

$$\dot{x}(t) = r_1(t)\Big[f_1(y(t-\tau_1(t)) - x(t)\Big], \qquad t \geq 0 \tag{1}$$

$$\dot{y}(t) = r_2(t)\Big[f_2(x(t-\tau_2(t)) - y(t)\Big], \qquad t \geq 0 \tag{2}$$

was considered as a special case of a more general two-dimensional system of nonlinear delay equations with distributed delays. Sufficient conditions were given implying that the solutions of the System (1)-(2) are *permanent*, i.e., there exist positive constants a, A, b and B such that $a \leq x(t) \leq A$ and $b \leq y(t) \leq B$ hold for $t \geq 0$.

Populations are frequently modeled in heterogeneous environments due to, e.g., different food-rich patches, different stages of a species according to age or size. In such models time delays appear naturally due to the time needed for species to disperse from one patch to another. We recall here the n-dimensional Nicholson's blowflies systems with patch structure

$$\dot{x}_i(t) = \sum_{\ell=1}^{n_0} \beta_{i\ell}x_i(t-\tau_{i\ell})e^{-x_i(t-\tau_{i\ell})} + \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}x_j(t) - d_ix_i(t), \quad 1 \leq i \leq n, \tag{3}$$

where $d_i > 0, \beta_{i\ell} \geq 0, a_{ij} \geq 0, \tau_{i\ell} \geq 0$ for $1 \leq i \neq j \leq n$, $\ell = 1,\ldots,n_0$. Asymptotic behavior, permanence of the solutions was investigated, e.g., in, [4, 7, 16, 26]. For the scalar case, this model reduces to the famous Nicholson's blowflies equation introduced in [18] to model the Australian sheep-blowfly population.

The n-dimensional population model with patch structure

$$\begin{aligned}\dot{x}_i(t) &= \sum_{\ell=1}^{n_0} \frac{\lambda_{i\ell}(t)x_i(t-\tau_{i\ell}(t))}{1+\gamma_{i\ell}(t)x_i(t-\tau_{i\ell}(t))} + \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(t)x_j(t-\sigma_{ij}(t)) \\ &\quad -\mu_i(t)x_i(t) - \kappa_i(t)x_i^2(t), \qquad t\geq 0, \quad 1\leq i\leq n \end{aligned} \tag{4}$$

was introduced in [15], and the permanence of the positive solutions was investigated. Here all functions are nonnegative. It is a generalization of a scalar modified logistic equation with several delays introduced in [2].

Motivated by the above models, in this paper we consider a system of nonlinear delay differential equations of the form

$$\dot{x}_i(t) = \sum_{j=1}^{n}\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)h_{ij}(x_j(t-\tau_{ij\ell}(t))) - \beta_i(t)f_i(x_i(t)) + \rho_i(t), \quad t\geq 0, 1\leq i\leq n, \tag{5}$$

where, $f_i, h_{ij}, \alpha_{ij\ell}, \beta_i, \rho_i$ and $\tau_{ij\ell}$ are nonnegative continuous functions. We associate the initial condition

$$x_i(t) = \varphi_i(t), \qquad -\tau\leq t\leq 0, \ 1\leq i\leq n \tag{6}$$

to our system, where $\tau>0$ and $0\leq\tau_{ij\ell}(t)\leq\tau$ hold for $t\geq 0, 1\leq i,j\leq n$ and $1\leq \ell\leq n_0$. We study positive solutions of the System (5), so we assume that $\varphi_i\in C_+ := \{\psi\in C([-\tau,0],\mathbb{R}_+) : \psi(0)>0\}$, $1\leq i\leq n$, where $\mathbb{R}_+ := [0,\infty)$. Our main result, Theorem 2.4 below shows that, under certain conditions, the solutions of the initial value problem (IVP) (5) and (6) is *uniformly permanent*, i.e., there exist positive constants $k_1,\dots,k_n, K_1,\dots,K_n$, such that for any initial functions $\varphi_i\in C_+$, $i=1,\dots,n$ the corresponding solution satisfies

$$0<k_i\leq \liminf_{t\to\infty} x_i(t) \leq \limsup_{t\to\infty} x_i(t)\leq K_i, \qquad 1\leq i\leq n.$$

Moreover, the constants $k_1,\dots,k_n$ and $K_1,\dots,K_n$ are given explicitly, as unique positive solutions of an associated nonlinear algebraic systems. As a consequence of the main result, we formulate conditions which imply that all the positive solutions converge to a constant limit (see Corollary 3.1 below). In Theorem 3.5, for nonlinear systems of the form

$$\dot{x}_i(t) = \sum_{j=1}^{n}\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)x_j(t-\tau_{ij\ell}(t)) - \beta_i(t)x_i^{q_i}(t) + \rho_i(t), \qquad t\geq 0, \quad 1\leq i\leq n,$$

we give sufficient conditions which imply that the positive solutions are *asymptotically equivalent*, i.e., the difference of any two positive solutions tends to 0 as the time goes to ∞.

Permanence of solutions of a differential equation model is especially important in mathematical biology and ecology [7, 16, 25]. Permanence of solutions of scalar delay population models was recently studied in [1, 2, 6, 14, 16]. In [21] we considered a scalar delay population model

$$\dot{x}(t) = r(t)\Big(g(t,x_t) - h(x(t))\Big), \qquad t\geq 0,$$

where $r,h\in C(\mathbb{R}_+,\mathbb{R}_+)$, $g\in C(\mathbb{R}_+\times C([-\tau,0],\mathbb{R}),\mathbb{R}_+)$, $\tau>0$ is fixed, and $x_t(s) = x(t+s)$ for $s\in[-\tau,0]$. This manuscript extends the method introduced for the scalar case in [21] to the nonlinear delay system (5). A key element of the proof of our Theorem 2.4 is a result proved in [22], where sufficient conditions are formulated implying that a certain nonlinear algebraic system associated to (5) has a unique positive solution (see Lemma 2.3 and Theorem 4.2 below).

The structure of our paper is the following. In Section 2 we formulate our main results: Theorem 2.4 below gives estimates for the limit inferior and limit superior of the positive solutions of System (5). In Section 3 we show several corollaries of our main results and numerical examples. In Section 4 we give the proofs of our main results, and in Section 5 we summarize our conclusions and formulate some open questions.

2. **Main results.** We start this section by listing all conditions on the parameters of the IVP (5) and (6) will be used in the rest of the manuscript. $\tau > 0$ is a fixed constant, and all delay functions are assumed to be uniformly bounded by τ.

($\mathbf{A_0}$) $\tau_{ij\ell} \in C(\mathbb{R}_+, \mathbb{R}_+)$ are such that $0 \le \tau_{ij\ell}(t) \le \tau$ for $t \ge 0$, $1 \le i, j \le n$ and $1 \le \ell \le n_0$;

($\mathbf{A_1}$) $\beta_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ are such that $\beta_i(t) > 0$ for $t > 0$, $1 \le i \le n$, and

$$\int_0^\infty \beta_i(s)\,ds = \infty, \qquad 1 \le i \le n;$$

($\mathbf{A_2}$) $\alpha_{ij\ell} \in C(\mathbb{R}_+, \mathbb{R}_+)$, for all $1 \le i, j \le n$ and $1 \le \ell \le n_0$ are such that

$$\sup_{t>0} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{\beta_i(t)} < \infty, \qquad 1 \le i, j \le n; \tag{7}$$

($\mathbf{A_3}$) $f_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, $1 \le i \le n$, are strictly increasing with $f_i(0) = 0$ and f_i are locally Lipschitz continuous;

($\mathbf{A_4}$) $h_{ij} \in C(\mathbb{R}_+, \mathbb{R}_+)$ are increasing, locally Lipschitz continuous, and $h_{ij}(u) > 0$ for $u > 0$ and $1 \le i, j \le n$;

($\mathbf{A_5}$) $\rho_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ and for each $i = 1, \dots, n$,

$$\text{either} \quad \liminf_{t\to\infty} \frac{\rho_i(t)}{\beta_i(t)} > 0 \quad \text{or} \quad \limsup_{u\to 0^+} \frac{f_i(u)}{h_{ii}(u)} < \liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ii\ell}(t)}{\beta_i(t)}, \tag{8}$$

$$\sup_{t>0} \frac{\rho_i(t)}{\beta_i(t)} < \infty, \qquad \lim_{u\to\infty} f_i(u) = \infty, \tag{9}$$

and

$$\sum_{j=1}^{n} \Big(\limsup_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{\beta_i(t)} \Big) \limsup_{u\to\infty} \frac{h_{ij}(u)}{f_i(u)} < 1; \tag{10}$$

($\mathbf{A_6}$) (i) $\frac{f_i(u)}{h_{ij}(u)}$ is increasing and $\frac{h_{jj}(u)}{h_{ij}(u)}$ is decreasing on $(0, \infty)$, for each $1 \le i, j \le n$;

(ii) for each $1 \le i \le n$, either $\frac{f_i(u)}{h_{ii}(u)}$ is strictly increasing on the interval $(0, \infty)$ or $\Big(\liminf_{t\to\infty} \frac{\rho_i(t)}{\beta_i(t)} > 0$ and $h_{ii}(u)$ is strictly increasing on $(0, \infty) \Big)$;

(iii) either $\liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{\beta_i(t)} = 0$ for all $i, j \in \{1, \dots, n\}$ satisfying $i \ne j$; or there exist $i, j \in \{1, \dots, n\}$, $i \ne j$ such that $\liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{\beta_i(t)} > 0$ and

$\Big[$either $\frac{f_j(u)}{h_{ij}(u)}$ is strictly increasing on $(0,\infty)$ or $\Big(\liminf\limits_{t\to\infty} \frac{\sum_{\ell=1}^{n_0}\alpha_{jj\ell}(t)}{\beta_j(t)} > 0$ and $\frac{h_{jj}(u)}{h_{ij}(u)}$ is strictly decreasing on $(0,\infty)\Big)$ or $\Big(\liminf\limits_{t\to\infty} \frac{\rho_j(t)}{\beta_j(t)} > 0$ and $h_{ij}(u)$ is strictly increasing on $(0,\infty)\Big)\Big]$;

(iv) for each $1 \le i \le n$, either $\frac{f_i(u)}{h_{ii}(u)}$ is strictly increasing on the interval $(0,\infty)$ or $\Big(\limsup\limits_{t\to\infty} \frac{\rho_i(t)}{\beta_i(t)} > 0$ and $h_{ii}(u)$ is strictly increasing on $(0,\infty)\Big)$;

(v) either $\limsup\limits_{t\to\infty} \frac{\sum_{\ell=1}^{n_0}\alpha_{ij\ell}(t)}{\beta_i(t)} = 0$ for all $i,j \in \{1,\dots,n\}$ satisfying $i \ne j$; or there exist $i,j \in \{1,\dots,n\}$, $i \ne j$ such that $\limsup\limits_{t\to\infty} \frac{\sum_{\ell=1}^{n_0}\alpha_{ij\ell}(t)}{\beta_i(t)} > 0$ and $\Big[$either $\frac{f_j(u)}{h_{ij}(u)}$ is strictly increasing on $(0,\infty)$ or $\Big(\limsup\limits_{t\to\infty} \frac{\sum_{\ell=1}^{n_0}\alpha_{jj\ell}(t)}{\beta_j(t)} > 0$ and $\frac{h_{jj}(u)}{h_{ij}(u)}$ is strictly decreasing on $(0,\infty)\Big)$ or $\Big(\limsup\limits_{t\to\infty} \frac{\rho_j(t)}{\beta_j(t)} > 0$ and $h_{ij}(u)$ is strictly increasing on $(0,\infty)\Big)\Big]$.

Clearly, under conditions $(\mathbf{A_0})$-$(\mathbf{A_5})$, the IVP (5) and (6) has a unique solution corresponding to any $\varphi = (\varphi_1,\dots,\varphi_n) \in C_+^n$. This solution is denoted by $x(\varphi) = (x_1(\varphi),\dots,x_n(\varphi))$. Note that in [21] a scalar version of (5) was studied where, instead of the local Lipschitz-continuity, it was assumed that f_i are such that for any nonnegative constants ϱ and L satisfying $L \ne \varrho$, one has

$$\int_L^{\varrho} \frac{ds}{f_i(\varrho) - f_i(s)} = +\infty. \tag{11}$$

Hence the solution studied in [21] was not necessary unique. It is easy to see that the locally Lipschitz-continuity of f_i implies condition (11). We assume the locally Lipschitz-continuity of f_i and h_{ij} to simplify the presentation, but it can be omitted as in [21].

