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Nonlinear variation of constants formula for differential equations with state-dependent delays*

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Abstract

In this paper we consider a class of differential equations with state-dependent delays. We show differentiability of the solution with respect to the initial function and the initial time for each fixed time value assuming that the state-dependent time lag function is piecewise monotone increasing. Based on these results, we prove a nonlinear variation of constants formula for differential equations with state-dependent delay. As an application, we discuss asymptotic properties of perturbed nonlinear differential equations with state-dependent delays.

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1 Introduction

In this manuscript we extend the nonlinear variation of constants formula of Alekseev to a class of delay differential equations with state-dependent delays (SD-DDEs). This relation was first proved for ODEs in [2], and later it was extended to several different classes of differential equations, including Volterra equations by Brauer [6], Hu, Lakshmikantham, Rao [27] and Agyingi, Baker [1], delay equations by Shanholt [31], and neutral differential equations by Izé and Ventura [28]. See also [33] for a related work in SD-DDEs with a perturbation without delays. The nonlinear variation of constants formula was used, e.g., to study stability or asymptotic behavior of perturbed nonlinear systems ([4], [5], [7], [14], [34]) and for convergence properties of certain numerical methods in Volterra equations ([3], [9]).

In Section 2 we consider the SD-DDE

$$\dot{x}(t) = f(t, x_t, x(t - \tau(t, x_t))),$$

and recall a well-posedness results for this class of SD-DDEs. The dependence on x_t in the second variable of f represents state-independent delays, since we will assume smooth dependence of f with respect to its second variable. The results of this manuscript can be easily extended to the case of multiple point state-dependent delays; single state-dependent delay is assumed here for simplicity of the notations.

*This manuscript is dedicated to John Mallet-Paret on the occasion of his 60th birthday.

The proof of our nonlinear variation of constants formula is based on the differentiability of the solutions with respect to (wrt) the initial time and initial function. Differentiability of the solutions of SD-DDEs wrt to the initial function (and other parameters) was studied in [10], [19], [21], [22], [24], [35], [36]. The only paper which proves differentiability of solutions of SD-DDEs wrt the initial time σ is [21]. In Section 3 we show the differentiability of the solutions wrt the initial function. This section is based on the results of [22], but in this case we consider the initial time also as a parameter in the equation, and our assumptions here are a little simpler than those used in [22]. In Section 4 we extend the results of [21] concerning the differentiability of the solutions wrt the initial time using weaker assumptions. In this manuscript we unify and simplify the sufficient conditions for differentiability. In Section 5 we formulate and prove a nonlinear variation of constants formula for differential equations with state-dependent delays. In Section 6, as an application of the nonlinear variation of constants formula, we study asymptotic behaviour of SD-DDEs with nonlinear perturbations.

2 Well-posedness and preliminaries

A fixed norm on $\mathbb{R}^n$ and its induced matrix norm on $\mathbb{R}^{n\times n}$ are both denoted by $|\cdot|$. C denotes the Banach space of continuous functions $\psi: [-r,0] \to \mathbb{R}^n$ equipped with the norm $|\psi|_C = \sup\{|\psi(s)|: s \in [-r,0]\}$. C^1 is the space of continuously differentiable functions $\psi: [-r,0] \to \mathbb{R}^n$ where the norm is defined by $|\psi|_{C^1} = \max\{|\psi|_C, |\dot\psi|_C\}$. L^∞ is the space of Lebesgue measurable functions $\psi: [-r,0] \to \mathbb{R}^n$ which are essentially bounded. The norm on L^∞ is denoted by $|\cdot|_{L^\infty}$. $W^{1,\infty}$ denotes the Banach-space of absolutely continuous functions $\psi: [-r,0] \to \mathbb{R}^n$ of finite norm defined by

$$|\psi|_{W^{1,\infty}} := \max\left\{|\psi|_C, |\dot\psi|_{L^\infty}\right\}.$$

We note that $W^{1,\infty}$ is equal to the space of Lipschitz continuous functions from $[-r,0]$ to $\mathbb{R}^n$. If the domain or the range of the functions is different from $[-r,0]$ and $\mathbb{R}^n$, respectively, we will use a more detailed notation. E.g., $C(X,Y)$ denotes the space of continuous functions mapping from X to Y. Finally, $\mathcal{L}(X,Y)$ denotes the space of bounded linear operators from X to Y, where X and Y are normed linear spaces. An open ball in the normed linear space $(X, |\cdot|)$ centered at a point $x \in X$ with radius δ is denoted by $\mathcal{B}_X(x;\delta) := \{y \in X: |x-y| < \delta\}$.

