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Existence and uniqueness of positive solutions of a system of nonlinear algebraic equations

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Abstract In this paper we consider the nonlinear system $\gamma_i(x_i) = \sum_{j=1}^m g_{ij}(x_j)$, $1 \le i \le m$. We give sufficient conditions which imply the existence and uniqueness of positive solutions of the system. Our theorem extends earlier results known in the literature. Several examples illustrate the main result.

Keywords Nonlinear algebraic system · Positive solution · Existence · Uniqueness

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1 Introduction

Nonlinear or linear algebraic systems appear as steady-state equations in continuous and discrete dynamical models (e.g., reaction-diffusion equations [14,19], neural networks [5,6,15,22] compartmental systems [2,4,11,12,16,17], population models [13,21]). Next we mention some typical models.

Compartmental systems are used to model many processes in pharmacokinetics, metabolism, epidemiology and ecology. We refer to [16,17] as surveys of basic theory and applications of linear and nonlinear compartmental system without and with delays. A standard form of a linear compartmental system with delays is

$$\dot{q}_i(t) = -k_{ii} q_i(t) + \sum_{\substack{j=1 \\ j \ne i}}^{m} k_{ij} q_j(t - \tau_{ij}) + I_i, \qquad i = 1, \ldots, m. \tag{1.1}$$

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Here $q_i(t)$ is the mass of the ith compartment at time t, $k_{ij} > 0$ represent the transfer or rate coefficients, $I_i \geq 0$ is the inflow to the ith compartment. A possible generalization of (1.1) used in several applications is a compartmental system, where it is assumed that the intercompartmental flows are functions of the state of the donor compartments only in the form $k_{ij} f_j(q_j)$ with some positive nonlinear function f_j. So we get the nonlinear donor-controlled compartmental system (see, e.g., [2,4])

$$\dot{q}_i(t) = -k_{ii} f_i(q_i(t)) + \sum_{\substack{j=1\\ j\neq i}}^{m} k_{ij} f_j(q_j(t-\tau_{ij})) + I_i, \qquad i = 1, \ldots, m. \tag{1.2}$$

Next we consider an ecological system of m species which are living in a symbiotic relationship with the other species (see [10]):

$$\dot{x}_i = x_i \left(-k_{ii} x_i + \sum_{\substack{j=1\\ j\neq i}}^{m} k_{ij} x_j + b_i \right), \qquad i = 1, \ldots, m. \tag{1.3}$$

Here $k_{ii} > 0$ represents the measure of the mortality due to intraspecific competition, the terms $b_i \geq 0$ represents the per capita growth due to external (inexhaustible) sources of energy, and the coefficients k_{ij} $(j \neq i)$ are nonnegative due to the symbiosis.

Cellular neural networks were introduced by Chua and Yang [7] in 1988, and since then they have been applied in many scientific and engineering applications. Here we consider the Hopfield neural network studied in [5]

$$C_i \dot{u}_i = \sum_{j=1}^{m} T_{ij} g_j(u_j) - \frac{u_i}{R_i} + I_i, \qquad i = 1, \ldots, m, \tag{1.4}$$

where $C_i > 0$, $R_i > 0$ and I_i are capacity, resistance, bias, respectively, T_{ij} is the interconnection weight, and g_i is a stricly monotone increasing nonlinear function with $g_i(0) = 0$.

Finally, we recall the delayed Cohen–Grossberg neural network model from [15]

$$\dot{x}_i(t) = -d_i(x_i(t)) \left(c_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij}(t))) + J_i \right) \tag{1.5}$$

for $i = 1, \ldots, n$.

A nonzero equilibrium of both (1.1) and (1.3) satisfies a linear system of the form

$$A\mathbf{x} = \mathbf{b}, \tag{1.6}$$

where $A \in \mathbb{R}^{m\times m}$ has elements

$$a_{ij} = \begin{cases} k_{ii}, & j = i, \\ -k_{ij}, & j \neq i, \end{cases}$$

and $\mathbf{b} \geq \mathbf{0}$, i.e., all coordinates of $\mathbf{b}$ are nonnegative. It is known (see, e.g., [1]) that if A is a nonsingular M-matrix and $\mathbf{b} \gg \mathbf{0}$, i.e, all coordinates of $\mathbf{b}$ are positive, then the System (1.6) has a positive solution $\mathbf{x} \gg \mathbf{0}$. The existence of positive solutions of various classes of linear systems have been studied in [10,18,20].

The existence and uniqueness of positive solutions of the nonlinear algebraic system

$$A\mathbf{u} = \lambda g(\mathbf{u}) \tag{1.7}$$

have been investigated in [3,23–27], where $A \in \mathbb{R}^{m\times m}$, $\mathbf{u} = (u_1, \dots, u_m)^T \in \mathbb{R}^m$, $\lambda > 0$ and $f(\mathbf{u}) = (f_1(u_1), \dots, f_m(u_m))^T$. It was demonstrated in [26] that positive solutions of such systems appear in several problems including finding positive solutions of a finite difference approximation of second-order differential equations with periodic boundary conditions, periodic solutions of fourth-order difference equations, second-order lattice dynamic systems, discrete neural networks.