We note that assumption $(\mathbf{A_3})$ is weaker than that used in the [2, 14], where, investigating permanence of a scalar population model, it was assumed that the coefficient function β_i is bounded below and above by positive constants.

The monotonicity assumptions of $(\mathbf{A_6})$ for the ratios $\frac{f_i(u)}{h_{ij}(u)}$ and $\frac{h_{jj}(u)}{h_{ij}(u)}$ are crucial for using Lemma 2.3 below. This assumption allows us to include examples when some ratios are constants, and only some of these functions are strictly monotone. This week form of the condition will be important when we apply our main results to the population models (3) and (4) (see Corollary 3.7 and 3.8 below).

The next lemma shows that all solutions of System (5) corresponding to any initial function $\varphi = (\varphi_1,\varphi_2,\dots,\varphi_n) \in C_+^n$ are positive on $\mathbb{R}_+$.

Lemma 2.1. *Assume that $\tau_{ij\ell}$ satisfies condition $(\boldsymbol{A_0})$, β_i satisfies condition $(\boldsymbol{A_1})$, f_i satisfies condition $(\boldsymbol{A_3})$ and $\alpha_{ij\ell}, h_{ij}, \rho_i \in C(\mathbb{R}_+,\mathbb{R}_+)$ for $1 \le i,j \le n$ and $1 \le \ell \le n_0$. Then for any initial function $\varphi = (\varphi_1,\varphi_2,\dots,\varphi_n) \in C_+^n$, the solution $x(\varphi)(t) = (x_1(\varphi)(t),\dots,x_n(\varphi)(t))$ of the IVP (5) and (6) obeys $x_i(\varphi)(t) > 0$ for $t \ge 0$ and $1 \le i \le n$.*

The following result implies that, under our conditions, the System (5) is persistent.

Lemma 2.2. *Assume that conditions* ($\boldsymbol{A_0}$)–($\boldsymbol{A_5}$) *are satisfied. Then for any* $\varphi \in C_+^n$, *the solution* $x(\varphi)(t) = (x_1(\varphi)(t), \ldots, x_n(\varphi)(t))$ *of the IVP (5) and (6) satisfies*

$$0 < \inf_{t \geq 0} x_i(\varphi)(t) \leq \sup_{t \geq 0} x_i(\varphi)(t) < \infty, \qquad 1 \leq i \leq n. \tag{12}$$

The next lemma displays several properties of the positive solutions of the algebraic system

$$f_i(x_i) = \sum_{j=1}^{n} m_{ij} h_{ij}(x_j) + l_i, \qquad 1 \leq i \leq n. \tag{13}$$

We say that $x = (x_1, \ldots, x_n)$ is a *positive solution* of (13) if $x_i > 0$ for $i = 1, \ldots, n$.

Lemma 2.3. *Assume that* $m_{ij} \geq 0$, $l_i \geq 0$ *for* $1 \leq i, j \leq n$, f_i *satisfies condition* ($\boldsymbol{A_3}$) *and* h_{ij} *satisfies condition* ($\boldsymbol{A_4}$). *Suppose that*

($\boldsymbol{H_1}$) $\frac{f_i(u)}{h_{ij}(u)}$ *is increasing and* $\frac{h_{jj}(u)}{h_{ij}(u)}$ *is decreasing on* $(0, \infty)$ *for each* $1 \leq i, j \leq n$;

($\boldsymbol{H_2}$) *for each* $1 \leq i \leq n$, *either* $\frac{f_i(u)}{h_{ii}(u)}$ *is strictly increasing on* $(0, \infty)$ *or* $\Big(l_i > 0$ *and* $h_{ii}(u)$ *is strictly increasing on* $(0, \infty)\Big)$;

($\boldsymbol{H_3}$) *either* $m_{ij} = 0$ *for all* $i, j \in \{1, \ldots, n\}$ *satisfying* $i \neq j$; *or there exist* $i, j \in \{1, \ldots, n\}$, $i \neq j$ *such that* $m_{ij} > 0$ *and* $\Big[$*either* $\frac{f_j(u)}{h_{ij}(u)}$ *is strictly increasing on* $(0, \infty)$ *or* $\Big(m_{jj} > 0$ *and* $\frac{h_{jj}(u)}{h_{ij}(u)}$ *is strictly decreasing on* $(0, \infty)\Big)$ *or* $\Big(l_j > 0$ *and* $h_{ij}(u)$ *is strictly increasing on* $(0, \infty)\Big)\Big]$;

($\boldsymbol{H_4}$) *the functions* f_i *and* h_{ij} *satisfy*

$$\text{either} \quad l_i > 0 \quad \text{or} \quad \lim_{u \to 0^+} \frac{f_i(u)}{h_{ii}(u)} < m_{ii}, \qquad 1 \leq i \leq n, \tag{14}$$

and

$$\sum_{j=1}^{n} m_{ij} \lim_{u \to \infty} \frac{h_{ij}(u)}{f_i(u)} < 1 \quad \text{and} \quad \lim_{u \to \infty} f_i(u) = \infty, \qquad 1 \leq i \leq n. \tag{15}$$

Then

(i) the System (13) has a unique positive solution $x^* = (x_1^*, \ldots, x_n^*)$.

(ii) For any $x = (x_1, \ldots, x_n)$ *satisfying*

$$x_i > 0, \qquad f_i(x_i) \geq \sum_{j=1}^{n} m_{ij} h_{ij}(x_j) + l_i, \qquad 1 \leq i \leq n, \tag{16}$$

one has

$$x_i \geq x_i^*, \qquad 1 \leq i \leq n.$$

(iii) For any $x = (x_1, \ldots, x_n)$ *satisfying*

$$x_i > 0, \qquad f_i(x_i) \leq \sum_{j=1}^{n} m_{ij} h_{ij}(x_j) + l_i, \qquad 1 \leq i \leq n,$$

one has

$$x_i \leq x_i^*, \qquad 1 \leq i \leq n.$$

We use the following notations in our main theorem:

$$\underline{m}_{ij} := \liminf_{t\to\infty} \frac{\sum\limits_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{\beta_i(t)}, \quad \overline{m}_{ij} := \limsup_{t\to\infty} \frac{\sum\limits_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{\beta_i(t)}, \quad 1 \le i, j \le n, \tag{17}$$

$$\underline{l}_i := \liminf_{t\to\infty} \frac{\rho_i(t)}{\beta_i(t)}, \qquad \bar{l}_i := \limsup_{t\to\infty} \frac{\rho_i(t)}{\beta_i(t)}, \qquad 1 \le i \le n. \tag{18}$$

We note that $(\mathbf{A_2})$, $(\mathbf{A_5})$ and Lemma 2.2 yield $0 \le \underline{m}_{ij} < \infty$, $0 \le \overline{m}_{ij} < \infty$, $0 \le \underline{l}_i < \infty$, $0 \le \bar{l}_i < \infty$ for $1 \le i, j \le n$, and

$$0 < \liminf_{t\to\infty} x_i(t) \le \limsup_{t\to\infty} x_i(t) < \infty, \qquad 1 \le i \le n.$$

Now, we are ready to formulate the main result of this paper.

Theorem 2.4. *Assume that conditions $(\boldsymbol{A_0})$–$(\boldsymbol{A_5})$ are satisfied.*

(i) If, in addition, $(\boldsymbol{A_6})$ (i), (ii) and (iii) hold, then for any initial function $\varphi = (\varphi_1, \dots, \varphi_n) \in C_+^n$, the solution $x(\varphi)(t) = (x_1(\varphi)(t), \dots, x_n(\varphi)(t))$ of the IVP (5) and (6) obeys

$$\underline{x}_i^* \le \liminf_{t\to\infty} x_i(\varphi)(t), \qquad 1 \le i \le n,$$

where $(\underline{x}_1^*, \dots, \underline{x}_n^*)$ is the unique positive solution of the algebraic system

$$f_i(x_i) = \sum_{j=1}^{n} \underline{m}_{ij} h_{ij}(x_j) + \underline{l}_i, \qquad 1 \le i \le n. \tag{19}$$

(ii) If, in addition, $(\boldsymbol{A_6})$ (i), (iv) and (v) hold, then for any initial function $\varphi = (\varphi_1, \dots, \varphi_n) \in C_+^n$, the solution $x(\varphi)(t) = (x_1(\varphi)(t), \dots, x_n(\varphi)(t))$ of the IVP (5) and (6) obeys

$$\limsup_{t\to\infty} x_i(\varphi)(t) \le \overline{x}_i^*, \qquad 1 \le i \le n,$$

where $(\overline{x}_1^*, \dots, \overline{x}_n^*)$ is the unique positive solution of the algebraic system

$$f_i(x_i) = \sum_{j=1}^{n} \overline{m}_{ij} h_{ij}(x_j) + \bar{l}_i, \qquad 1 \le i \le n. \tag{20}$$

3. **Corollaries and examples.**

In this section we present several corollaries to our main result and illustrative numerical examples.

Corollary 3.1. *Assume that conditions $(\boldsymbol{A_0})$–$(\boldsymbol{A_6})$ are satisfied, moreover, the finite limits*

$$m_{ij} := \lim_{t\to\infty} \frac{\sum\limits_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{\beta_i(t)} \quad \text{and} \quad l_i := \lim_{t\to\infty} \frac{\rho_i(t)}{\beta_i(t)}, \qquad 1 \le i, j \le n$$

exist. Then, for any initial function $\varphi = (\varphi_1, \dots, \varphi_n) \in C_+^n$, the solution $x(\varphi)(t) = (x_1(\varphi)(t), \dots, x_n(\varphi)(t))$ of the IVP (5) and (6) satisfies

$$\lim_{t\to\infty} x_i(\varphi)(t) = x_i^*, \qquad 1 \le i \le n,$$

where $(x_1^*, \dots, x_n^*)$ *is the unique positive solution of the algebraic system*

$$f_i(x_i) = \sum_{j=1}^{n} m_{ij} h_{ij}(x_j) + l_i, \qquad 1 \le i \le n.$$

Now, we study a special form of (5). We consider the IVP

$$\dot{x}_i(t) = \sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) x_j^{p_{ij}}(t - \tau_{ij\ell}(t)) - \beta_i(t) x_i^{q_i}(t) + \rho_i(t), \qquad t \ge 0, \quad 1 \le i \le n, \tag{21}$$

with the initial condition

$$x_i(t) = \varphi_i(t), \qquad -\tau \le t \le 0, \quad 1 \le i \le n, \tag{22}$$

where $\tau > 0$, $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n) \in C_+^n$ and $\alpha_{ij\ell}, \beta_i, \tau_{ij\ell} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $p_{ij}, q_i \in \mathbb{R}_+$ for $1 \le i, j \le n$ and $1 \le \ell \le n_0$. We remark that $(\mathbf{A_3})$, $(\mathbf{A_4})$, $(\mathbf{A_5})$ and $(\mathbf{A_6})$ hold if

$$q_i > p_{ij} > 1, \quad \text{and} \quad p_{ij} \ge p_{jj}, \qquad 1 \le i, j \le n \tag{23}$$

and

$$\text{either} \quad \liminf_{t\to\infty} \frac{\rho_i(t)}{\beta_i(t)} > 0 \quad \text{or} \quad \liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} \alpha_{ii\ell}(t)}{\beta_i(t)} > 0, \quad i = 1, \dots, n \tag{24}$$

are satisfied. Therefore Theorem 2.4 has the following consequence.

Corollary 3.2. *Assume that* $\tau_{ij\ell}$ *satisfies* $(\boldsymbol{A_0})$, β_i *and* $\alpha_{ij\ell}$ *satisfy* $(\boldsymbol{A_1})$ *and* $(\boldsymbol{A_2})$, $\rho_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ *satisfies* $\sup_{t>0} \frac{\rho_i(t)}{\beta_i(t)} < \infty$, $1 \le i \le n$, *and (23) and (24) hold. Then, for any initial function* $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n) \in C_+^n$ *, the solution* $x(\varphi)(t) = (x_1(\varphi)(t), \dots, x_n(\varphi)(t))$ *of the IVP (21) and (22) satisfies*

$$\underline{x}_i^* \le \liminf_{t\to\infty} x_i(\varphi)(t) \le \limsup_{t\to\infty} x_i(\varphi)(t) \le \overline{x}_i^*, \qquad 1 \le i \le n,$$

where $(\underline{x}_1^*, \dots, \underline{x}_n^*)$ *is the unique positive solution of the algebraic system*

$$x_i^{q_i} = \sum_{j=1}^{n} \underline{m}_{ij} x_j^{p_{ij}} + \underline{l}_i, \qquad 1 \le i \le n, \tag{25}$$

and $(\overline{x}_1^*, \dots, \overline{x}_n^*)$ *is the unique positive solution of the algebraic system*

$$x_i^{q_i} = \sum_{j=1}^{n} \overline{m}_{ij} x_j^{p_{ij}} + \overline{l}_i, \qquad 1 \le i \le n, \tag{26}$$

respectively, where $\underline{m}_{ij}, \overline{m}_{ij}, \underline{l}_i$ *and* $\overline{l}_i$ *are defined in (17) and (18) for* $1 \le i, j \le n$.