The partial derivatives of a function $g: X \times Y \to Z$ wrt the first and second variable will be denoted by $D_1 g$ and $D_2 g$, respectively. All derivatives in this paper are Fréchet-derivatives.

Consider the nonlinear SD-DDE

$$\dot x(t) = f(t, x_t, x(t-\tau(t,x_t))), \qquad t \in [\sigma, T], \tag{2.1}$$

and the corresponding initial condition

$$x(t) = \varphi(t-\sigma), \qquad t \in [\sigma - r, \sigma]. \tag{2.2}$$

Here and throughout this paper $r > 0$ is a fixed finite number, and $x_t: [-r,0] \to \mathbb{R}^n$, $x_t(\theta) := x(t+\theta)$ is the segment function. Let $\Omega_1 \subset C$, $\Omega_2 \subset \mathbb{R}^n$ be open subsets of the respective spaces. $T > 0$ is finite or $T = \infty$, in which case $[0,T]$ denotes the interval $[0,\infty)$.

We assume

(A1) $f: \mathbb{R} \times C \times \mathbb{R}^n \supset [0,T] \times \Omega_1 \times \Omega_2 \to \mathbb{R}^n$ is continuous and it is continuously differentiable wrt its second and third arguments, (A2) $\tau\colon \mathbb{R}\times C\supset[0,T]\times\Omega_1\to[0,r]$ is continuously differentiable wrt both arguments.

We introduce the set of admissible parameters

$$\Pi := \Big\{(\sigma,\varphi)\in[0,T)\times W^{1,\infty}\colon\ \varphi\in\Omega_1,\ \varphi(-\tau(\sigma,\varphi))\in\Omega_2\Big\}.$$

The next theorem shows that every admissible parameter $(\hat\sigma,\hat\varphi)\in\Pi$ has a neighborhood $P\subset\mathbb{R}\times W^{1,\infty}$ and there exists a time $\alpha>\hat\sigma$ such that the IVP (2.1)-(2.2) has a unique solution on $[\sigma-r,\alpha]$ corresponding to all parameters $(\sigma,\varphi)\in P$. This solution will be denoted by $x(t,\sigma,\varphi)$, and its segment function at t is denoted by $x_t(\cdot,\sigma,\varphi)$.

The well-posedness of several classes of SD-DDEs was studied in many papers (see, e.g., [15, 23, 24, 32]). The next result about the well-posedness of solutions is proved in [21] under a weaker assumption when only Lipschitz continuity of f and τ is assumed wrt the second and third, and wrt the second arguments, respectively, instead of the continuous differentiability. The notations and estimates introduced in the next theorem will be essential in the following sections.

Theorem 2.1 ([21]) *Assume (A1), (A2), and let $(\hat\sigma,\hat\varphi)\in\Pi$. Then there exist $\delta>0$, $0\le\sigma_0\le\hat\sigma$, $\hat\sigma<\alpha\le T$ finite numbers such that $0\le\sigma_0<\hat\sigma$ if $\hat\sigma>0$, and $\sigma_0=0$ if $\hat\sigma=0$, and*

(i) for all $(\sigma,\varphi)\in P$ the IVP (2.1)-(2.2) has a unique solution $x(t,\sigma,\varphi)$ on $[\sigma-r,\alpha]$, where

$$P := [\sigma_0,\alpha)\times\mathcal{B}_{W^{1,\infty}}(\hat\varphi;\,\delta); \tag{2.3}$$

(ii) there exist $M_1\subset C$ and $M_2\subset\mathbb{R}^n$ compact subsets of the respective spaces such that $x_t(\cdot,\sigma,\varphi)\in M_1$ and $x(t-\tau(t,x_t(\cdot,\sigma,\varphi)),\sigma,\varphi)\in M_2$ for $(\sigma,\varphi)\in P$ and $t\in[\sigma,\alpha]$; and

(iii) $x_t(\cdot,\sigma,\varphi)\in W^{1,\infty}$ for $(\sigma,\varphi)\in P$ and $t\in[\sigma,\alpha]$, and there exist constants $N=N(\sigma_0,\alpha,\delta)$ and $L=L(\sigma_0,\alpha,\delta)$ such that

$$|x_t(\cdot,\sigma,\varphi)|_{W^{1,\infty}}\le N,\qquad(\sigma,\varphi)\in P,\ t\in[\sigma,\alpha], \tag{2.4}$$

and

$$|x_t(\cdot,\sigma,\varphi)-x_t(\cdot,\bar\sigma,\bar\varphi)|_{W^{1,\infty}}\le L(|\sigma-\bar\sigma|+|\varphi-\bar\varphi|_{W^{1,\infty}}) \tag{2.5}$$

for $(\sigma,\varphi),(\bar\sigma,\bar\varphi)\in P$ and $t\in[\max\{\sigma,\bar\sigma\},\alpha]$.