If A is invertible, we can rewrite (1.7) as $\mathbf{u} = \lambda A^{-1} g(\mathbf{u})$. Then, assuming g is also invertible, using $f_i(u) = g_i^{-1}(u)$, and introducing the new variables $x_i = g_i(u_i)$, we get a nonlinear system of the form

$$f_i(x_i) = \sum_{j=1}^{m} c_{ij} x_j, \qquad 1 \le i \le m. \tag{1.8}$$

In many applications (see [28]) we have that A^{-1} is a positive matrix, i.e., all its coefficients are positive, hence we assume $c_{ij} > 0$ for all $i, j = 1, \dots, m$. The existence and uniqueness of the positive solutions of the System (1.8) was investigated in [7,28] for the special case $f_i(u) = u^\gamma$, and in [8] for the case when all the functions f_i are equal to a given function f.

Recently, in [9] the existence and uniqueness of positive solutions of the nonlinear system

$$f_i(x_i) = \sum_{j=1}^{m} c_{ij} x_j + p_i, \qquad 1 \le i \le m \tag{1.9}$$

was investigated under the conditions $c_{ij} > 0$ for all $i, j = 1, \dots, m$ and $p_i \ge 0$. The main tool used in proving the existence in [9] was Brouwer's fixed point theorem.

The goal of this manuscript is to study the existence and uniqueness of the positive solutions of the general nonlinear system

$$\gamma_i(x_i) = \sum_{j=1}^{m} g_{ij}(x_j), \qquad 1 \le i \le m. \tag{1.10}$$

Note that the System (1.10) includes the steady-state equations of a nonzero equilibrium of the dynamical systems (1.2), (1.4) and (1.5), respectively. Our main result, Theorem 2.1 below, uses a monotone iterative method to prove existence of a positive solution, and an extension of the method used in [9] to prove uniqueness under a weaker condition than that assumed in [9].

The structure of our paper is the following. In Section 2 we formulate our main results. Theorem 2.1 below gives sufficient conditions to imply the existence and uniqueness of the positive solutions of the System (1.10). In Section 3 we show several examples including the Equations (1.6) and (1.9), where Theorem 2.1 is applicable.

2 Main results

Consider the nonlinear system

$$\gamma_i(x_i) = \sum_{j=1}^{m} g_{ij}(x_j), \qquad 1 \le i \le m, \tag{2.1}$$

where $\gamma_i \in C(\mathbb{R}_+, \mathbb{R})$, $g_{ij} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $1 \le i, j \le m$ and $\mathbb{R}_+ := [0, \infty)$. By a positive solution of the System (2.1) we mean a column vector $\mathbf{x} := (x_1, ..., x_m)^T$ which satisfies (2.1), and $x_1 > 0, ..., x_m > 0$.

Next we formulate the main result of this manuscript.

Theorem 2.1 *Let $\gamma_i : \mathbb{R}_+ \to \mathbb{R}$ and $g_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$, $1 \le i, j \le m$ be continuous functions such that for each $1 \le i \le m$,*

(A) *there exists a $u_i^* > 0$ satisfying*

$$\gamma_i(u) \begin{cases} < 0, & \text{if} \quad 0 < u < u_i^*, \\ = 0, & \text{if} \quad u = u_i^*, \\ > 0, & \text{if} \quad u > u_i^*, \end{cases} \tag{2.2}$$

and γ_i is strictly increasing on $[u_i^*, \infty)$.

(B) *g_{ij}, $1 \le i, j \le m$ is increasing on $\mathbb{R}_+$, and there exists a $u_i^{**} \ge u_i^*$ such that*

$$\sum_{j=1}^{m} g_{ij}(u) < \gamma_i(u), \qquad u > u_i^{**}, \qquad 1 \le i \le m. \tag{2.3}$$

Then the System (2.1) has a positive solution.

Moreover, assume that

(C) *for each $1 \le i, j \le m$, either $g_{ij}(u) > 0$ for $u > 0$ or $g_{ij}(u) = 0$ for $u > 0$;*

(D) *for each $1 \le i, j \le m$, $\frac{\gamma_j(u)}{g_{ij}(u)}$ is strictly monotone increasing on $(0, \infty)$, assuming $g_{ij}(u) > 0$ for $u > 0$.*

Then the System (2.1) has a unique positive solution.

Proof Let $B_i := \lim_{u \to \infty} \gamma_i(u)$, $i = 1, \ldots, m$. Then either B_i is positive finite or it is ∞. Note that assumption (2.3) yields that $\sum_{j=1}^{m} g_{ij}(u) \le B_i$ for $u \ge 0$ and $i = 1, \ldots, m$. Assumption **(A)** implies that, for each $i = 1, ..., m$, γ_i restricted to $[u_i^*, \infty)$ has an inverse, i.e., there exists a continuous strictly increasing function $h_i : [0, B_i) \to [u_i^*, \infty)$ satisfying

$$\gamma_i(h_i(u)) = u, \ u \in [0, B_i), \qquad h_i(\gamma_i(u)) = u, \ u \ge u_i^* \quad \text{and} \quad h_i(0) = u_i^*. \tag{2.4}$$

Now we have from (2.1) and the definition of h_i that (2.1) has a positive solution $(x_1, ..., x_m)^T$ if and only if

$$x_i = h_i\left(\sum_{j=1}^{m} g_{ij}(x_j)\right), \qquad 1 \le i \le m.$$

Fix any $\underline{u} > 0$ and $\overline{u} > 0$ such that

$$\underline{u} < \min_{1 \le i \le m} u_i^* \le \max_{1 \le i \le m} u_i^{**} < \overline{u}.$$