We remark that the condition (23) in Corollary 3.2 can be weakened.

Example 3.3. Consider the following system of nonlinear differential equations in three dimensions, for $t \geq 0$,

$$\begin{aligned}
\dot{x}_1(t) &= t^{0.1}(1+\cos t)x_1(t-2) + t^{0.1}x_1(t-1.5) + t^{0.1}x_2^2(t-0.05) \\
&\quad + t^{0.1}x_2^2(t-3) + t^{0.1}(2+2\sin t)x_3^3(t-0.5) \\
&\quad + t^{0.1}x_3^3(t-2.4) + t^{0.1}x_3^3(t-2.5) - 2t^{0.1}x_1^4(t) \\
&\quad + 0.2t^{0.1}(1.2+\sin t), \\
\dot{x}_2(t) &= x_1(t-1.5) + 2x_1(t-0.5) + x_1(t-0.4) \\
&\quad + 6(10+\cos t)x_2(t-0.05) + (3+3\cos t)x_3^2(t-0.09) \\
&\quad + 2x_3^2(t-1.3) - x_2^3(t) + 4.5 + \cos t, \\
\dot{x}_3(t) &= 5x_1^2(t-1.9) + 2x_1^3(t-0.2) + x_1^3(t-0.3) + 10x_2(t-1.2) \\
&\quad + (2+5\sin t)x_2(t-5) + 6x_3^2(t-0.01) + 4x_3^2(t-1) \\
&\quad - 2x_3^3(t) + 4.5 + 2\cos t.
\end{aligned} \tag{27}$$

Note that the conditions of Corollary 3.2 are satisfied for (27). So, we see from Corollary 3.2 that

$$\liminf_{t\to\infty} x_1(t) \geq \underline{x}_1^*, \quad \liminf_{t\to\infty} x_2(t) \geq \underline{x}_2^* \quad \text{and} \quad \liminf_{t\to\infty} x_1(t) \geq \underline{x}_3^*,$$

where $(\underline{x}_1^*, \underline{x}_2^*, \underline{x}_3^*)$ is the unique positive solution of the algebraic system

$$\begin{aligned}
x_1^4 &= 0.5x_1 + x_2^2 + x_3^3 + 0.02, \\
x_2^3 &= 4x_1 + 54x_2 + 2x_3^2 + 3.5, \\
x_3^3 &= 4x_1^2 + 3.5x_2 + 5x_3^2 + 1.25.
\end{aligned} \tag{28}$$

We solve the System (28) numerically by the fixed point iteration

$$\begin{aligned}
\underline{x}_1^{(k+1)} &= \sqrt[4]{0.5\underline{x}_1^{(k)} + (\underline{x}_2^{(k)})^2 + (\underline{x}_3^{(k)})^3 + 0.02}, \\
\underline{x}_2^{(k+1)} &= \sqrt[3]{4\underline{x}_1^{(k)} + 54\underline{x}_2^{(k)} + 2(\underline{x}_3^{(k)})^2 + 3.5}, \\
\underline{x}_3^{(k+1)} &= \sqrt[3]{4(\underline{x}_1^{(k)})^2 + 3.5\underline{x}_2^{(k)} + 5(\underline{x}_3^{(k)})^2 + 1.25}.
\end{aligned} \tag{29}$$

We compute the sequence $(\underline{x}_1^{(k)}, \underline{x}_2^{(k)}, \underline{x}_3^{(k)})$ defined by (29) starting from the initial value $(\underline{x}_1^{(0)}, \underline{x}_2^{(0)}, \underline{x}_3^{(0)}) = (0,0,0)$. The first ten terms of this sequence are displayed in Table 1. We can observe that the sequence is convergent, and its limit is $(\underline{x}_1^*, \underline{x}_2^*, \underline{x}_3^*) = (4.5960\ldots, 8.3147\ldots, 7.2095\ldots)$.

Similarly, we can see that

$$\limsup_{t\to\infty} x_1(t) \leq \overline{x}_1^*, \quad \limsup_{t\to\infty} x_2(t) \leq \overline{x}_2^* \quad \text{and} \quad \limsup_{t\to\infty} x_1(t) \leq \overline{x}_3^*,$$

where $(\overline{x}_1^*, \overline{x}_2^*, \overline{x}_3^*)$ is the unique positive solution of the algebraic system

$$\begin{aligned}
x_1^4 &= 1.5x_1 + x_2^2 + 3x_3^3 + 0.22, \\
x_2^3 &= 4x_1 + 66x_2 + 8x_3^2 + 5.5, \\
x_3^3 &= 4x_1^2 + 8.5x_2 + 5x_3^2 + 3.25.
\end{aligned} \tag{30}$$

We solve the System (30) numerically by a fixed point iteration defined similarly to (29) from the starting value $(0,0,0)$. The numerical results can be seen in Table 2. We conclude that $(\overline{x}_1^*, \overline{x}_2^*, \overline{x}_3^*) = (6.7840\ldots, 11.1161\ldots, 8.7126\ldots)$. Therefore Corollary 3.2 yields

$$\begin{aligned}
4.5960\ldots \leq \liminf_{t\to\infty} x_1(t) \leq \limsup_{t\to\infty} x_1(t) \leq 6.7840\ldots, \\
8.3147\ldots \leq \liminf_{t\to\infty} x_2(t) \leq \limsup_{t\to\infty} x_2(t) \leq 11.1161\ldots, \\
7.2095\ldots \leq \liminf_{t\to\infty} x_3(t) \leq \limsup_{t\to\infty} x_3(t) \leq 8.7126\ldots.
\end{aligned} \tag{31}$$

We plotted the numerical solution of (27) in Figure 1 corresponding to the constant initial functions $(\varphi_1(t), \varphi_2(t), \varphi_3(t)) = (2.5, 6, 2.5)$ and $(\varphi_1(t), \varphi_2(t), \varphi_3(t)) = (3.5, 8, 4)$. The horizontal lines in Figure 1 correspond to the upper and lower bounds listed in (31), respectively. We also observe that the difference of the components of the two solutions converges to zero, i.e., the positive solutions are asymptotically equivalent. The numerical results demonstrate the theoretical bounds (31).

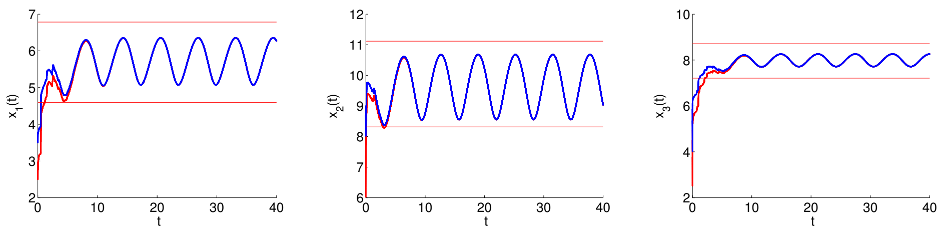

FIGURE 1. Numerical solution of the System (27).

k	$\underline{x}_1^{(k)}$	$\underline{x}_2^{(k)}$	$\underline{x}_3^{(k)}$
0	0	0	0
1	0.3761	1.7105	1.9834
2	1.8185	4.8060	3.7077
3	3.6353	7.5553	5.9214
4	4.0406	7.9252	6.4602
5	4.4130	8.1962	6.9628
6	4.5364	8.2765	7.1294
7	4.5767	8.3023	7.1836
8	4.5958	8.3146	7.2092
9	4.5960	8.3147	7.2095
10	4.5960	8.3147	7.2095

TABLE 1. Numerical solution of the System (28)

k	$\overline{x}_1^{(k)}$	$\overline{x}_2^{(k)}$	$\overline{x}_3^{(k)}$
0	0	0	0
1	0.6849	2.0198	2.8145
2	2.9151	5.9799	5.0354
3	5.5288	9.7858	7.5194
4	6.4086	10.7362	8.3557
5	6.6740	11.0053	8.6081
6	6.7520	11.0838	8.6822
7	6.7747	11.1067	8.7038
8	6.7839	11.1159	8.7125
9	6.7840	11.1161	8.7126
10	6.7840	11.1161	8.7126

TABLE 2. Numerical solution of the System (30)

Next we study the asymptotic equivalence of positive solutions for a special case of the System (21). We consider the IVP

$$\dot{x}_i(t) = \sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) x_j(t - \tau_{ij\ell}(t)) - \beta_i(t) x_i^{q_i}(t) + \rho_i(t), \qquad t \geq 0, \quad 1 \leq i \leq n, \tag{32}$$

with the initial condition

$$x_i(t) = \varphi_i(t), \qquad -\tau \leq t \leq 0, \qquad 1 \leq i \leq n, \tag{33}$$

where $\tau > 0$, $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \in C_+^n$, $\alpha_{ij\ell}, \tau_{ij\ell}, \beta_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, $0 \leq \tau_{ij\ell}(t) \leq \tau$ for $t \geq 0$, $1 \leq i, j \leq n$, $1 \leq \ell \leq n_0$ and $q_i \in \mathbb{N}$, $q_i > 1$, $1 \leq i \leq n$.

Remark 3.4. Equation (25) corresponding to (32) has the form

$$x_i^{q_i} = \sum_{j=1}^{n} \underline{m}_{ij} x_j + \underline{l}_i, \qquad 1 \le i \le n.$$

Therefore

$$x_i(x_i^{q_i-1} - \underline{m}_{ii}) = \sum_{\substack{j=1\\ j\neq i}}^{n} \underline{m}_{ij} x_j + \underline{l}_i, \qquad 1 \le i \le n.$$

So its positive solution $(\underline{x}_1^*, \dots, \underline{x}_n^*)$ satisfies $\underline{x}_i^* \ge \underline{m}_{ii}^{\frac{1}{q_i-1}}$, hence Corollary 3.2 yields that for every $\varphi \in C_+^n$ the solution $x_i(\varphi)(t)$ of (32)-(33) satisfies

$$\liminf_{t\to\infty} x_i(\varphi)(t) \ge \underline{x}_i^* \ge \underline{m}_{ii}^{\frac{1}{q_i-1}}, \qquad 1 \le i \le n. \tag{34}$$

Theorem 3.5. *Suppose that $\tau_{ij\ell}$, β_i and $\alpha_{ij\ell}$ satisfy $(\boldsymbol{A_0})$, $(\boldsymbol{A_1})$ and $(\boldsymbol{A_2})$, $q_i \in \mathbb{N}$, $\rho_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies $\sup\limits_{t>0} \frac{\rho_i(t)}{\beta_i(t)} < \infty$ for $1 \le i \le n$, and*

$$\sum_{j=1}^{n} \overline{m}_{ij} < q_i \underline{m}_{ii}, \qquad q_i > 1, \quad 1 \le i \le n. \tag{35}$$

Then, for any initial functions $\varphi, \psi \in C_+^n$, the corresponding solutions $x(\varphi)(t)$ and $x(\psi)(t)$ of the IVP (32) and (33) satisfy

$$\lim_{t\to\infty} \Big(x_i(\varphi)(t) - x_i(\psi)(t) \Big) = 0, \qquad 1 \le i \le n,$$

i.e., any positive solutions of Equation (32) are asymptotically equivalent.