Let $(\hat\sigma,\hat\varphi)\in\Pi$ be fixed, and throughout Sections 3 and 4 let the parameter set P and time α be defined by Theorem 2.1 (i). We introduce the following set:

$$H := \{(t,\sigma,\varphi)\in\mathbb{R}\times\mathbb{R}\times W^{1,\infty}\colon\ (\sigma,\varphi)\in P,\ t\in[\sigma,\alpha]\}. \tag{2.6}$$

Then the solution $x(t,\sigma,\varphi)$ exists and it is continuous on H.

The following result is obvious.

Remark 2.2 *Suppose the conditions of Theorem 2.1 hold, P and α are defined by Theorem 2.1, and let*

$$\mathcal{P} := \Big\{(\sigma,\varphi)\in P\colon\ \varphi\in C^1,\quad \dot\varphi(0-)=f(\sigma,\varphi,\varphi(\sigma-\tau(\sigma,\varphi)))\Big\}. \tag{2.7}$$

Then for all parameter values $(\sigma,\varphi)\in\mathcal{P}$ the corresponding solution $x(t,\sigma,\varphi)$ is continuously differentiable wrt t for $t\in[\sigma-r,\alpha]$.

We note that the compatibility condition used in the definition of $\mathcal{P}$ was essential in [35], [36] to prove the existence of a semiflow of continuously differentiable solution operators. Note that an analogous set was used for neutral FDEs in order to guarantee the existence of a continuous semiflow on a subset of C^1 in [29]. Similar compatibility assumption was also used in [19] to prove differentiability of $x(t,\sigma,\varphi)$ wrt φ (and other parameters of the equation).

Let M_1 and M_2 be defined by Theorem 2.1 (ii). It follows from (A1) and (A2) that there exist constants $L_1 \geq 0$ and $L_2 \geq 0$ such that

$$|f(t,\psi,y) - f(t,\tilde{\psi},\tilde{y})| \leq L_1(|\psi - \tilde{\psi}|_C + |y - \tilde{y}|), \qquad t \in [0,\alpha],\ \psi,\tilde{\psi} \in M_1, y,\tilde{y} \in M_2, \tag{2.8}$$

and

$$|\tau(t,\psi) - \tau(t,\tilde{\psi})| \leq L_2|\psi_t - \tilde{\psi}|_C, \qquad t \in [0,\alpha],\ \psi,\tilde{\psi} \in M_1. \tag{2.9}$$

3 Differentiability wrt initial function

In this section we summarize the results for differentiability of $x(t,\sigma,\varphi)$ wrt φ and σ. To obtain such a result we have to consider the linearization of the right-hand-side of Eq. (2.1) about a fixed solution $x(t) = x(t,\sigma,\varphi)$ for some $(\sigma,\varphi) \in P$, where the parameter set P is defined in (2.3). This question was first studied in [8]. The formal linearization yields the linear operator $L(t,x)\colon C \to \mathbb{R}^n$,

$$\begin{aligned} L(t,x)\psi \quad := \quad & D_2 f(t,x_t,x(t-\tau(t,x_t)))\psi \\ & + D_3 f(t,x_t,x(t-\tau(t,x_t)))\Big(-\dot{x}(t-\tau(t,x_t))D_2\tau(t,x_t)\psi + \psi(-\tau(t,x_t))\Big) \end{aligned} \tag{3.1}$$

for $\psi \in C$. Since we do not assume the compatibility condition, and $\varphi \in W^{1,\infty}$ only, not a C^1-function, therefore $\dot{x}(t-\tau(t,x_t))$ may not be defined for such t when $t-\tau(t,x_t) \in [-r,0]$. In order to handle this technical difficulty, we need to guarantee the measurability and the integrability of the function $t \mapsto \dot{\varphi}(t-\tau(t,x_t))$ for the case when $\varphi \in W^{1,\infty}$, i.e., when $\dot{\varphi} \in L^\infty$.