Then (2.3) and (2.4) yield

$$\underline{u} \le h_i\left(\sum_{j=1}^{m} g_{ij}(\underline{u})\right) \le h_i\left(\sum_{j=1}^{m} g_{ij}(\overline{u})\right) \le \overline{u}, \qquad 1 \le i \le m. \tag{2.5}$$

Now, for each $i = 1, ..., m$, we construct a sequence $(x_i^{(0)}, ..., x_i^{(n)}, ...)$ by the definition

$$x_i^{(0)} = \underline{u} \text{ and } x_i^{(n+1)} = h_i\left(\sum_{j=1}^{m} g_{ij}(x_j^{(n)})\right), \qquad n \ge 0, \tag{2.6}$$

and we prove that the sequence $(x_i^{(0)}, ..., x_i^{(n)}, ...)$ is convergent. For this aim, we prove that the sequence $(x_i^{(0)}, ..., x_i^{(n)}, ...)$ is monotone increasing and bounded from above. First we show, for each fixed $i = 1, ..., m$, that

$$x_i^{(n+1)} \ge x_i^{(n)}, \qquad \text{for all } \; n \ge 0. \tag{2.7}$$

We use the mathematical induction. At $n = 0$ we have, by (2.5) and (2.6),

$$x_i^{(1)} = h_i\left(\sum_{j=1}^{m} g_{ij}(x_j^{(0)})\right) = h_i\left(\sum_{j=1}^{m} g_{ij}(\underline{u})\right) \ge \underline{u} = x_i^{(0)}.$$

Next, we assume that for some $n \ge 1$

$$x_i^{(n)} \ge x_i^{(n-1)}. \tag{2.8}$$

Then, by (2.6) and (2.8) and the monotonicity of g_{ij} and h_i, we have

$$x_i^{(n+1)} = h_i\left(\sum_{j=1}^{m} g_{ij}(x_j^{(n)})\right) \ge h_i\left(\sum_{j=1}^{m} g_{ij}(x_j^{(n-1)})\right) = x_i^{(n)}.$$

Hence the sequence $(x_i^{(0)}, ..., x_i^{(n)}, ...)$ is monotone increasing.

Now to prove that the sequence $(x_i^{(0)}, ..., x_i^{(n)}, ...)$ is bounded from above for all $1 \le i \le m$, we show that

$$x_i^{(n+1)} \le \overline{u}, \qquad \text{for all } \; n \ge 0, \qquad 1 \le i \le m. \tag{2.9}$$

Again we use the mathematical induction. So, for a fixed $i = 1, ..., m$, at $n = 0$ we have by (2.5) and (2.6) that

$$x_i^{(1)} = h_i\left(\sum_{j=1}^{m} g_{ij}(x_j^{(0)})\right) = h_i\left(\sum_{j=1}^{m} g_{ij}(\underline{u})\right) \le h_i\left(\sum_{j=1}^{m} g_{ij}(\overline{u})\right) \le \overline{u}.$$

Next, we assume for some $n \ge 0$ that

$$x_i^{(n)} \le \overline{u}. \tag{2.10}$$

Then, by (2.5) and (2.10) and the monotonicity of g_{ij} and h_i, we have

$$x_i^{(n+1)} = h_i\left(\sum_{j=1}^m g_{ij}(x_j^{(n)})\right) \le h_i\left(\sum_{j=1}^m g_{ij}(\overline{u})\right) \le \overline{u},$$

and hence the sequence $(x_i^{(0)}, ..., x_i^{(n)}, ...)$ is bounded from above for all $1 \le i \le m$. Now since the sequence is monotone increasing and bounded from above, then it converges to a finite limit, i.e., there exist positive constants x_i such that

$$\lim_{n\to\infty} x_i^{(n)} = x_i, \qquad 1 \le i \le m.$$

On the other hand,

$$x_i = \lim_{n\to\infty} x_i^{(n+1)} = \lim_{n\to\infty} h_i\left(\sum_{j=1}^m g_{ij}(x_j^{(n)})\right) = h_i\left(\sum_{j=1}^m g_{ij}(x_j)\right), \qquad 1 \le i \le m,$$

and hence (2.1) has a positive solution.

Now, we show the uniqueness of the solution of the System (2.1). Suppose that $(u_1, ..., u_m)$ and $(v_1, ..., v_m)$ are two positive solutions of the System (2.1). Then for each $1 \le i \le m$, we have

$$\gamma_i(u_i) = \sum_{j=1}^m g_{ij}(u_j), \qquad \text{and} \qquad \gamma_i(v_i) = \sum_{j=1}^m g_{ij}(v_j). \tag{2.11}$$

Since

$$\gamma_i(u_i) = \sum_{j=1}^m g_{ij}(u_j) \ge 0, \qquad \text{and} \qquad \gamma_i(v_i) = \sum_{j=1}^m g_{ij}(v_j) \ge 0,$$

it follows from **(A)** that $u_i \ge u_i^*$ and $v_i \ge u_i^*$ for $i = 1, ..., m$. Let $H = \{(i,j) : 1 \le i, j \le m, g_{ij}(u) > 0 \text{ for } u > 0\}$. If the set H is empty, then (2.11) reduces to

$$\gamma_i(u_i) = 0, \qquad \text{and} \qquad \gamma_i(v_i) = 0,$$

and hence **(A)** implies that $u_i = u_i^* = v_i$ for $i = 1, ..., m$, and so the uniqueness is proved. Therefore, for the rest of the proof, we assume that $H \ne \emptyset$. Define $(l, s), (k, r) \in H$ such that

$$\frac{g_{ls}(u_s)}{g_{ls}(v_s)} \le \frac{g_{ij}(u_j)}{g_{ij}(v_j)} \le \frac{g_{kr}(u_r)}{g_{kr}(v_r)}, \qquad (i,j) \in H. \tag{2.12}$$