Proof. Let $\varphi, \psi \in C_+^n$ be fixed and define $\nu_i(t) := x_i(\varphi)(t)$ and $\omega_i(t) := x_i(\psi)(t)$. Then

$$\dot{\nu}_i(t) = \sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) \nu_j(t - \tau_{ij\ell}(t)) - \beta_i(t) \nu_i^{q_i}(t) + \rho_i(t), \qquad t \ge 0, \quad 1 \le i \le n,$$

and

$$\dot{\omega}_i(t) = \sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) \omega_j(t - \tau_{ij\ell}(t)) - \beta_i(t) \omega_i^{q_i}(t) + \rho_i(t), \qquad t \ge 0, \quad 1 \le i \le n.$$

Now, introduce $z_i(t) := \nu_i(t) - \omega_i(t)$, then for $t \ge 0$

$$\dot{z}_i(t) = \sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) z_j(t - \tau_{ij\ell}(t)) - \beta_i(t) z_i(t) \sum_{r=0}^{q_i-1} \nu_i^r(t) \omega_i^{q_i-1-r}(t), \quad 1 \le i \le n,$$

or equivalently

$$\dot{z}_i(t) = -a_i(t) z_i(t) + \sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) z_j(t - \tau_{ij\ell}(t)), \qquad t \ge 0, \ 1 \le i \le n, \tag{36}$$

where $a_i(t) := \beta_i(t) \sum\limits_{r=0}^{q_i-1} \nu_i^r(t) \omega_i^{q_i-1-r}(t)$. We can consider (36) as the perturbation of the scalar ordinary differential equation

$$\dot{y}_i(t) = -a_i(t) y_i(t), \qquad t \ge 0, \quad 1 \le i \le n.$$

Thus, for any $T \geq 0$ and $1 \leq i \leq n$, the solution of (36) satisfies

$$z_i(t) = z_i(T)e^{-\int_T^t a_i(u)du} + \int_T^t e^{-\int_s^t a_i(u)du} \sum_{j=1}^n \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(s) z_j(s - \tau_{ij\ell}(s))\, ds, \ t \geq T. \tag{37}$$

The definition of $a_i(t)$, (34) and assumption (35) yield, for each $i = 1, \dots, n$,

$$\begin{aligned}
\limsup_{t\to\infty} \frac{\sum\limits_{j=1}^n \sum\limits_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{a_i(t)} &\leq \limsup_{t\to\infty} \frac{1}{\sum\limits_{r=0}^{q_i-1} \nu_i^r(t)\omega_i^{q_i-1-r}(t)} \limsup_{t\to\infty} \frac{\sum\limits_{j=1}^n \sum\limits_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{\beta_i(t)} \\
&\leq \frac{1}{q_i \underline{m}_{ii}} \sum_{j=1}^n \limsup_{t\to\infty} \frac{\sum\limits_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{\beta_i(t)} \\
&\leq \frac{\sum\limits_{j=1}^n \overline{m}_{ij}}{q_i \underline{m}_{ii}} \\
&< 1.
\end{aligned}$$

Thus, there exist $0 < \eta < 1$ and $T_1 \geq 0$ such that

$$\frac{\sum\limits_{j=1}^n \sum\limits_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{a_i(t)} < \eta < 1, \qquad t \geq T_1,$$

or equivalently

$$\sum_{j=1}^n \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) \leq \eta a_i(t), \qquad t \geq T_1, \quad 1 \leq i \leq n. \tag{38}$$

We introduce $\overline{z}_j(\infty) := \limsup\limits_{t\to\infty} z_j(t)$, $1 \leq j \leq n$. For every $\varepsilon > 0$, there exists a $T \geq T_1$ such that

$$|z_j(s - \tau_{ij\ell}(s))| \leq \overline{z}_j(\infty) + \varepsilon \leq \max_{1\leq l\leq n} \overline{z}_l(\infty) + \varepsilon, \quad s \geq T, \ 1 \leq i, j \leq n, \ 1 \leq \ell \leq n_0. \tag{39}$$

Using (37), (38) and (39), we get

$$\begin{aligned}
|z_i(t)| &\leq |z_i(T)|e^{-\int_T^t a_i(u)\,du} + \int_T^t e^{-\int_s^t a_i(u)\,du} \sum_{j=1}^n \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(s) |z_j(s - \tau_{ij\ell}(s))|\, ds \\
&\leq |z_i(T)|e^{-\int_T^t a_i(u)\,du} + \Big(\max_{1\leq j\leq n} \overline{z}_j(\infty) + \varepsilon\Big)\eta \int_T^t e^{-\int_s^t a_i(u)\,du} a_i(s)\, ds \\
&= |z_i(T)|e^{-\int_T^t a_i(u)\,du} + \Big(\max_{1\leq j\leq n} \overline{z}_j(\infty) + \varepsilon\Big)\eta(1 - e^{-\int_T^t a_i(u)\,du})
\end{aligned}$$

for $t \geq T$ and $1 \leq i \leq n$. Taking the limit supremum for both sides as $t \to \infty$, and using $(\mathbf{A_1})$ and Lemma 2.2, and that

$$\begin{aligned}
\int_T^\infty a_i(u)du &= \int_T^\infty \beta_i(u) \sum_{r=0}^{q_i-1} \nu_i^r(u)\omega_i^{q_i-1-r}(u)du \\
&\geq \left(\inf_{t\geq T} \sum_{r=0}^{q_i-1} \nu_i^r(t)\omega_i^{q_i-1-r}(t)\right) \int_T^\infty \beta_i(u)du \\
&= \infty,
\end{aligned}$$

we obtain

$$\overline{z}_i(\infty) \leq \eta(\max_{1\leq l\leq n} \overline{z}_l(\infty) + \varepsilon), \qquad 1 \leq i \leq n.$$

Thus

$$\max_{1\leq i\leq n} \overline{z}_i(\infty) \leq \eta \max_{1\leq i\leq n} \overline{z}_i(\infty) + \eta\varepsilon,$$

which implies

$$\max_{1\leq i\leq n} \overline{z}_i(\infty) \leq \frac{\eta\varepsilon}{1-\eta}.$$

Since $\varepsilon > 0$ can be arbitrary small, we get $\max\limits_{1\leq i\leq n} \overline{z}_i(\infty) = 0$ and consequently $\lim\limits_{t\to\infty} z_i(t) = 0,\ 1 \leq i \leq n$. Hence the proof is completed. □

Example 3.6. Consider the following system of nonlinear differential equations in two dimensions, for $t \geq 0$,

$$\begin{aligned}
\dot{x}_1(t) &= (1.7 + 0.2\cos t)x_1(t-2) + (0.25 + 0.1\sin t)x_2(t-1.5) \\
&\quad -0.5x_1^2(t) + 8 + 2\cos t, \\
\dot{x}_2(t) &= (0.02 + 0.01\sin t)x_1(t-0.3) + (1.2 + 0.2\cos t)x_2(t-10) \\
&\quad -0.2x_2^2(t) + 2.2 + 2\sin t.
\end{aligned} \tag{40}$$

Note that the conditions of Theorem 3.5 are satisfied for (40), where $\underline{m}_{11} = 3, \overline{m}_{11} = 3.8, \overline{m}_{12} = 0.7, \underline{m}_{22} = 5, \overline{m}_{21} = 0.15$ and $\overline{m}_{22} = 7$ satisfy (35) for $i, j = 1, 2$. Also, using Corollary 3.2, we see that

$$\liminf_{t\to\infty} x_1(t) \geq \underline{x}_1^* \quad \text{and} \quad \liminf_{t\to\infty} x_2(t) \geq \underline{x}_2^*,$$

where $(\underline{x}_1^*, \underline{x}_2^*)$ is the unique positive solution of the algebraic system

$$\begin{aligned}
x_1^2 &= 3x_1 + 0.3x_2 + 12, \\
x_2^2 &= 0.05x_1 + 5x_2 + 1.
\end{aligned} \tag{41}$$

We solve the System (41) numerically by a fixed point iteration

$$\begin{aligned}
\underline{x}_1^{(k+1)} &= \sqrt{3\underline{x}_1^{(k)} + 0.3\underline{x}_2^{(k)} + 12}, \\
\underline{x}_2^{(k+1)} &= \sqrt{0.05\underline{x}_1^{(k)} + 5\underline{x}_2^{(k)} + 1}.
\end{aligned} \tag{42}$$

We compute the sequence defined by (42) starting from the initial value $(0, 0)$. The first ten terms of this sequence are displayed in Table 3. We can observe that the sequence is convergent and its limit is $(\underline{x}_1^*, \underline{x}_2^*) = (5.4778\ldots, 5.2430\ldots)$.

Similarly, we can see that

$$\limsup_{t\to\infty} x_1(t) \leq \overline{x}_1^* \quad \text{and} \quad \limsup_{t\to\infty} x_2(t) \leq \overline{x}_2^*,$$

where $(\overline{x}_1^*, \overline{x}_2^*)$ is the unique positive solution of the algebraic system

$$\begin{aligned} x_1^2 &= 3.8x_1 + 0.7x_2 + 20, \\ x_2^2 &= 0.15x_1 + 7x_2 + 21. \end{aligned} \tag{43}$$

We solve the System (43) numerically by a fixed point iteration defined similarly to (42) from the starting value $(0,0)$. The numerical results can be seen in Table 4. We conclude that $(\overline{x}_1^*, \overline{x}_2^*) = (7.3921\ldots, 9.3616\ldots)$. Therefore Corollary 3.2 yields

$$\begin{aligned} 5.4778\ldots \le \liminf_{t\to\infty} x_1(t) \le \limsup_{t\to\infty} x_1(t) \le 7.3921\ldots, \\ 5.2430\ldots \le \liminf_{t\to\infty} x_2(t) \le \limsup_{t\to\infty} x_2(t) \le 9.3616\ldots. \end{aligned} \tag{44}$$

We plotted the numerical solution of (40) in Figure 2 corresponding to the initial functions $(\varphi_1(t), \varphi_2(t)) = (3, 2)$, $(\varphi_1(t), \varphi_2(t)) = (7, 7)$ and $(\varphi_1(t), \varphi_2(t)) = (9, 10)$. The horizontal lines in Figure 2 correspond to the upper and lower bounds listed in (44), respectively. We also observe that the difference of the components of every two solutions converges to zero, i.e., the positive solutions are asymptotically equivalent which coincide (3.5) in Theorem 3.5.

k	$\underline{x}_1^{(k)}$	$\underline{x}_2^{(k)}$
0	0	0
1	3.4641	1.0831
2	4.7663	2.5795
3	5.2031	3.7627
4	5.4246	4.8659
5	5.4659	5.1549
6	5.4721	5.2008
7	5.4751	5.2419
8	5.4777	5.2429
9	5.4778	5.2430
10	5.4778	5.2430

TABLE 3. Numerical solution of the System (41)

k	$\overline{x}_1^{(k)}$	$\overline{x}_2^{(k)}$
0	0	0
1	4.4721	4.6552
2	6.3445	7.3850
3	7.0199	8.5877
4	7.2586	9.0666
5	7.3436	9.2503
6	7.3744	9.3198
7	7.3918	9.3608
8	7.3920	9.3615
9	7.3921	9.3616
10	7.3921	9.3616

TABLE 4. Numerical solution of the System (43)

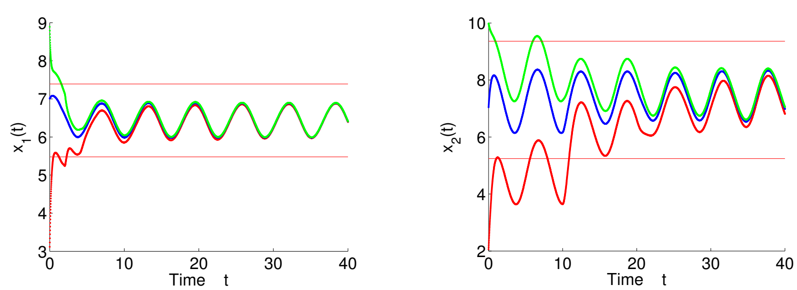

FIGURE 2. Numerical solution of the System (40).

Next, we consider again the population model (4):

$$\begin{aligned}\dot{x}_i(t) &= \sum_{\ell=1}^{n_0} \frac{\lambda_{i\ell}(t)x_i(t-\tau_{i\ell}(t))}{1+\gamma_{i\ell}(t)x_i(t-\tau_{i\ell}(t))} + \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(t)x_j(t-\sigma_{ij}(t)) \\ &\quad -\mu_i(t)x_i(t) - \kappa_i(t)x_i^2(t), \qquad t\ge 0, \quad 1\le i\le n, \end{aligned} \tag{45}$$

with the initial condition

$$x_i(t) = \varphi_i(t), \qquad -\tau \le t \le 0, \qquad 1\le i\le n. \tag{46}$$

We assume that $\varphi = (\varphi_1,\varphi_2,\dots,\varphi_n) \in C_0^n$, where $C_0 := \{\psi \in C([-\tau,0],\mathbb{R}_+) : \psi(t) > 0,\ -\tau \le t \le 0\}$. Note that $C_0 \subset C_+$.