Let $x(t) = x(t,\sigma,\varphi)$ be a fixed solution the IVP (2.1)-(2.2) for $(\sigma,\varphi) \in P$, then x is, in general, only a $W^{1,\infty}$-function on the interval $[\sigma - r,\sigma]$, but it is continuously differentiable for $t > \sigma$. In [21] (see also [24]) it was assumed that $(\sigma,\varphi) \in P$ is such that for $x(t) = x(t,\sigma,\varphi)$ the corresponding time lag function $t \mapsto t - \tau(t,x_t)$ is strictly monotone increasing, more precisely,

$$\operatorname{ess\,inf}\left\{\frac{d}{dt}(t-\tau(t,x_t))\colon\ t \in [\sigma,\alpha_\sigma^*]\right\} > 0, \tag{3.2}$$

where $\alpha_\sigma^* := \min\{\sigma + r,\alpha\}$. Note that similar condition was used in [8]. In [21] the parameter set

$$P_1 := \{(\sigma,\varphi) \in P\colon\ x(\cdot,\sigma,\varphi) \text{ satisfies (3.2)}\} \tag{3.3}$$

was introduced, and it was proved that P_1 is an open subset of P.

Lemma 2.5 in [22] yields that if $(\sigma,\varphi) \in P_1$, then for $x(t) = x(t,\sigma,\varphi)$, the function $t \mapsto \dot{x}(t-\tau(t,x_t))$ is measurable on $[0,\alpha_\sigma^*]$, and $\operatorname{ess\,sup}\{|\dot{x}(t-\tau(t,x_t))|\colon\ t \in [0,\alpha_\sigma^*]\} \leq \operatorname{ess\,sup}\{|\dot{x}(t)|\colon t \in [-r,\alpha_\sigma^*]\}$. Then the linear operator $L(t,x)$ is defined for a.e. $t \in [0,\alpha]$ and for all $(\sigma,\varphi) \in P_1$, and it is easy to check that $L(t,\varphi)\colon C \to \mathbb{R}^n$ is a bounded linear operator for all t for which $\dot{x}(t-\tau(t,x_t))$ exists (see [21]).

Another key element of the proof of differentiability in [21] (see also [24]) is the following lemma, which was stated and proved in [8] in a slightly different form.

Lemma 3.1 ([8]) *Let $g \in L^1([c,d],\mathbb{R}^n)$, $\varepsilon > 0$, $u \in \mathcal{A}(\varepsilon)$ and $u_k \in \mathcal{A}(\varepsilon)$ be a sequence such that $|u_k - u|_{C([a,b],\mathbb{R})} \to 0$ as $k \to \infty$, where*

$$\mathcal{A}(\varepsilon) := \{v \in W^{1,\infty}([a,b],[c,d]) \,:\, \dot{v}(s) \geq \varepsilon \text{ for a.e. } s \in [a,b]\}.$$

Then

$$\lim_{k\to\infty} \int_a^b \Big|g(u_k(s)) - g(u(s))\Big|\,ds = 0. \tag{3.4}$$

In [22] differentiability wrt the initial function was proved in the case when the strict monotonicity of the time lag function was relaxed to the weaker assumption that the time lag function is piecewise monotone. Recall the following definition.

Definition 3.2 ([22]) *$\mathcal{PM}([a,b],[c,d])$ denotes the set of absolutely continuous functions $u : [a,b] \to [c,d]$ which are piecewise strictly monotone on $[a,b]$ in the sense that there exists a finite mesh $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$ of $[a,b]$ such that for all $i = 0,1,\ldots,m-1$ either*

$$\operatorname{ess\,inf}\{\dot{u}(s)\colon\ s \in [a',b']\} > 0, \qquad \text{for all} \ \ [a',b'] \subset (t_i,t_{i+1})$$

or

$$\operatorname{ess\,sup}\{\dot{u}(s)\colon\ s \in [a',b']\} < 0, \qquad \text{for all} \ \ [a',b'] \subset (t_i,t_{i+1}).$$

The next result shows that $t \mapsto \dot{x}(t-\tau(t,x_t))$ is measurable and integrable on $[0,\alpha]$ if the time lag function $u(t) := t - \tau(t,x_t)$ is piecewise monotone in the sense of Definition 3.2.

Lemma 3.3 ([22]) *Suppose $g \in L^\infty([c,d],\mathbb{R}^n)$ and $u \in \mathcal{PM}([a,b],[c,d])$. Then the composite function $g \circ u \in L^\infty([a,b],\mathbb{R})$, and $|g\circ u|_{L^\infty([a,b],\mathbb{R})} \leq |g|_{L^\infty([c,d],\mathbb{R})}$.*

It was shown in [22] that Lemma 3.1 can be extended to the above class of piecewise monotone functions instead of the class $\mathcal{A}(\varepsilon)$ of strictly monotone increasing functions. Note that the price for it was that in (3.5) $W^{1,\infty}$-nom is used instead of the C-norm that was used in Lemma 3.1.