We consider two cases:
(i) Suppose first that

$$\frac{g_{ls}(u_s)}{g_{ls}(v_s)} = \frac{g_{kr}(u_r)}{g_{kr}(v_r)}.$$

Then (2.12) yields that there exists a $\lambda > 0$ such that $g_{ij}(u_j) = \lambda g_{ij}(v_j)$ for $(i,j) \in H$. But then $g_{ij}(u_j) = \lambda g_{ij}(v_j)$ for all $1 \le i, j \le m$. Therefore, from (2.11), we have

$$\gamma_i(u_i) - \lambda\gamma_i(v_i) = \sum_{j=1}^m [g_{ij}(u_j) - \lambda g_{ij}(v_j)] = 0, \qquad 1 \le i \le m.$$

It follows that

$$\frac{\gamma_j(u_j)}{\gamma_j(v_j)} = \lambda, \qquad 1 \le j \le m, \qquad \text{and} \qquad \lambda = \frac{g_{ij}(u_j)}{g_{ij}(v_j)}, \qquad (i,j) \in H,$$

which implies that

$$\frac{\gamma_j(u_j)}{g_{ij}(u_j)} = \frac{\gamma_j(v_j)}{g_{ij}(v_j)}, \qquad (i,j) \in H,$$

and so the strict monotonicity of $\frac{\gamma_j}{g_{ij}}$ yields that $u_j = v_j$ and thus $\lambda = 1$. Hence $\gamma_i(u_i) = \gamma_i(v_i)$, $1 \le i \le m$, which implies $u_i = v_i$, $1 \le i \le m$. Therefore the solution of the System (2.1) is unique.

(ii) Suppose now that

$$\frac{g_{ls}(u_s)}{g_{ls}(v_s)} < \frac{g_{kr}(u_r)}{g_{kr}(v_r)}. \tag{2.13}$$

Note that (2.12) yields

$$g_{ij}(u_j)g_{ls}(v_s) - g_{ij}(v_j)g_{ls}(u_s) \ge 0, \qquad 1 \le i,j \le m, \tag{2.14}$$

and

$$g_{ij}(v_j)g_{kr}(u_r) - g_{ij}(u_j)g_{kr}(v_r) \ge 0, \qquad 1 \le i,j \le m. \tag{2.15}$$

With $i = s$, (2.11) implies

$$\gamma_s(u_s) = \sum_{j=1}^{m} g_{sj}(u_j), \qquad \text{and} \qquad \gamma_s(v_s) = \sum_{j=1}^{m} g_{sj}(v_j),$$

hence

$$\gamma_s(u_s)g_{ls}(v_s) - \gamma_s(v_s)g_{ls}(u_s) = \sum_{j=1}^{m}[g_{sj}(u_j)g_{ls}(v_s) - g_{sj}(v_j)g_{ls}(u_s)].$$

Using (2.14) and that $g_{ls}(u_s) > 0, g_{ls}(v_s) > 0$, we get

$$0 \le \gamma_s(u_s)g_{ls}(v_s) - \gamma_s(v_s)g_{ls}(u_s) = g_{ls}(u_s)g_{ls}(v_s)\left(\frac{\gamma_s(u_s)}{g_{ls}(u_s)} - \frac{\gamma_s(v_s)}{g_{ls}(v_s)}\right).$$

Since $\frac{\gamma_s(u)}{g_{ls}(u)}$ is monotone increasing, it follows $u_s \ge v_s$. Similarly, with $i = r$, (2.11) implies

$$\gamma_r(u_r)g_{kr}(v_r) - \gamma_r(v_r)g_{kr}(u_r) = \sum_{j=1}^{m}[g_{rj}(u_j)g_{kr}(v_r) - g_{rj}(v_j)g_{kr}(u_r)].$$

Using (2.15) and that $g_{kr}(u_r) > 0, g_{kr}(v_r) > 0$, we get

$$0 \ge \gamma_r(u_r)g_{kr}(v_r) - \gamma_r(v_r)g_{kr}(u_r) = g_{kr}(u_r)g_{kr}(v_r)\left(\frac{\gamma_r(u_r)}{g_{kr}(u_r)} - \frac{\gamma_r(v_r)}{g_{kr}(v_r)}\right).$$

Since $\frac{\gamma_r(u)}{g_{kr}(u)}$ is monotone increasing, we get $u_r \le v_r$. The monotonicity of the functions g_{ij} implies that $g_{ls}(u_s) \ge g_{ls}(v_s)$ and $g_{kr}(u_r) \le g_{kr}(v_r)$, and therefore $g_{ls}(v_s)g_{kr}(u_r) - g_{ls}(u_s)g_{kr}(v_r) \le 0$, which contradicts with (2.13). Hence the System (2.1) has a unique solution, and the proof is completed. □

3 Applications

In this section we investigate special cases of the general System (2.1). We show several examples which demonstrate that Theorem 2.1 generalizes known existence and uniqueness results of the literature.