The permanence of positive solutions of (45) was investigated in [15] for the case when the delays in the model can be unbounded. Next, we show that, for the bounded delay case, our Theorem 2.4 gives permanence of the positive solutions for this model under weak conditions. We note that we do not need the boundedness of the functions $\lambda_{i\ell}$, a_{ij}, μ_i and κ_i which was assumed in [15].

Corollary 3.7. *Assume $\lambda_{i\ell},\gamma_{i\ell},a_{ij},\mu_i,\kappa_i \in C(\mathbb{R}_+,\mathbb{R}_+)$, and $\tau_{i\ell},\sigma_{ij} \in C(\mathbb{R}_+,\mathbb{R}_+)$ with $0\le\tau_{i\ell}(t)\le\tau$ and $0\le\sigma_{ij}(t)\le\tau$ for $t\ge 0$, $1\le i\ne j\le n$ and $\ell=1,\dots,n_0$. Moreover, we assume that there exist positive constants $\underline{\gamma}_i,\overline{\gamma}_i,\underline{\pi}_i$ and $\overline{\pi}_i$ such that, for all $1\le i\ne j\le n$ and $1\le\ell\le n_0$,*

$$0<\underline{\gamma}_i \le \gamma_{i\ell}(t) \le \overline{\gamma}_i, \quad 0<\underline{\pi}_i \le \frac{\kappa_i(t)}{\mu_i(t)} \le \overline{\pi}_i,\ t>0 \quad \text{and} \quad \int_0^\infty \mu_i(t)\,dt = \infty, \tag{47}$$

and

$$\sup_{t>0} \frac{\sum_{\ell=1}^{n_0}\lambda_{i\ell}(t)}{\mu_i(t)} < \infty, \quad \sup_{t>0}\frac{a_{ij}(t)}{\mu_i(t)} < \infty,\ j\ne i, \quad \text{and} \quad \liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0}\lambda_{i\ell}(t)}{\mu_i(t)} > 1. \tag{48}$$

Then, for any initial function $\varphi = (\varphi_1,\varphi_2,\dots,\varphi_n)\in C_0^n$, the solution $x(\varphi)(t) = (x_1(\varphi)(t),\dots,x_n(\varphi)(t))$ of the IVP (45) and (46) satisfies

$$\underline{x}_i^* \le \liminf_{t\to\infty} x_i(\varphi)(t) \le \limsup_{t\to\infty} x_i(\varphi)(t) \le \overline{x}_i^*, \qquad 1\le i\le n,$$

where $(\underline{x}_1^*,\dots,\underline{x}_n^*)$ is the unique positive solution of the algebraic system

$$x_i + \overline{\pi}_i x_i^2 = \frac{\underline{m}_{ii}x_i}{1+\overline{\gamma}_i x_i} + \sum_{\substack{j=1\\ j\neq i}}^{n} \underline{m}_{ij}x_j, \qquad 1\le i\le n, \tag{49}$$

and $(\overline{x}_1^*,\dots,\overline{x}_n^*)$ is the unique positive solution of the algebraic system

$$x_i + \underline{\pi}_i x_i^2 = \frac{\overline{m}_{ii}x_i}{1+\underline{\gamma}_i x_i} + \sum_{\substack{j=1\\ j\neq i}}^{n} \overline{m}_{ij}x_j, \qquad 1\le i\le n, \tag{50}$$

respectively, where $\underline{m}_{ii} := \liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0}\lambda_{i\ell}(t)}{\mu_i(t)}$, $\overline{m}_{ii} := \limsup_{t\to\infty} \frac{\sum_{\ell=1}^{n_0}\lambda_{i\ell}(t)}{\mu_i(t)}$, $1\le i\le n$, and $\underline{m}_{ij} := \liminf_{t\to\infty}\frac{a_{ij}(t)}{\mu_i(t)}$, $\overline{m}_{ij} := \limsup_{t\to\infty}\frac{a_{ij}(t)}{\mu_i(t)}$ for $1\le i\ne j\le n$.

Proof. All conditions of Lemma 2.1 hold for the System (45), therefore it implies that $x_i(t) = x_i(\varphi)(t) > 0$ for $t \geq 0$ and $i = 1, \dots, n$. Since we assumed that $\varphi_i \in C_0$ for all $i = 1, \dots, n$, it follows $x_i(t - \tau_{i\ell}(t)) > 0$ for $t \geq 0$ and $i = 1, \dots, n$. From (47), we have $\gamma_{i\ell}(t) \leq \overline{\gamma}_i$ and $\frac{\kappa_i(t)}{\mu_i(t)} \leq \overline{\pi}_i$, for $t > 0$. Thus, we get from (45) for $t \geq 0$ and $i = 1, \dots, n$ that

$$\dot{x}_i(t) \geq \sum_{\ell=1}^{n_0} \frac{\lambda_{i\ell}(t) x_i(t - \tau_{i\ell}(t))}{1 + \overline{\gamma}_i x_i(t - \tau_{i\ell}(t))} + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}(t) x_j(t - \sigma_{ij}(t)) - \mu_i(t)[x_i(t) + \overline{\pi}_i x_i^2(t)].$$

By comparison theorem of differential inequalities, we have $x_i(t) \geq y_i(t)$ for $t \geq 0$ and $i = 1, \dots, n$, where $y_i(t)$ is the positive solution of the differential equation

$$\begin{aligned} \dot{y}_i(t) \quad &= \quad \sum_{\ell=1}^{n_0} \frac{\lambda_{i\ell}(t) y_i(t - \tau_{i\ell}(t))}{1 + \overline{\gamma}_i y_i(t - \tau_{i\ell}(t))} + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}(t) y_j(t - \sigma_{ij}(t)) \\ &\quad -\mu_i(t)[y_i(t) + \overline{\pi}_i y_i^2(t)], \qquad 1 \leq i \leq n, \end{aligned} \tag{51}$$

with the initial condition

$$y_i(t) = \varphi_i(t), \qquad -\tau \leq t \leq 0, \quad 1 \leq i \leq n. \tag{52}$$

Next, we check that conditions $(\mathbf{A_0})$–$(\mathbf{A_6})$ of Theorem 2.4 are satisfied for the System (51). First note that we can rewrite (51) in the form (5) with

$$\begin{aligned} \alpha_{ij\ell}(t) \quad &:= \quad \begin{cases} \lambda_{i\ell}(t), & j = i, \quad \ell = 1, \dots, n_0, \\ a_{ij}(t), & j \neq i, \quad \ell = 1, \\ 0, & j \neq i, \quad \ell \neq 1, \end{cases} \\ h_{ij}(u) \quad &:= \quad \begin{cases} \frac{u}{1 + \overline{\gamma}_i u}, & j = i, \\ u, & j \neq i, \end{cases} \\ \tau_{ij\ell}(t) \quad &:= \quad \begin{cases} \tau_{i\ell}(t), & j = i, \quad \ell = 1, \dots, n_0, \\ \sigma_{ij}(t), & j \neq i, \quad \ell = 1, \\ 0, & j \neq i, \quad \ell \neq 1, \end{cases} \end{aligned}$$

and $\beta_i(t) := \mu_i(t)$, $f_i(u) := u + \overline{\pi}_i u^2$ and $\rho_i(t) := 0$, $1 \leq i, j \leq n$. We have $\lim\limits_{u \to 0^+} \frac{f_i(u)}{h_{ii}(u)} = \lim\limits_{u \to 0^+} \frac{(u + \overline{\pi}_i u^2)(1 + \overline{\gamma}_i u)}{u} = 1$ and $\lim\limits_{u \to \infty} \frac{h_{ij}(u)}{f_i(u)} = 0$ for all $1 \leq i, j \leq n$.

Therefore, by our assumptions (47) and (48), we can see that conditions $(\mathbf{A_0})$–$(\mathbf{A_5})$ hold. To check condition $(\mathbf{A_6})$, we observe that

$$\frac{f_i(u)}{h_{ij}(u)} = \begin{cases} (1 + \overline{\pi}_i u)(1 + \overline{\gamma}_i u), & j = i, \\ 1 + \overline{\pi}_i u, & j \neq i, \end{cases}$$

is strictly increasing and

$$\frac{h_{jj}(u)}{h_{ij}(u)} = \frac{u}{u(1 + \overline{\gamma}_i u)} = \frac{1}{1 + \overline{\gamma}_i u}$$

is strictly decreasing on $(0, \infty)$, for each $1 \leq i \neq j \leq n$. We see that $\underline{m}_{jj} = \liminf\limits_{t \to \infty} \frac{\sum_{\ell=1}^{n_0} \lambda_{j\ell}(t)}{\mu_j(t)} > 1$ by (48), and $\frac{h_{jj}(u)}{h_{ij}(u)}$ is strictly decreasing on $(0, \infty)$, for all $j \neq i$. Hence conditions $(\mathbf{A_6})$ (i), (ii) and (iii) are satisfied, and we can apply Theorem 2.4 (i) to the System (51). Therefore we get the lower estimates $\liminf\limits_{t \to \infty} x_i(\varphi)(t) \geq \liminf\limits_{t \to \infty} y_i(\varphi)(t) \geq \underline{x}_i^*$, $1 \leq i \leq n$, where $(\underline{x}_1^*, \dots, \underline{x}_n^*)$ is the unique positive solution of the algebraic system (49). Similarly, we can get the upper estimates $\limsup_{t\to\infty} x_i(\varphi)(t) \le \overline{x}_i^*$, $1 \le i \le n$, where $(\overline{x}_1^*, \ldots, \overline{x}_n^*)$ is the unique positive solution of the algebraic system (50). □

Now, we consider a time-dependent version of the n-dimensional Nicholson's blowflies system (3) for $t \ge 0$:

$$\dot{x}_i(t) = \sum_{\ell=1}^{n_0} b_{i\ell}(t)x_i(t-\sigma_{i\ell}(t))e^{-x_i(t-\sigma_{i\ell}(t))} + \sum_{\substack{j=1\\ j\ne i}}^{n} a_{ij}(t)x_j(t) - d_i(t)x_i(t),\ 1 \le i \le n \tag{53}$$

with the initial condition

$$x_i(t) = \varphi_i(t), \qquad -\tau \le t \le 0, \qquad 1 \le i \le n, \tag{54}$$

where $\tau > 0$, $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \in C_+^n$, $b_{i\ell}, a_{ij}, d_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, and $\sigma_{i\ell} \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $0 \le \sigma_{i\ell}(t) \le \tau$ for $t \ge 0$, $1 \le i \ne j \le n$, $\ell = 1, \ldots, n_0$. The the persistence and permanence of the autonomous system (3) was investigated in [16]. Unfortunately, our method does not work for this population model, since the function ue^{-u} is not monotone increasing, and so condition $(\mathbf{A_4})$ of our main Theorem 2.4 is not satisfied for (53). But we can apply our method to get an upper bound of the limit superior of the solutions of (53). We formulate this result next.