Lemma 3.4 ([22]) *Suppose $g \in L^\infty([c,d],\mathbb{R}^n)$ and $u, u_k \in \mathcal{PM}([a,b],[c,d])$ $(k \in \mathbb{N})$ are such that*

$$|u_k - u|_{W^{1,\infty}([a,b],\mathbb{R})} \to 0, \qquad \text{as } k \to \infty. \tag{3.5}$$

Then

$$\lim_{k\to\infty} \int_a^b |g(u_k(s)) - g(u(s))|\,ds = 0. \tag{3.6}$$

In [22] the following parameter set was introduced

$$P_2 := \{(\sigma,\varphi)\in P\colon \text{the map } [\sigma,\alpha^*_\sigma]\to\mathbb{R},\ t\mapsto t-\tau(t,x_t(\cdot,\sigma,\varphi)) \text{ belongs to } \mathcal{PM}([\sigma,\alpha^*_\sigma],[\sigma-r,\alpha^*_\sigma])\}. \tag{3.7}$$

We have $P_1 \subset P_2 \subset P$.

Let $(\sigma,\varphi)\in P_2$ be fixed, and let $x(t) := x(t,\sigma,\varphi)$. Then Lemma 3.4 yields that the linear operator defined by (3.1) is well-defined for a.e. $t\in[\sigma,\alpha]$, and it is easy to check that it is a bounded linear operator for all t for which $\dot x(t-\tau(t,x_t))$ exists.

For $(\sigma,\varphi)\in P_2$ we define the variational equation associated to $x = x(\cdot,\sigma,\varphi)$ as

$$\dot z(t) = L(t,x)z_t, \qquad \text{a.e. } t\in[\sigma,\alpha], \tag{3.8}$$

$$z(t) = h(t-\sigma), \qquad t\in[\sigma-r,\sigma], \tag{3.9}$$

where the initial function is $h\in C$. The IVP (3.8)-(3.9) is a Carathéodory type linear delay equation. By its solution we mean a continuous function $z:\ [\sigma-r,\alpha]$ which is absolutely continuous on $[\sigma,\alpha]$, and it satisfies (3.8) for a.e. $t\in[\sigma,\alpha]$ and (3.9) for all $t\in[\sigma-r,\sigma]$. Standard argument ([11], [18]) shows that the IVP (3.8)-(3.9) has a unique solution $z(t) = z(t,\sigma,\varphi,h)$ for $(\sigma,\varphi)\in P_2$, $h\in C$ and $t\in[\sigma-r,\alpha]$.

Now we state the result concerning differentiability of $x(t,\sigma,\varphi)$ wrt φ for the IVP (2.1)-(2.2). This derivative is denoted by $D_3x(t,\sigma,\varphi)$. Note that the existence of $D_3x(t,\sigma,\varphi)$ was proved in [22] for the case when the initial time was fixed to be $\sigma=0$. It is easy to extend the results to the case when the initial time is fixed to be $\sigma>0$ (see also the proof of the analogous Theorem 4.7 in [21] for the case when $(\sigma,\varphi)\in P_1$). We also remark that in [22] other parameters on the right-hand-side of (2.1) was considered, and differentiability was obtained wrt to these parameters too. We formulate the result in the form we need it later.

First we formulate an additional assumption on the delay function. We need a specific form of the partial derivative $D_2\tau$, and also Lipschitz continuity of the partial derivatives.

(A3) For every $(\sigma,\varphi)\in P_2$ there exist continuous functions $b_1,\dots,b_\ell:\ [\sigma,\alpha]\to\mathbb{R}^{1\times n}$, $b:\ [\sigma,\alpha]\times[-r,0]\to\mathbb{R}^{1\times n}$, and $\xi_1,\dots,\xi_\ell\in W^{1,\infty}([\sigma,\alpha],[0,r])$ such that

- (i) for $x = x(\cdot,\sigma,\varphi)$ and for all $\psi\in C$, $t\in[\sigma,\alpha]$

$$D_2\tau(t,x_t)\psi = \sum_{j=1}^{\ell} b_j(t)\psi(-\xi_j(t)) + \int_{-r}^{0} b(t,\theta)\psi(\theta)\,d\theta;$$