3.1 A linear system

First, we consider a system of linear equations given by

$$a_i x_i = \sum_{j=1}^{m} c_{ij} x_j + p_i, \qquad 1 \le i \le m. \tag{3.1}$$

We show that Theorem 2.1 is applicable for this linear system, too.

Corollary 3.1 *Assume that $a_i > 0$, $p_i > 0$ and $c_{ij} \ge 0$ for each $1 \le i, j \le m$ are such that $a_i > \sum_{j=1}^{m} c_{ij}$. Then the System (3.1) has a unique positive solution.*

Proof Equation (3.1) can be written in the form (2.1) with $\gamma_i(u) := a_i u - p_i$ and $g_{ij}(u) := c_{ij} u$ for each $1 \le i, j \le m$. Now, to prove the existence of a positive solution for System (3.1), we check that conditions **(A)** and **(B)** of Theorem 2.1 are satisfied. Our assumptions yield that $u_i^* = \frac{p_i}{a_i} > 0$ satisfies (2.2). Also, it is clear that $\gamma_i(u)$ is strictly increasing on $[u_i^*, \infty)$, hence condition **(A)** holds. To check condition **(B)**, we see that $g_{ij}(u) := c_{ij} u$, $1 \le i, j \le m$, is increasing on $\mathbb{R}_+$ and (2.3) is satisfied if and only if

$$\sum_{j=1}^{m} g_{ij}(u) < \gamma_i(u) \Leftrightarrow \sum_{j=1}^{m} c_{ij} u < a_i u - p_i \Leftrightarrow u > \frac{p_i}{a_i - \sum_{j=1}^{m} c_{ij}} > 0,$$

therefore (2.3) holds with $u_i^{**} = \frac{p_i}{a_i - \sum_{j=1}^{m} c_{ij}} \ge u_i^*$. Hence (3.1) has a positive solution.

Now, to show the uniqueness of the positive solution of the System (3.1), we check that conditions **(C)** and **(D)** of Theorem 2.1 are satisfied. By our assumption that $c_{ij} \ge 0$ for each $1 \le i, j \le m$, we see that $g_{ij}(u) = c_{ij} u > 0$ for $u > 0$ if $c_{ij} > 0$, and $g_{ij}(u) = 0$ for $u > 0$ if $c_{ij} = 0$, and hence condition **(C)** holds. If $c_{ij} > 0$ for some $1 \le i, j \le m$, then we have

$$\frac{\gamma_j(u)}{g_{ij}(u)} = \frac{a_j u - p_j}{c_{ij} u} = \frac{a_j}{c_{ij}} - \frac{p_j}{c_{ij} u}$$

is strictly increasing on $(0, \infty)$ and so condition **(D)** is satisfied. Hence the System (3.1) has a unique positive solution and the proof is completed. □

Note that the conditions of Corollary 3.1 imply that the matrix $A \in \mathbb{R}^{m \times m}$ with elements

$$a_{ij} = \begin{cases} a_i - c_{ii}, & i = j, \\ -c_{ij}, & i \ne j, \end{cases}$$

is positive definite, so A is a nonsingular M-matrix (see [1]). Therefore the existence and uniqueness of the positive solution of (1.6) with $\mathbf{b} = (p_1, \ldots, p_m)^T$ follows immediately using the results of [1].

3.2 Nonlinear systems

Next we consider the nonlinear system

$$a_i x_i^{\alpha_i} = \sum_{j=1}^{m} c_{ij} x_j^{\beta_{ij}} + p_i, \qquad 1 \le i \le m. \tag{3.2}$$

If we set $\beta_{ij} = 1$ for all i, j, then the corresponding Equation (3.2) will be a special case of (1.8) with $f_i(u) = a_i u^{\alpha_i}$. For this case it was shown in [9] that if $a_i > 0$, $\alpha_i > 1$, $p_i \ge 0$, $\beta_{ij} = 1$ and $c_{ij} > 0$ for $1 \le i, j \le m$, then (3.2) has a unique positive solution. Now in the next result we show the existence and uniqueness of the solution of (3.2) under weaker assumption even in the above special case, since c_{ij} is allowed to be 0, and we suppose that one of the parameters c_{ii} or p_i is positive for all $i = 1, \dots, m$.

Corollary 3.2 *Assume that $a_i > 0$, $p_i \ge 0$ and $c_{ij} \ge 0$ for each $1 \le i, j \le m$ are such that $c_{ii} + p_i > 0$ for $1 \le i \le m$. Then the System (3.2) has a unique positive solution assuming that $\alpha_i > \beta_{ij} \ge 0$ for all $1 \le i, j \le m$.*

Proof Equation (3.2) can be written in the form (2.1) with $\gamma_i(u) := a_i u^{\alpha_i} - c_{ii} u^{\beta_{ii}} - p_i$, $g_{ij}(u) := c_{ij} u^{\beta_{ij}}$ for each $1 \le i \ne j \le m$ and $g_{ii}(u) = 0$. Now, we check that conditions **(A)** and **(B)** of Theorem 2.1 are satisfied. For condition **(A)**, we have $\gamma_i(u) = 0$, $1 \le i \le m$, if and only if

$$a_i u^{(\alpha_i - \beta_{ii})} = c_{ii} + \frac{p_i}{u^{\beta_{ii}}}, \qquad 1 \le i \le m. \tag{3.3}$$