Corollary 3.8. *Assume $b_{i\ell}, a_{ij}, d_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, and $\sigma_{i\ell} \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $0 \le \sigma_{i\ell}(t) \le \tau$ for $t \ge 0$, $1 \le i \ne j \le n$ and $\ell = 1, \ldots, n_0$. Moreover, we assume that, for all $1 \le i, j \le n$,*

$$d_i(t) > 0, \quad t > 0 \qquad \text{and} \qquad \int_0^\infty d_i(t)\,dt = \infty, \tag{55}$$

$$\sup_{t>0} \frac{\sum_{\ell=1}^{n_0} b_{i\ell}(t)}{d_i(t)} < \infty \qquad \text{and} \qquad \sup_{t>0} \frac{a_{ij}(t)}{d_i(t)} < \infty, \quad j \ne i, \tag{56}$$

and

$$\liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} b_{i\ell}(t)}{d_i(t)} > 1 \qquad \text{and} \qquad \sum_{\substack{j=1\\ j\ne i}}^{n} \limsup_{t\to\infty} \frac{a_{ij}(t)}{d_i(t)} < 1. \tag{57}$$

Then, for any initial function $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \in C_+^n$, the solution $x(\varphi)(t) = (x_1(\varphi)(t), \ldots, x_n(\varphi)(t))$ of the IVP (53) and (54) satisfies

$$x_i(\varphi)(t) > 0, \quad t \ge 0, \quad \text{and} \quad \limsup_{t\to\infty} x_i(\varphi)(t) \le \overline{x}_i^*, \qquad 1 \le i \le n,$$

where $(\overline{x}_1^*, \ldots, \overline{x}_n^*)$ is the unique positive solution of the algebraic system

$$x_i = \overline{m}_{ii} H(x_i) + \sum_{\substack{j=1\\ j\ne i}}^{n} \overline{m}_{ij} x_j, \qquad 1 \le i \le n, \tag{58}$$

where $\overline{m}_{ii} := \limsup_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} b_{i\ell}(t)}{d_i(t)}$, $1 \le i \le n$, and $\overline{m}_{ij} := \limsup_{t\to\infty} \frac{a_{ij}(t)}{d_i(t)}$ for $1 \le i \ne j \le n$, and $H(u) := \begin{cases} ue^{-u}, & u \le 1, \\ \frac{1}{e}, & u > 1. \end{cases}$

Proof. All conditions of Lemma 2.1 hold for the System (53), therefore it implies that $x_i(\varphi)(t) > 0$ for $t \geq 0$ and $i = 1, \dots, n$. We have $ue^{-u} \leq H(u)$ for $u \geq 0$, therefore (53) yields

$$\dot{x}_i(t) \leq \sum_{\ell=1}^{n_0} b_{i\ell}(t) H(x_i(t - \sigma_{i\ell}(t))) + \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}(t)x_j(t) - d_i(t)x_i(t), \qquad 1 \leq i \leq n.$$

By comparison theorem of differential inequalities, we have $x_i(t) \leq y_i(t)$ for $t \geq 0$, $i = 1, \dots, n$, where $y_i(t)$ is the positive solution of the differential equation

$$\dot{y}_i(t) = \sum_{\ell=1}^{n_0} b_{i\ell}(t) H(y_i(t - \sigma_{i\ell}(t))) + \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}(t)y_j(t) - d_i(t)y_i(t), \qquad 1 \leq i \leq n, \tag{59}$$

with the initial condition

$$y_i(t) = \varphi_i(t), \qquad -\tau \leq t \leq 0, \quad 1 \leq i \leq n. \tag{60}$$

Next, we check that $(\mathbf{A_0})$–$(\mathbf{A_6})$ of Theorem 2.4 are satisfied for the System (59). First note that we can rewrite (59) in the form (5) with

$$\begin{aligned}
\alpha_{ij\ell}(t) &:= \begin{cases} b_{i\ell}(t), & j = i, \quad \ell = 1, \dots, n_0, \\ a_{ij}(t), & j \neq i, \quad \ell = 1, \\ 0, & j \neq i, \quad \ell \neq 1, \end{cases} \\
h_{ij}(u) &:= \begin{cases} H(u), & j = i, \\ u, & j \neq i, \end{cases} \\
\tau_{ij\ell}(t) &:= \begin{cases} \sigma_{i\ell}(t), & j = i, \quad \ell = 1, \dots, n_0, \\ 0, & \text{otherwise}, \end{cases}
\end{aligned}$$

and $\beta_i(t) := d_i(t)$, $f_i(u) := u$ and $\rho_i(t) := 0$, $1 \leq i, j \leq n$. We have

$$\lim_{u\to 0^+} \frac{f_i(u)}{h_{ii}(u)} = \lim_{u\to 0^+} \frac{u}{H(u)} = 1 \quad \text{and} \quad \lim_{u\to\infty} \frac{h_{ij}(u)}{f_i(u)} = \begin{cases} 0, & j = i, \\ 1, & j \neq i \end{cases}$$

for $1 \leq i, j \leq n$. Thus, by our assumptions (55), (56) and (57), we can see that conditions $(\mathbf{A_0})$–$(\mathbf{A_5})$ hold. To check condition $(\mathbf{A_6})$, we observe that

$$\frac{f_i(u)}{h_{ij}(u)} = \begin{cases} e^u, & u \leq 1, \ j = i, \\ eu, & u > 1, \ j = i, \\ 1, & u > 1, \ j \neq i, \end{cases}$$

is increasing and

$$\frac{h_{jj}(u)}{h_{ij}(u)} = \frac{H(u)}{h_{ij}(u)} = \begin{cases} e^{-u}, & u \leq 1, \ j \neq i, \\ \frac{1}{eu}, & u > 1, \ j \neq i, \end{cases}$$

is strictly decreasing on $(0, \infty)$, for each $1 \leq i, j \leq n$. Moreover, for each $1 \leq i \leq n$, $\frac{f_i(u)}{h_{ii}(u)}$ is strictly increasing on $(0, \infty)$. For each $j = 1, \dots, n$, $\overline{m}_{jj} \geq \liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0} b_{j\ell}(t)}{d_j(t)} > 1$ by (57), and $\frac{h_{jj}(u)}{h_{ij}(u)}$ is strictly decreasing on $(0, \infty)$, for all $j \neq i$. Hence conditions $(\mathbf{A_6})$ (i), (iv) and (v) are satisfied, and we can apply Theorem 2.4 (ii) to the System (59). Therefore we get the upper estimates $\limsup_{t\to\infty} x_i(\varphi)(t) \leq \limsup_{t\to\infty} y_i(\varphi)(t) \leq \overline{x}_i^*$, $1 \leq i \leq n$, where $(\overline{x}_1^*, \dots, \overline{x}_n^*)$ is the unique positive solution of the algebraic system (58). $\square$

4. **Proof of the main results.** In this section we provide the proofs of our main results formulated in Section 2. First we recall the following result from [21].

Lemma 4.1. *(see [21]) Let $1 \le i \le n$ be fixed and $y(T, y_0, c, \beta_i, f_i)(t)$ be the solution of the IVP*

$$\dot{y}(t) = \beta_i(t)\Big(c - f_i(y(t))\Big), \qquad t \ge T \ge 0, \tag{61}$$

$$y(T) = y_0, \tag{62}$$

where $c \ge 0$, β_i satisfies condition ($\boldsymbol{A_1}$) and f_i satisfies condition ($\boldsymbol{A_3}$). Then for any $T \ge 0, y_0 > 0$, and $c \ge 0$ the corresponding solution $y(T, y_0, c, \beta_i, f_i)(t)$ of the IVP (61) and (62) is uniquely defined on $[T, \infty)$, moreover we have

(i) $c > 0$ and $0 < y_0 < f_i^{-1}(c)$ yield that

$$0 < y(T, y_0, c, \beta_i, f_i)(t) < f_i^{-1}(c), \qquad \dot{y}(T, y_0, c, \beta_i, f_i)(t) > 0, \quad t \ge T$$

and

$$\lim_{t\to\infty} y(T, y_0, c, \beta_i, f_i)(t) = f_i^{-1}(c);$$

(ii) $y_0 = f_i^{-1}(c)$ yields that $y(T, y_0, c, \beta_i, f_i)(t) = f_i^{-1}(c), \quad t \ge T$;

(iii) $c \ge 0$ and $y_0 > f_i^{-1}(c)$ yield that

$$y(T, y_0, c, \beta_i, f_i)(t) > f_i^{-1}(c), \qquad \dot{y}(T, y_0, c, \beta_i, f_i)(t) < 0, \quad t \ge T$$

and

$$\lim_{t\to\infty} y(T, y_0, c, \beta_i, f_i)(t) = f_i^{-1}(c).$$

Proof of Lemma 2.1 Since $x_i(0) = \varphi_i(0) > 0$, $1 \le i \le n$, then if $x_i(t) > 0$ for $t \ge 0, 1 \le i \le n$, then we are done. Otherwise at least one of $x_1(t), \dots, x_n(t)$ is equal to zero for some positive t. Since the functions $x_1(t), \dots, x_n(t)$ are continuous, then in the last case there exists a $t_1 \in (0, \infty)$ such that $x_i(t) > 0$ for $0 \le t < t_1$, $1 \le i \le n$ and $\min\{x_1(t_1), \dots, x_n(t_1)\} = 0$. Since $\alpha_{ij\ell}(t) \ge 0, \tau_{ij\ell}(t) \ge 0$, $\rho_i(t) \ge 0$, $t \ge 0$, $1 \le i, j \le n$, $1 \le \ell \le n_0$, and $h_{ij}(u) \ge 0$, $u \ge 0$, $1 \le i, j \le n$, then from (5) we have

$$\dot{x}_i(t) \ge -\beta_i(t) f_i(x_i(t)), \qquad 1 \le i \le n, \qquad 0 \le t \le t_1.$$

But from the comparison theorem of the differential inequalities (see [10]), we have

$$x_i(t) \ge y_i(t), \qquad 1 \le i \le n, \quad 0 \le t \le t_1,$$

where $y_i(t) = y(0, \varphi_i(0), 0, \beta_i, f_i)(t)$, $1 \le i \le n$ is the unique positive solution of (61) with $c = 0$ and with the initial condition

$$y_i(0) = x_i(0) = \varphi_i(0) > 0, \qquad 1 \le i \le n.$$

Lemma 4.1 yields $y_i(t) > 0$, for all $t \ge 0$. Then at $t = t_1$ we get $x_i(t_1) \ge y_i(t_1) > 0$, $1 \le i \le n$, which contradicts our assumption that $\min\{x_1(t_1), \dots, x_n(t_1)\} = 0$. Hence $x_i(t) > 0$, $1 \le i \le n$ for $t \in \mathbb{R}_+$. $\square$

Proof of Lemma 2.2 First we show that

$$\inf_{t\ge 0} x_i(t) > 0, \qquad 1 \le i \le n. \tag{63}$$

Let $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n) \in C_+^n$ be an arbitrary fixed initial function. Then, by Lemma 2.1, the solution $x(t) = x(\varphi)(t) = (x_1(\varphi)(t), \dots, x_n(\varphi)(t))$ obeys $x_i(t) > 0$,

$1 \le i \le n$, $t \ge 0$. We claim that there exist $T > 0$ and $c > 0$ such that the following inequalities are satisfied, for every $i = 1, \dots, n$,

$$\min_{0\le t\le T} x_i(t) > c \quad \text{and} \quad \frac{\sum_{j=1}^{n}\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) h_{ij}(c)}{\beta_i(t)} + \frac{\rho_i(t)}{\beta_i(t)} > f_i(c), \qquad t \ge T. \tag{64}$$

From (8), we have two cases:
(i) if i is such that $\liminf\limits_{t\to\infty} \frac{\rho_i(t)}{\beta_i(t)} > 0$, then fix a $\xi_i > 0$ such that

$$\liminf_{t\to\infty} \frac{\rho_i(t)}{\beta_i(t)} > \xi_i > 0.$$

Thus there exists $T_i > 0$ such that

$$\frac{\rho_i(t)}{\beta_i(t)} > \xi_i > 0, \quad \text{for} \quad t \ge T_i.$$

Lemma 2.1 and $(\mathbf{A_3})$ imply that there exists a $c_i > 0$ such that

$$\min_{0\le t\le T} x_i(t) > c_i \quad \text{and} \quad f_i(u) < \xi_i \quad \text{for} \quad 0 < u \le c_i.$$

Therefore (64) is satisfied for such i.

(ii) if i is such that $\limsup\limits_{u\to 0^+} \frac{f_i(u)}{h_{ii}(u)} < \liminf\limits_{t\to\infty} \frac{\sum_{\ell=1}^{n_0}\alpha_{ii\ell}(t)}{\beta_i(t)}$, then let $K_i > 0$ be such that

$$\limsup_{u\to 0^+} \frac{f_i(u)}{h_{ii}(u)} < K_i < \liminf_{t\to\infty} \frac{\sum_{\ell=1}^{n_0}\alpha_{ii\ell}(t)}{\beta_i(t)}.$$

Thus there exists $T_i > 0$ such that

$$K_i < \frac{\sum_{\ell=1}^{n_0}\alpha_{ii\ell}(t)}{\beta_i(t)}, \qquad t \ge T_i.$$

Also, there exists $c_i > 0$ such that

$$\frac{f_i(u)}{h_{ii}(u)} < K_i, \quad \text{for} \quad 0 < u \le c_i \quad \text{and} \quad \min_{0\le t\le T} x_i(t) > c_i.$$

Then we have

$$\frac{\sum_{j=1}^{n}\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) h_{ij}(c)}{\beta_i(t)} \ge \frac{1}{K_i} f_i(u) \frac{\sum_{\ell=1}^{n_0}\alpha_{ii\ell}(t)}{\beta_i(t)} > f_i(u), \qquad t \ge T_i, \quad 0 < u \le c_i,$$

and hence (64) holds for such i. Therefore (64) is satisfied, for all $i = 1, \dots, n$, with $T = \max\{T_1, \dots, T_n\}$ and $c = \min\{c_1, \dots, c_n\}$.