- (ii) the map $[\sigma,\alpha^*_\sigma]\colon\to\mathbb{R}$, $t\mapsto t-\xi_j(t)$ belongs to $\mathcal{PM}([\sigma,\alpha^*_\sigma],[\sigma-r,\alpha^*_\sigma]\}$ for $j=1,\dots,\ell$;
- (iii) for $\alpha>0$ and M_1 compact subset of C there exists $L_3 = L_3(\alpha,M_1)\ge 0$ such that

$$|D_1\tau(t,\psi)-D_1\tau(t,\tilde\psi)| \le L_3|\psi-\tilde\psi|_C, \qquad t\in[0,\alpha],\ \psi,\tilde\psi\in M_1$$

and

$$|D_2\tau(t,\psi)-D_2\tau(t,\tilde\psi)|_{\mathcal{L}(C,\mathbb{R})} \le L_3|\psi-\tilde\psi|_C, \qquad t\in[0,\alpha],\ \psi,\tilde\psi\in M_1.$$

Assumption (A3) can be naturally satisfied for delay functions of the form

$$\tau(t,\psi)=\bar\tau\Big(t,\psi(-\xi_1(t)),\ldots,\psi(-\xi_\ell(t)),\int_{-r}^0 \bar b(t,\theta)\psi(t+\theta)\,d\theta\Big)$$

where $\bar\tau:\ [\sigma,\alpha]\times\mathbb{R}^{\ell\times n}\times\mathbb{R}^n\to[0,r]$ and $\bar b:\ [\sigma,\alpha]\times[-r,0]\to\mathbb{R}^{n\times n}$ are twice continuously differentiable functions.

In order to apply Lemma 3.4 we need the following relation.

Lemma 3.5 *Suppose τ satisfies (A2) and (A3). Let $(\sigma,\varphi),(\sigma_k,\varphi_k)\in P_2$ for $k\in\mathbb{N}$ be such that $|\sigma_k-\sigma|+|\varphi_k-\varphi|_{W^{1,\infty}}\to 0$ as $k\to\infty$, let $x(t)=x(t,\sigma,\varphi)$ and $x^k(t)=x(t,\sigma_k,\varphi_k)$ be the solutions of the corresponding IVP (2.1)-(2.2) on $[\sigma-r,\alpha]$ and $[\sigma_k-r,\alpha]$, respectively, and let $u(t):=t-\tau(t,x_t)$, $u_k(t):=t-\tau(t,x^k_t)$ for $t\in[\sigma-r,\alpha]$ and $t\in[\sigma_k-r,\alpha]$, respectively. Then*

$$\lim_{k\to\infty}|u_k-u|_{W^{1,\infty}([\nu_k,\alpha],\mathbb{R}^n)}=0, \tag{3.10}$$

where $\nu_k:=\max\{\sigma,\sigma_k\}$.

Proof It follows from Theorem 2.1 (ii) and (A3) (i) that L_2 in (2.9) can be selected so that for $(\sigma,\varphi)\in P_2$

$$|D_2\tau(t,x_t)\psi|\le\Big(\sum_{j=1}^{\ell}|b_j(t)|+\int_{-r}^0|b(t,\theta)|\,d\theta\Big)|\psi|_C\le L_2|\psi|_C,\quad t\in[\sigma,\alpha],\ \psi\in C. \tag{3.11}$$

Therefore Theorem 2.1 (iii) and (2.9) yield

$$|u_k(t)-u(t)|=|\tau(t,x^k_t)-\tau(t,x_t)|\le L_2L(|\sigma_k-\sigma|+|\varphi_k-\varphi|_{W^{1,\infty}}),\quad t\in[\nu_k,\alpha],\ k\in\mathbb{N}. \tag{3.12}$$

Next we show that for $(\sigma,\varphi)\in P_2$ it follows

$$\frac{d}{dt}\tau(t,x_t)=D_1\tau(t,x_t)+\sum_{j=1}^{\ell}b_j(t)\dot x(t-\xi_j(t))+\int_{-r}^0 b(t,\theta)\dot x(t+\theta)\,d\theta,\quad \text{a.e. } t\in[\sigma,\alpha]. \tag{3.13}$$

Fix $t\in[\sigma,\alpha]$, and let h be small enough that $t+h\in[\sigma,\alpha]$. Then the assumed continuous differentiability of τ implies that there exists a function ω_τ such that

$$\tau(t+h,x_{t+h})-\tau(t,x_t)=D_1\tau(t,x_t)h+D_2\tau(t,x_t)(x_{t+h}-x_t)+\omega_\tau(t,x_t,t+h,x_{t+h}),$$

where

$$\frac{|\omega_\tau(t,\psi,\tilde t,\tilde\psi)|}{|\tilde t-t|+|\tilde\psi-\psi|_C}\to 0,\qquad \text{as } |\tilde t-t|+|\tilde\psi-\psi|_C\to 0.$$