It is clear that the left hand side of (3.3) is an increasing function and the right hand side of (3.3) is a decreasing function if and only if $\alpha_i > \beta_{ii} \ge 0$ for all $1 \le i \le m$. So it is easy to see, using the assumed conditions, that their graphs intersect in a unique point $u_i^* > 0$, therefore there exists a $u_i^* > 0$ which satisfies (2.2). Note that

$$\gamma_i'(u) = \alpha_i a_i u^{(\alpha_i - 1)} - c_{ii}\beta_{ii} u^{(\beta_{ii} - 1)} = u^{(\beta_{ii} - 1)} \left(\alpha_i a_i u^{(\alpha_i - \beta_{ii})} - c_{ii}\beta_{ii} \right) > 0,$$

if

$$u > \bar{u}_i := \left(\frac{c_{ii}\beta_{ii}}{a_i \alpha_i} \right)^{\frac{1}{\alpha_i - \beta_{ii}}} \ge 0, \qquad 1 \le i \le m.$$

Since $\gamma_i(\bar{u}_i) < 0$, we have $u_i^* > \bar{u}_i$, and therefore $\gamma_i(u)$ is strictly increasing on $[u_i^*, \infty)$ and condition **(A)** is satisfied. To check condition **(B)**, we see that $g_{ij}(u) := c_{ij} u^{\beta_{ij}}$, $1 \le i \ne j \le m$, and $g_{ii}(u) = 0$ are increasing on $\mathbb{R}_+$, and (2.3) is satisfied if and only if

$$\sum_{\substack{j=1 \\ j \ne i}}^{m} c_{ij} u^{\beta_{ij}} < a_i u^{\alpha_i} - c_{ii} u^{\beta_{ii}} - p_i \Leftrightarrow \sum_{j=1}^{m} c_{ij} u^{(\beta_{ij} - \alpha_i)} < a_i - \frac{p_i}{u^{\alpha_i}},$$

therefore (2.3) is satisfied with a large enough u_i^{**}. Therefore (3.2) has a positive solution.

Now, we check conditions **(C)** and **(D)** of Theorem 2.1. Since $c_{ij} \geq 0$ for each $1 \leq i, j \leq m$, then condition **(C)** holds. If $c_{ij} = 0$ for all $1 \leq i, j \leq m$, then **(D)** is satisfied. Assuming that $c_{ij} > 0$ for some $1 \leq i, j \leq m$, then

$$\frac{\gamma_j(u)}{g_{ij}(u)} = \frac{a_j u^{\alpha_j} - c_{jj} u^{\beta_{jj}} - p_j}{c_{ij} u^{\beta_{ij}}} = \frac{a_j u^{(\alpha_j - \beta_{ij})}}{c_{ij}} - \frac{c_{jj}}{c_{ij}} u^{(\beta_{jj} - \beta_{ij})} - \frac{p_j}{c_{ij} u^{\beta_{ij}}}. \quad (3.4)$$

If $\beta_{jj} < \beta_{ij}$, then each term in (3.4) is strictly monotone increasing on $(0, \infty)$, and hence so is $\frac{\gamma_j(u)}{g_{ij}(u)}$. If $\beta_{jj} \geq \beta_{ij}$, then it follows from (3.4) that

$$\frac{\gamma_j(u)}{g_{ij}(u)} = \frac{u^{(\beta_{jj} - \beta_{ij})}}{c_{ij}} \left(a_j u^{(\alpha_j - \beta_{jj})} - c_{jj} \right) - \frac{p_j}{c_{ij} u^{\beta_{ij}}},$$

which is also strictly monotone increasing on $(0, \infty)$, so condition **(D)** is satisfied. Hence, by Theorem 2.1, the System (3.2) has a unique positive solution, and the proof is completed. □

Now we consider the system

$$f_i(x_i) = \sum_{j=1}^{m} c_{ij} x_j + p_i, \qquad 1 \leq i \leq m \quad (3.5)$$

which was studied in [9]. It was assumed in [9] that the function $\frac{f_i(u)}{u}$ is strictly increasing for all $i = 1, \ldots, m$, $c_{ij} > 0$ for all $1 \leq i, j \leq m$, and for every $i = 1, \ldots, m$ and $s_i = c_{i1} + \cdots + c_{im}$ there exists $t_i > 0$ such that $\frac{f_i(t_i)}{t_i} = s_i$. Then the System (3.5) has a unique positive solution. Our main result of Theorem 2.1 gives back this results under a weaker assumption that c_{ij} can take the values 0, and only either c_{ii} or p_i is assumed to be positive for all $i = 1, \ldots, m$.

Corollary 3.3 *Assume that, for each $i = 1, ..., m$, $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, such that $\frac{f_i(u)}{u}$ is strictly increasing, and*

$$\lim_{u \to 0^+} \frac{f_i(u)}{u} \begin{cases} < \infty, & \text{if} \quad p_i > 0, \\ = 0, & \text{if} \quad p_i = 0, \end{cases} \qquad \text{and} \qquad \lim_{u \to \infty} \frac{f_i(u)}{u} > \sum_{j=1}^{m} c_{ij}, \qquad i = 1, ..., m.$$

Furthermore, assume that $p_i \geq 0$ and $c_{ij} \geq 0$ for each $1 \leq i, j \leq m$ are such that $c_{ii} + p_i > 0$ for $1 \leq i \leq m$. Then the System (3.5) has a unique positive solution.