Now, in virtue of (64), either $x_i(t) > c$ for all $t \ge 0$, $1 \le i \le n$, or there exists $t_2 \in (T, \infty)$ such that $\min\{x_1(t_2), \dots, x_n(t_2)\} = c$ and $x_i(t) > c$ for $t \in [0, t_2)$, $1 \le i \le n$. In this case at least one of the values of $x_1(t_2), \dots, x_n(t_2)$ is equal to c. Assume, e.g., that $x_1(t_2) = c$, then $\dot{x}_1(t_2) \le 0$. On the other hand, the monotonicity of h_{1j} and (64) yield that

$$\begin{aligned}\dot{x}_1(t_2) &= \beta_1(t_2)\left[\frac{\sum_{j=1}^{n}\sum_{\ell=1}^{n_0}\alpha_{1j\ell}(t_2)h_{1j}(x_j(t_2-\tau_{1j\ell}(t_2)))}{\beta_1(t_2)} - f_1(x_1(t_2)) + \frac{\rho_1(t_2)}{\beta_1(t_2)}\right]\\ &\geq \beta_1(t_2)\left[\frac{\sum_{j=1}^{n}\sum_{\ell=1}^{n_0}\alpha_{1j\ell}(t_2)h_{1j}(c)}{\beta_1(t_2)} + \frac{\rho_1(t_2)}{\beta_1(t_2)} - f_1(c)\right]\\ &> 0,\end{aligned}$$

which is a contradiction, since $\dot{x}_1(t_2) \leq 0$. Hence $x_1(t) > c$ for all $t \geq 0$. Similarly, we can show that $x_i(t) > c$, for all $t \geq 0$, $2 \leq i \leq n$, and therefore (63) holds.

Now we show that

$$\sup_{t\geq 0} x_i(t) < \infty, \qquad 1 \leq i \leq n. \tag{65}$$

We claim that there exist $T > 0$ and $M > 0$ such that the following inequalities are satisfied, for every $i = 1, \ldots, n$,

$$\max_{0\leq t\leq T} x_i(t) < M \quad \text{and} \quad \left(\frac{\sum_{j=1}^{n}\sum_{\ell=1}^{n_0}\alpha_{ij\ell}(t)h_{ij}(M)}{\beta_i(t)} + \frac{\rho_i(t)}{\beta_i(t)}\right) < f_i(M), \quad t \geq T. \tag{66}$$

The second relation of (66) holds if

$$\left(\frac{\sum_{j=1}^{n}\sum_{\ell=1}^{n_0}\alpha_{ij\ell}(t)\frac{h_{ij}(M)}{f_i(M)}}{\beta_i(t)} + \frac{1}{f_i(M)}\frac{\rho_i(t)}{\beta_i(t)}\right) < 1, \qquad t \geq T. \tag{67}$$

Using (10), there exists a $\mu_i > 0$ such that

$$\sum_{j=1}^{n}\Big(\limsup_{t\to\infty}\frac{\sum_{\ell=1}^{n_0}\alpha_{ij\ell}(t)}{\beta_i(t)}\Big)\limsup_{u\to\infty}\frac{h_{ij}(u)}{f_i(u)} < \mu_i < 1,$$

then there exists an $\delta > 0$ such that

$$\sum_{j=1}^{n}\Big(\limsup_{t\to\infty}\frac{\sum_{\ell=1}^{n_0}\alpha_{ij\ell}(t)}{\beta_i(t)} + \delta\Big)\Big(\limsup_{u\to\infty}\frac{h_{ij}(u)}{f_i(u)} + \delta\Big) < \mu_i.$$

Thus there exist $T_i > 0$ and $V_{1i} > 0$ such that

$$\sum_{j=1}^{n}\Big(\sup_{t\geq T_i}\frac{\sum_{\ell=1}^{n_0}\alpha_{ij\ell}(t)}{\beta_i(t)}\Big)\frac{h_{ij}(u)}{f_i(u)} < \mu_i, \quad u \geq V_{1i}.$$

Moreover, using (9), there exists a $V_{2i} > 0$ such that

$$\frac{1}{f_i(u)}\sup_{t\geq T_i}\frac{\rho_i(t)}{\beta_i(t)} < 1 - \mu_i, \quad u \geq V_{2i},$$

and so there exists a large $M > 0$ such that (67) holds and $\max_{0\le t\le T} x_i(t) < M$, with $T = \max\{T_1, \dots, T_n\}$, for all $1 = 1, \dots, n$. Hence inequality (66) is satisfied for each $i = 1, \dots, n$. Now, in virtue of (66), either $x_i(t) < M$ for all $t \ge 0$, $1 \le i \le n$, or there exists $t_3 \in (T, \infty)$ such that $\max\{x_1(t_3), \dots, x_n(t_3)\} = M$, and $x_i(t) < M$ for $t \in [0, t_3)$ and $i = 1, \dots, n$. In this case at least one of the values of $x_1(t_3), \dots, x_n(t_3)$ is equal to M. Assume, e.g., that $x_1(t_3) = M$, then $\dot{x_1}(t_3) \ge 0$. On the other hand, using (66) and the monotonicity of h_{1j}, we have

$$\begin{aligned}
\dot{x}_1(t_3) &= \beta_1(t_3)\left[\frac{\sum_{j=1}^{n}\sum_{\ell=1}^{n_0}\alpha_{1j\ell}(t_3)h_{1j}(x_j(t_3-\tau_{1j\ell}(t_3)))}{\beta_1(t_3)} - f_1(x_1(t_3)) + \frac{\rho_1(t_3)}{\beta_1(t_3)}\right] \\
&\le \beta_1(t_3)\left[\frac{\sum_{j=1}^{n}\sum_{\ell=1}^{n_0}\alpha_{1j\ell}(t_3)h_{1j}(M)}{\beta_1(t_3)} - f_1(M) + \frac{\rho_1(t_3)}{\beta_1(t_3)}\right] \\
&< 0,
\end{aligned}$$

which is a contradiction, since $\dot{x}_1(t_3) \ge 0$. Hence $x_1(t) < M$, for all $t \ge 0$. Similarly, we can show that $x_i(t) < M$, for all $t \ge 0$, $2 \le i \le n$, and therefore (65) holds. □

Now, we consider the system of nonlinear algebraic equations

$$\gamma_i(x_i) = \sum_{j=1}^{n} g_{ij}(x_j), \qquad 1 \le i \le n, \tag{68}$$

where $\gamma_i \in C(\mathbb{R}_+, \mathbb{R})$, $g_{ij} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $1 \le i, j \le n$.

In [22] sufficient conditions were given to guarantee the existence of a unique positive solution of (68). We recall this result next.

Theorem 4.2. *([22]) Let $\gamma_i \in C(\mathbb{R}_+, \mathbb{R})$ and $g_{ij} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $1 \le i, j \le n$ be such that*

(A) *for each $1 \le i \le n$, there exists a $u_i^* > 0$ satisfying*

$$\gamma_i(u)\begin{cases} < 0, & \text{if} \quad 0 < u < u_i^*, \\ = 0, & \text{if} \quad u = u_i^*, \\ > 0, & \text{if} \quad u > u_i^*, \end{cases}$$

and γ_i is strictly increasing on $[u_i^*, \infty)$.

(B) *g_{ij}, $1 \le i, j \le n$ is increasing on $\mathbb{R}_+$, and there exists a $u_i^{**} \ge u_i^*$ such that*

$$\sum_{j=1}^{n} g_{ij}(u) < \gamma_i(u), \qquad u > u_i^{**}, \quad 1 \le i \le n. \tag{69}$$

Then the System (68) has a positive solution.

Moreover, assume that

(C) *for each $1 \le i, j \le n$, either $g_{ij}(u) > 0$ for $u > 0$ or $g_{ij}(u) = 0$ for $u > 0$;*

(D) *for each $1 \le i, j \le n$, $\frac{\gamma_j(u)}{g_{ij}(u)}$ is monotone increasing on (u_j^*, ∞), assuming $g_{ij}(u) > 0$ for $u > 0$, and there exist i, j such that $g_{ij}(u) > 0$ for $u > 0$ and $\frac{\gamma_j(u)}{g_{ij}(u)}$ is strictly monotone increasing on (u_j^*, ∞).*

Then the System (68) has a unique positive solution.

We note that uniqueness of Theorem 4.2 was proved in [22] under the condition that

(D*) for each $1 \le i,j \le n$, $\frac{\gamma_j(u)}{g_{ij}(u)}$ is strictly monotone increasing on (u_j^*, ∞), assuming $g_{ij}(u) > 0$ for $u > 0$.

We note that the uniqueness of the positive solutions of (68) also holds if we assume **(D)** instead of **(D*)**, and the proof is an obvious extension of that presented in [22].

Proof of Lemma 2.3 The proof of part (i) is obtained directly from Theorem 4.2, where we can rewrite (13) in the form (68) with $\gamma_i(u) := f_i(u) - m_{ii}h_{ii}(u) - l_i$ and $g_{ij}(u) := m_{ij}h_{ij}(u)$ for each $1 \le i \ne j \le n$ and $g_{ii}(u) = 0$. Now, to prove the existence of a positive solution for System (13), we check that conditions **(A)** and **(B)** of Theorem 4.2 are satisfied. For condition **(A)**, we have that $\gamma_i(u) = 0$ if and only if

$$\frac{f_i(u)}{h_{ii}(u)} = \frac{l_i}{h_{ii}(u)} + m_{ii}, \qquad 1 \le i \le n. \tag{70}$$

The left hand side of (70) is increasing and the right hand side of (70) is decreasing, moreover, assumption $(\mathbf{H_2})$ yields that either the left hand side or the right hand side is a strictly monotone function. Therefore, condition **(A)** of Theorem 4.2 holds, if we show

$$\lim_{u\to 0^+} \frac{f_i(u)}{h_{ii}(u)} < \lim_{u\to 0^+} \frac{l_i}{h_{ii}(u)} + m_{ii}, \qquad 1 \le i \le n, \tag{71}$$

and

$$\lim_{u\to \infty} \frac{f_i(u)}{h_{ii}(u)} > \lim_{u\to \infty} \frac{l_i}{h_{ii}(u)} + m_{ii}, \qquad 1 \le i \le n. \tag{72}$$

If $l_i > 0$ and $h_{ii}(0) = 0$, then (71) follows, since the left hand side of (71) is always finite, since $\frac{f_i(u)}{h_{ii}(u)}$ is monotone increasing. If $l_i > 0$ and $h_{ii}(0) > 0$, then the right hand side of (71) is finite and positive, but $\lim\limits_{u\to 0^+} \frac{f_i(u)}{h_{ii}(u)} = 0$ using $(\mathbf{A_3})$. If $l_i = 0$, then assumption (14) yields (71). Relation (72) follows immediately from (15). Hence condition **(A)** is satisfied.

To check condition **(B)**, we see that $g_{ij}(u) := m_{ij}h_{ij}(u)$, $1 \le i \ne j \le n$, and $g_{ii}(u) = 0$ are increasing on $\mathbb{R}_+$, and relation (69) is equivalent to

$$\sum_{\substack{j=1\\ j\ne i}}^{n} m_{ij}h_{ij}(u) < f_i(u) - m_{ii}h_{ii}(u) - l_i,$$

which is satisfied if and only if

$$\sum_{j=1}^{n} m_{ij}\frac{h_{ij}(u)}{f_i(u)} + \frac{l_i}{f_i(u)} < 1.$$

Therefore, using (15), (69) is satisfied when u is large enough and hence condition **(B)** is satisfied. Therefore (13) has a positive solution. For the proof of the uniqueness of the positive solution of the System (13), we check that conditions **(C)** and **(D)** of Theorem 4.2 are satisfied. Since $m_{ij} \ge 0$ and $h_{ij}(u) > 0$ for $u > 0$, for each $1 \le i,j \le n$, then condition **(C)** is satisfied. To check condition **(D)** assume

$m_{ij} > 0$. Then the function

$$\begin{aligned} \frac{\gamma_j(u)}{g_{ij}(u)} &= \frac{f_j(u) - m_{jj}h_{jj}(u) - l_j}{m_{ij}h_{ij}(u)} \\ &= \frac{f_j(u)}{m_{ij}h_{ij}(u)} - \frac{m_{jj}h_{jj}(u)}{m_{ij}h_{ij}(u)} - \frac{l_j}{m_{ij}h_{ij}(u)} \end{aligned}$$

is monotone increasing on $(0,\infty)$, by $(\mathbf{A_4})$ and $(\mathbf{H_1})$. By assumption $(\mathbf{H_3})$, there exists $i \neq j$ such that $\frac{\gamma_j(u)}{g_{ij}(u)}$ is strictly monotone increasing on $(0,\infty)$, and so condition **(D)** is satisfied. Hence the System (13) has a unique positive solution.