Then (A3) (i) yields

$$\begin{aligned}\tau(t+h,x_{t+h})-\tau(t,x_t) &= D_1\tau(t,x_t)h+\sum_{j=1}^{\ell}b_j(t)(x(t+h-\xi_j(t))-x(t-\xi_j(t)))\\ &\quad+\int_{-r}^0 b(t,\theta)(x(t+h+\theta)-x(t+\theta))\,d\theta\\ &\quad+\omega_\tau(t,x_t,t+h,x_{t+h}).\end{aligned} \tag{3.14}$$

It follows from (2.4) that $|x_{t+h} - x_t|_C \le Nh$, hence (3.14) implies (3.13) for every $t \in [\sigma, \alpha]$ for which x is differentiable at $t - \xi_j(t)$ for all $j = 1, \dots, \ell$. Therefore, by (A3) (ii), (3.13) holds for a.e. $t \in [\sigma, \alpha]$.

Relation (3.13) can be shortly written as

$$\frac{d}{dt}\tau(t, x_t) = D_1\tau(t, x_t) + D_2\tau(t, x_t)\dot{x}_t, \quad t \in [0, \alpha], \tag{3.15}$$

where $D_2\tau(t, x_t)$ is understood here as the extension of the linear operator $D_2\tau(t, x_t)$ from the space C to the larger space L^∞. The extended linear operator is denoted by the same notation for simplicity, and it satisfies

$$|D_2\tau(t, x_t)|_{\mathcal{L}(L^\infty,\mathbb{R})} \le L_2, \qquad t \in [0, \alpha],\ (\sigma, \varphi) \in P_2. \tag{3.16}$$

Finally, (A3) (iii), (2.4), (2.5), (3.15) and (3.16) have the consequence for a.e. $t \in [\nu_k, \alpha]$ that

$$\begin{aligned}
\left|\frac{d}{dt}\tau(t, x_t^k) - \frac{d}{dt}\tau(t, x_t)\right| &\le |D_1\tau(t, x_t^k) - D_1\tau(t, x_t)| + |D_2\tau(t, x_t^k)(\dot{x}_t^k - \dot{x}_t)| \\
&\quad + |(D_2\tau(t, x_t^k) - D_2\tau(t, x_t))\dot{x}_t| \\
&\le L_3|x_t^k - x_t|_C + L_2|\dot{x}_t^k - \dot{x}_t|_{L^\infty} + L_3|x_t^k - x_t|_C|\dot{x}_t|_{L^\infty} \\
&\le (L_3L + L_2L + L_3LN)(|\sigma_k - \sigma| + |\varphi_k - \varphi|_{W^{1,\infty}}).
\end{aligned}$$

This relation and (3.12) complete the proof of the lemma. □

Similarly to (2.6) we introduce the following set of parameters:

$$H_2 := \{(t, \sigma, \varphi) \in \mathbb{R} \times \mathbb{R} \times W^{1,\infty} \colon (\sigma, \varphi) \in P_2,\ t \in [\sigma, \alpha]\}. \tag{3.17}$$

Now we can state our main theorem about continuous differentiability of the solutions wrt the initial function.

Theorem 3.6 *Assume f satisfies (A1), and τ satisfies (A2) and (A3). Then the functions*

$$\mathbb{R} \times \mathbb{R} \times W^{1,\infty} \supset H_2 \to \mathbb{R}^n, \qquad (t, \sigma, \varphi) \mapsto x(t, \sigma, \varphi)$$

and

$$\mathbb{R} \times \mathbb{R} \times W^{1,\infty} \supset H_2 \to C, \qquad (t, \sigma, \varphi) \mapsto x_t(\cdot, \sigma, \varphi)$$

are both continuously differentiable wrt φ, and

$$D_3x(t, \sigma, \varphi)h = z(t, \sigma, \varphi, h), \qquad (t, \sigma, \varphi) \in H_2,\ h \in W^{1,\infty}, \tag{3.18}$$

and

$$D_3x_t(\cdot, \sigma, \varphi)h = z_t(\cdot, \sigma, \varphi, h), \qquad (t, \sigma, \varphi) \in H_2,\ h \in W^{1,\infty}, \tag{3.19}$$

where $z(t, \sigma, \varphi, h)$ is the solution of the IVP (3.8)-(3.9) *for $t \in [0, \alpha]$, $(\sigma, \varphi) \in P_2$ and $h \in W^{1,\infty}$.*