Proof We can rewrite (3.5) in the form (2.1) with $\gamma_i(u) := f_i(u) - c_{ii} u - p_i$ and $g_{ij}(u) := c_{ij} u$ for each $1 \leq i \neq j \leq m$ and $g_{ii}(u) = 0$. Now, we check that conditions **(A)** and **(B)** of Theorem 2.1 are satisfied. For condition **(A)**, we have with $u_i^* > 0$ that $\gamma_i(u_i^*) = 0$, if

$$\frac{f_i(u_i^*)}{u_i^*} = \frac{p_i}{u_i^*} + c_{ii}, \qquad 1 \leq i \leq m. \quad (3.6)$$

It is clear that the left hand side of (3.6) is an increasing function and the right hand side of (3.6) is a decreasing function, so the assumed conditions yield that their graphs intersect in a unique point $u_i^* > 0$, therefore there exists a $u_i^* > 0$ satisfying (2.2). We have that

$$\gamma_i(u) = u\left[\frac{f_i(u)}{u} - c_{ii}\right] - p_i, \qquad 1 \le i \le m,$$

is strictly increasing on $(0,\infty)$, and hence condition **(A)** is satisfied. To check condition **(B)**, we see that $g_{ij}(u) := c_{ij}u$, $1 \le i \ne j \le m$, and $g_{ii}(u) = 0$ are increasing on $\mathbb{R}_+$, and (2.3) is satisfied if and only if

$$\sum_{\substack{j=1\\ j\neq i}}^{m} c_{ij}u < f_i(u) - c_{ii}u - p_i \Leftrightarrow \sum_{j=1}^{m} c_{ij} < \frac{f_i(u)}{u} - \frac{p_i}{u},$$

therefore (2.3) is satisfied when u is large enough. Hence condition **(B)** holds. Therefore (3.5) has a positive solution.

For the proof of the uniqueness of the positive solution of the System (3.5), we check conditions **(C)** and **(D)** of Theorem 2.1. Since $c_{ij} \ge 0$ for each $1 \le i,j \le m$, condition **(C)** is satisfied. Assuming that $c_{ij} > 0$ for some $1 \le i,j \le m$, we get

$$\frac{\gamma_j(u)}{g_{ij}(u)} = \frac{f_j(u) - c_{jj}u - p_j}{c_{ij}u} = \frac{f_j(u)}{c_{ij}u} - \frac{c_{jj}}{c_{ij}} - \frac{p_j}{c_{ij}u}$$

is strictly increasing on $(0,\infty)$ and so condition **(D)** is satisfied. Hence the System (3.5) has a unique positive solution. □

Now, we consider a more general system of nonlinear algebraic equations

$$\gamma_i(x_i) = \sum_{j=1}^{m} c_{ij}\sigma_j(x_j), \qquad 1 \le i \le m. \tag{3.7}$$

The System (3.7) includes the steady-state equations of the donor-controlled compartmental system (1.2) and the Cohen–Grossberg neural network model (1.5).

Corollary 3.4 *Assume that $c_{ij} \ge 0$, for each $1 \le i,j \le m$, $\gamma_i : (0,\infty) \to (0,\infty)$ and $\sigma_i : (0,\infty) \to (0,\infty)$ are continuous and strictly increasing for $i = 1,...,m$, such that*

- ($\boldsymbol{A}^*$) *the function γ_i, $i = 1,...,m$, satisfies condition* **(A)** *of Theorem 2.1;*
- ($\boldsymbol{B}^*$) *the functions γ_i and σ_j, $1 \le i,j \le m$ satisfy $\sum_{j=1}^{m} c_{ij}\sigma_j(u) < \gamma_i(u)$ for large enough u.*

Then the System (3.7) has a positive solution.

Furthermore, assume that $\frac{\gamma_i(u)}{\sigma_i(u)}$ is continuous and strictly increasing on $(0,\infty)$, for all $1 \le i \le m$. Then the System (3.7) has a unique positive solution.

Proof Equation (3.7) can be written in the form (2.1) with $g_{ij}(u) := c_{ij}\sigma_j(u)$ for each $1 \le i,j \le m$. Assumptions ($\mathbf{A}^*$) and ($\mathbf{B}^*$) show that conditions **(A)** and **(B)** of Theorem 2.1 are satisfied. Therefore (3.7) has a positive solution.