Now we prove (ii). From (16) we have

$$x_i \geq f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j) + l_i\right), \qquad 1 \leq i \leq n. \tag{73}$$

Assumption $(\mathbf{A_3})$ and (14) yield that there exists a small u^* such that

$$0 < u^* < x_i, \qquad 1 \leq i \leq n. \tag{74}$$

and

$$1 \leq \sum_{j=1}^{n} m_{ij}\frac{h_{ij}(u^*)}{f_i(u^*)} + \frac{l_i}{f_i(u^*)},$$

or equivalently,

$$0 < u^* \leq f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(u^*) + l_i\right), \qquad 1 \leq i \leq n. \tag{75}$$

Now we construct a sequence $(x_i^{(0)}, \dots, x_i^{(k)}, \dots)$ such that

$$x_i^{(0)} = u^* \quad \text{and} \quad x_i^{(k+1)} = f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j^{(k)}) + l_i\right), \qquad k \geq 0,\ 1 \leq i \leq n, \tag{76}$$

and we prove that the sequence $(x_i^{(0)}, \dots, x_i^{(k)}, \dots)$ converges. For this, we prove that the sequence $(x_i^{(0)}, \dots, x_i^{(k)}, \dots)$ is monotone increasing and bounded from above. First we show

$$x_i^{(k+1)} \geq x_i^{(k)}, \qquad \text{for all} \quad k \geq 0, \quad 1 \leq i \leq n. \tag{77}$$

For this aim, we use the mathematical induction. At $k = 0$ we have, by (75) and (76),

$$\begin{aligned} x_i^{(1)} &= f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j^{(0)}) + l_i\right) \\ &= f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(u^*) + l_i\right) \\ &\geq u^* \\ &= x_i^{(0)}, \qquad 1 \leq i \leq n. \end{aligned}$$

Next, we assume, for some $k \geq 0$, that

$$x_i^{(k)} \geq x_i^{(k-1)}, \qquad 1 \leq i \leq n. \tag{78}$$

Then, by (76) and (78),

$$\begin{aligned} x_i^{(k+1)} &= f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j^{(k)}) + l_i\right) \\ &\geq f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j^{(k-1)}) + l_i\right) \\ &= x_i^{(k)}, \qquad 1 \leq i \leq n. \end{aligned}$$

Hence the sequence $(x_i^{(0)}, \ldots, x_i^{(k)}, \ldots)$ is monotone increasing for all $k \geq 0$, $1 \leq i \leq n$. Now, to prove that the sequence $(x_i^{(0)}, \ldots, x_i^{(k)}, \ldots)$ is bounded from above for all $k \geq 0$, $1 \leq i \leq n$, we show that

$$x_i^{(k+1)} \leq x_i, \qquad \text{for all} \qquad k \geq 0, \qquad 1 \leq i \leq n.$$

Again, we use the mathematical induction. At $k = 0$ we have, by (73), (74) and (76),

$$\begin{aligned} x_i^{(1)} &= f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j^{(0)}) + l_i\right) \\ &= f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(u^*) + l_i\right) \\ &\leq f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j) + l_i\right) \\ &\leq x_i, \qquad 1 \leq i \leq n. \end{aligned}$$

Next, we assume, for some $k \geq 0$, that

$$x_i^{(k)} \leq x_i, \qquad 1 \leq i \leq n. \tag{79}$$

Then, by (73), (76) and (79), we have

$$\begin{aligned} x_i^{(k+1)} &= f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j^{(k)}) + l_i\right) \\ &\leq f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}h_{ij}(x_j) + l_i\right) \\ &\leq x_i, \qquad 1 \leq i \leq n, \end{aligned}$$

and hence the sequence $(x_i^{(0)}, \ldots, x_i^{(k)}, \ldots)$ is bounded from above for all $k \geq 0$, $1 \leq i \leq n$. Now, since the sequence is monotone increasing and bounded from above, it converges and has a finite limit, i.e.,

$$\lim_{k\to\infty} x_i^{(k)} = x_i^*, \qquad 1 \leq i \leq n,$$

and clearly, $x^* = (x_1^*, \ldots, x_n^*)$ is the unique positive solution of (13). On the other hand, we know that

$$x_i^{(k)} \leq x_i, \qquad k \geq 0, \qquad 1 \leq i \leq n,$$

which implies

$$x_i^* \le x_i, \qquad 1 \le i \le n,$$

and hence the proof of (ii) is completed.

The proof of part (iii) is similar to that of part (ii), so it is omitted here. $\square$

Now, we are ready to prove the main result of the manuscript.

Proof of Theorem 2.4 In the proof we will use the notations

$$\underline{x}_i^\varphi(\infty) := \liminf_{t\to\infty} x_i(\varphi)(t) \quad \text{and} \quad \overline{x}_i^\varphi(\infty) := \limsup_{t\to\infty} x_i(\varphi)(t).$$

By conditions (7), (8), (9) and relation (12), we have for any $T \ge \tau$ that

$$0 \le m_{ij}(T) := \inf_{t\ge T} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{\beta_i(t)} \le \sup_{t\ge T} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{\beta_i(t)} =: M_{ij}(T) < \infty, \quad 1 \le i,j \le n; \tag{80}$$

$$0 \le l_i(T) := \inf_{t\ge T} \frac{\rho_i(t)}{\beta_i(t)} \le \sup_{t\ge T} \frac{\rho_i(t)}{\beta_i(t)} =: L_i(T) < \infty, \qquad 1 \le i \le n; \tag{81}$$

and

$$0 < \underline{x}_i(T) := \inf_{t\ge T-\tau} x_i(t) \le \sup_{t\ge T-\tau} x_i(t) =: \overline{x}_i(T) < \infty, \qquad 1 \le i \le n. \tag{82}$$

Thus from (80), (81), (82) in (5) we get

$$\begin{aligned}
\dot{x}_i(t) &\ge \beta_i(t)\left[\sum_{j=1}^n \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{\beta_i(t)} h_{ij}(\underline{x}_j(T)) + l_i(T) - f_i(x_i(t))\right] \\
&\ge \beta_i(t)\left[\sum_{j=1}^n \inf_{t\ge T} \frac{\sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t)}{\beta_i(t)} h_{ij}(\underline{x}_j(T)) + l_i(T) - f_i(x_i(t))\right] \\
&\ge \beta_i(t)\left[\sum_{j=1}^n m_{ij}(T) h_{ij}(\underline{x}_j(T)) + l_i(T) - f_i(x_i(t))\right], \quad t \ge T,\ 1 \le i \le n,
\end{aligned}$$

or equivalently

$$\dot{x}_i(t) \ge \beta_i(t)\left[C_i(T) - f_i(x_i(t))\right], \qquad t \ge T, \quad 1 \le i \le n, \tag{83}$$

where $C_i(T) := \sum_{j=1}^n m_{ij}(T) h_{ij}(\underline{x}_j(T)) + l_i(T)$. From (83) and the comparison theorem of differential inequalities we get

$$x_i(t) \ge y_i(t), \qquad t \ge T, \quad 1 \le i \le n,$$

where $y_i(t) = y(T, \varphi_i(T), C_i(T), \beta_i, f_i)(t)$, $1 \le i \le n$ are the solutions of the differential equations (61) with $c = C_i(T)$ and with the initial condition

$$y_i(T) = x_i(T), \qquad 1 \le i \le n. \tag{84}$$

So, from Lemma 4.1, we see that

$$\lim_{t\to\infty} y_i(t) = f_i^{-1}\left(C_i(T)\right), \qquad 1 \le i \le n.$$

Thus, for any $T \geq \tau$,

$$\underline{x}_i^{\varphi}(\infty) := \liminf_{t\to\infty} x_i(\varphi)(t) \geq \lim_{t\to\infty} y_i(t) = f_i^{-1}\left(C_i(T)\right), \qquad 1 \leq i \leq n.$$

But

$$\begin{aligned}
\lim_{T\to\infty} f_i^{-1}\left(C_i(T)\right) &= \lim_{T\to\infty} f_i^{-1}\left(\sum_{j=1}^{n} m_{ij}(T) h_{ij}(\underline{x}_j(T)) + l_i(T)\right) \\
&= f_i^{-1}\left(\sum_{j=1}^{n} \lim_{T\to\infty} m_{ij}(T) h_{ij}(\underline{x}_j(T)) + \lim_{T\to\infty} l_i(T)\right) \\
&= f_i^{-1}\left(\sum_{j=1}^{n} \underline{m}_{ij} h_{ij}(\underline{x}_j^{\varphi}(\infty)) + \underline{l}_i\right), \qquad 1 \leq i \leq n.
\end{aligned}$$

Therefore

$$\underline{x}_i^{\varphi}(\infty) \geq f_i^{-1}\left(\sum_{j=1}^{n} \underline{m}_{ij} h_{ij}(\underline{x}_j^{\varphi}(\infty)) + \underline{l}_i\right), \qquad 1 \leq i \leq n,$$

or equivalently

$$f_i(\underline{x}_i^{\varphi}(\infty)) \geq \sum_{j=1}^{n} \underline{m}_{ij} h_{ij}(\underline{x}_j^{\varphi}(\infty)) + \underline{l}_i, \qquad 1 \leq i \leq n.$$

Since all the conditions of Lemma 2.3 are satisfied with $m_{ij} = \underline{m}_{ij}$ and $l_i = \underline{l}_i$, it can be applied, and we obtain

$$\underline{x}_i^{\varphi}(\infty) \geq \underline{x}_i^*, \qquad 1 \leq i \leq n,$$

where $\underline{x}^* = (\underline{x}_1^*, \ldots, \underline{x}_n^*)$ is the unique positive solution of the System (19). In a similar way we can get

$$\overline{x}_i^{\varphi}(\infty) \leq \overline{x}_i^*, \qquad 1 \leq i \leq n,$$

where $\overline{x}^* = (\overline{x}_1^*, \ldots, \overline{x}_n^*)$ is the unique positive solution of the System (20). Hence the proof is completed. □

5. **Conclusions.** In this manuscript we obtained sufficient conditions for the uniform permanence of the positive solutions of a system of nonlinear differential equations with delays of the form

$$\dot{x}_i(t) = \sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) h_{ij}(x_j(t - \tau_{ij\ell}(t))) - \beta_i(t) f_i(x_i(t)) + \rho_i(t), \ t \geq 0, \ 1 \leq i \leq n.$$

It is an interesting future question to extend our method under weaker conditions, e.g., when some functions h_{ij} are decreasing. The key technical result we used in the proof of the main result is Theorem 4.2, where we gave sufficient conditions to guarantee that the algebraic system

$$\gamma_i(x_i) = \sum_{j=1}^{n} g_{ij}(x_j), \qquad 1 \leq i \leq n,$$

has a unique positive solution. It is challenging to extend this result to the case when some functions g_{ij} are increasing, but some others are decreasing.

In Corollary 3.2 we formulated explicit conditions for the uniform permanence of the special system when $h_{ij}(u) = u^{p_{ij}}$ and $f_i(u) = u^{q_i}$, $i, j = 1, \dots, n$. We also gave explicit conditions implying that all positive solutions of the system

$$\dot{x}_i(t) = \sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) x_j(t - \tau_{ij\ell}(t)) - \beta_i(t) x_i^{q_i}(t) + \rho_i(t), \qquad t \geq 0, \quad 1 \leq i \leq n$$

are asymptotically equivalent (see Theorem 3.5). It is an open problem to extend this result to a broader class of nonlinear delay systems, even for equations of the form

$$\dot{x}_i(t) = \sum_{j=1}^{n} \sum_{\ell=1}^{n_0} \alpha_{ij\ell}(t) x_j^{p_{ij}}(t - \tau_{ij\ell}(t)) - \beta_i(t) x_i^{q_i}(t) + \rho_i(t), \qquad t \geq 0, \quad 1 \leq i \leq n$$

such that $q_i > p_{ij} > 0$. We investigated some population models and presented several examples to illustrate our main result.

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