Now, we show that the positive solution the System (3.7) is unique. Since $c_{ij} \ge 0$ for each $1 \le i,j \le m$, then we see that $g_{ij}(u) = c_{ij}\sigma_j(u) > 0$ for $u > 0$ if

$c_{ij} > 0$ and $g_{ij}(u) = 0$ for $u > 0$ if $c_{ij} = 0$, and hence condition **(C)** of Theorem 2.1 is satisfied. Assuming that $c_{ij} > 0$ for some $1 \leq i, j \leq m$, then

$$\frac{\gamma_j(u)}{g_{ij}(u)} = \frac{\gamma_j(u)}{c_{ij}\sigma_j(u)} = \frac{1}{c_{ij}}\frac{\gamma_j(u)}{\sigma_j(u)}$$

is strictly increasing on $(0, \infty)$, and so condition **(D)** of Theorem 2.1 holds. Hence the System (3.5) has a unique positive solution and the proof is completed. □

3.3 Two dimensional systems

We consider the System (2.1) in the special case when $m = 2$:

$$\begin{aligned} \psi_1(x_1) &= g_{11}(x_1) + g_{12}(x_2), \\ \psi_2(x_2) &= g_{21}(x_1) + g_{22}(x_2). \end{aligned} \tag{3.8}$$

Introducing $\gamma_i(u) = \psi_i(u) - g_{ii}(u), \ \ i = 1, 2$, we get the equivalent system

$$\begin{aligned} \gamma_1(x_1) &= g_{12}(x_2), \\ \gamma_2(x_2) &= g_{21}(x_1). \end{aligned} \tag{3.9}$$

The following result shows that in this two dimensional case we can reduce the study of existence and uniqueness of solutions of the System (3.9) to that of a scalar equation.

Corollary 3.5 *Assume that, for each* $1 \leq i, j \leq 2$, $\gamma_i, g_{ij} \in C(\mathbb{R}_+, \mathbb{R}_+)$, *such that*

($\boldsymbol{H}_1$) *the functions* γ_1 *and* γ_2 *satisfy condition* **(A)** *of Theorem 2.1;*
($\boldsymbol{H}_2$) *the functions* g_{12} *and* g_{21} *satisfy condition* **(B)** *of Theorem 2.1.*

Then

(i) *the System (3.9) has a positive solution;*
(ii) *the positive vector* (u_1, u_2) *is a solution of (3.9) if and only if* u_1 *and* u_2 *are the solutions of the scalar equations*

$$u = h_1(g_{12}(h_2(g_{21}(u)))) \tag{3.10}$$

and

$$u = h_2(g_{21}(h_1(g_{12}(u)))) \tag{3.11}$$

respectively, where h_1 *and* h_2 *are defined by (2.4);*
(iii) *the positive solution of System (3.9) is unique if at least one of the Equations (3.10) or (3.11) (or equivalently both of them) has only a unique positive solution.*

Proof The proof of **(i)** is the consequence of Theorem 2.1. For the proof of **(ii)**, we see that the Equations (3.10) and (3.11) follow from System (3.9) using the inverse of the functions $\gamma_i, \ \ i = 1, 2$. For the proof of **(iii)** we consider the case when, e.g., x_1 is a unique solution of (3.10), then clearly $(x_1, h_2(g_{21}(h_1(g_{12}(x_1)))))$ is the unique solution of the System (3.9). □

Example 3.6 As an example on the two dimensional case, we consider the system

$$\begin{aligned} 2x_1 - 1 &= x_2, \\ x_2 - 0.5 &= g_{21}(x_1), \end{aligned} \tag{3.12}$$

where

$$g_{21}(u) = \begin{cases} 0.5, & \text{if} \quad u \in [0,1], \\ 2u - 1.5, & \text{if} \quad u \in [1,2], \\ 2.5, & \text{if} \quad u \in [2,\infty). \end{cases}$$

Define $\gamma_1(u) = 2u - 1$, $\gamma_2(u) = u - 0.5$, $g_{12}(u) = u$. Then, clearly, we can see that condition **(A)** of Theorem 2.1 is satisfied with $u_1^* = 0.5$ and $u_2^* = 0.5$. Also, condition **(B)** of Theorem 2.1 holds for the System (3.12), and so the System (3.12) has a positive solution. Condition **(C)** of Theorem 2.1 holds too. We have, from the definition of γ_1 and γ_2, that

$$h_1(u) = \frac{u+1}{2}, \quad u \in \mathbb{R}_+, \quad \textit{and} \quad h_2(u) = u + 0.5, \quad u \in \mathbb{R}_+.$$

Then Equation (3.11) reduces to

$$u = h_2(g_{21}(h_1(g_{12}(u)))) = h_2\left(g_{21}\left(\frac{u+1}{2}\right)\right) = \begin{cases} h_2(0.5), & \text{if} \quad u \in [0,1], \\ h_2(u-0.5), & \text{if} \quad u \in [1,3], \\ h_2(2.5), & \text{if} \quad u \in [3,\infty), \end{cases}$$

or equivalently,

$$u = \begin{cases} 1, & \text{if} \quad u \in [0,1], \\ u, & \text{if} \quad u \in [1,3], \\ 3, & \text{if} \quad u \in [3,\infty). \end{cases}$$

This shows that (3.11) has infinitely many solutions, say, $u_2 = t,\ t \in [1,3]$, then $(\frac{t+1}{2}, t),\ t \in [1,3]$ is a solution of the System (3.12). On the other hand, we have

$$\frac{\gamma_1(u)}{g_{21}(u)} = \frac{2u-1}{2u-1.5} = 1 + \frac{0.5}{2u-1.5}, \qquad u \in [1,2],$$

which is decreasing on $[1,2]$. Also, we have

$$\frac{\gamma_2(u)}{g_{12}(u)} = \frac{u-0.5}{u} = 1 - \frac{0.5}{u}, \qquad u \in [1,2],$$

which is increasing on $[1,2]$. So condition **(D)** of Theorem 2.1 is not satisfied in this case. This shows that if condition **(D)** of Theorem 2.1 does not hold, we may loose the uniqueness.

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