Proof Let $x(t) = x(t, \sigma, \varphi)$ be the solution of the IVP (2.1)-(2.2) on $[\sigma, \alpha]$. Then the function $y(t) = x(t + \sigma)$ satisfies

$$\dot{y}(t) = f(t + \sigma, y_t, y(t - \tau(t + \sigma, y_t))), \qquad t \in [0, \alpha - \sigma]. \tag{3.20}$$

All assumptions of Theorem 4.9 in [22] for Eq. (3.20) follow from the assumptions (A1)–(A3) of this theorem (property (A2) (iii) of [22] is shown in the proof of Lemma 3.5). It is also easy to check that the variational equation associated to (3.20) can be transformed to (3.9) by a time shift. Therefore relations (3.18) and (3.19) and the continuity of the derivatives wrt t and φ follow using Theorem 4.9 of [22]. Only the continuity of $D_3x(t,\sigma,\varphi)$ and $D_3x_t(\cdot,\sigma,\varphi)$ wrt to σ is left to prove.

Note that the continuity of the maps

$$\mathbb{R}\times\mathbb{R}\times W^{1,\infty}\supset H_1\to\mathcal{L}(W^{1,\infty},\mathbb{R}^n),\quad (t,\sigma,\varphi)\mapsto z(t,\sigma,\varphi,\cdot)$$

and

$$\mathbb{R}\times\mathbb{R}\times W^{1,\infty}\supset H_1\to\mathcal{L}(W^{1,\infty},C),\quad (t,\sigma,\varphi)\mapsto z_t(\cdot,\sigma,\varphi,\cdot)$$

with

$$H_1:=\{(t,\sigma,\varphi)\in\mathbb{R}\times\mathbb{R}\times W^{1,\infty}\colon\ (\sigma,\varphi)\in P_1,\ t\in[\sigma,\alpha]\} \tag{3.21}$$

was proved in Lemma 4.6 of [21]. The proof is identical in the case when H_1 is replaced by H_2, except when reference to Lemma 3.1 was used in [21], we have to refer to Lemma 3.4 here. Note that Lemma 3.4 can be applied since Lemma 3.5 guarantees relation (3.10). □

Note that condition $(\sigma,\varphi)\in P_2$ is essential, since an example is shown in [22] where a solution of an SD-DDE is not differentiable wrt a parameter at which parameter value the corresponding time lag $t-\tau(t,x_t)$ function is constant.

4 Differentiability wrt the initial time

In this section we investigate the Fréchet-differentiability of $x(t,\sigma,\varphi)$ and $x_t(\cdot,\sigma,\varphi)$ with respect to σ. We denote these derivatives by $D_2x(t,\sigma,\varphi)$ and $D_2x_t(\cdot,\sigma,\varphi)$, respectively. We show that the results proved in [21] for the existence of these partial derivatives for $(t,\sigma,\varphi)\in H_1$ can be extended to the case when $(t,\sigma,\varphi)\in H_2$. For this result we assume that the partial derivatives of f wrt the second argument has a specific form.

(A4) For every $(\sigma,\varphi)\in P$ there exist continuous functions $A_1,\dots,A_m:\ [\sigma,\alpha]\to\mathbb{R}^{n\times n}$, $A:[\sigma,\alpha]\times[-r,0]\to\mathbb{R}^{n\times n}$, and $\lambda_1,\dots,\lambda_m\in W^{1,\infty}([\sigma,\alpha],[0,r])$ such that

(i) for $x=x(\cdot,\sigma,\varphi)$ and for all $\psi\in C$, $s\in[\sigma,\alpha]$

$$D_2f(s,x_s,x(s-\tau(s,x_s)))\psi=\sum_{i=1}^m A_i(s)\psi(-\lambda_i(s))+\int_{-r}^0 A(s,\theta)\psi(\theta)\,d\theta;$$

and

(ii) the functions $[\sigma,\alpha^*_\sigma]\to\mathbb{R},\ t\mapsto t-\lambda_i(t)$ belong to $\mathcal{PM}([\sigma,\alpha^*_\sigma],[\sigma-r,\alpha^*_\sigma]\}$ for $i=1,\dots,m$.

Our additional assumptions can be naturally satisfied for equations of the form

$$\dot x(t)=\bar f\Big(t,x(t-\lambda_1(t)),\dots,x(t-\lambda_m(t)),\int_{-r}^0 A(t,\theta)x(t+\theta)\,d\theta,x(t-\tau(t,x_t))\Big).$$

We remark that in [21] instead of the piecewise monotonicity property of the functions $t \mapsto t - \lambda_i(t)$ and $t \mapsto t - \xi_j(t)$ required in (A3) (ii) and (A4) (ii), the strict monotonicity of the respective functions was assumed.

In virtue of (A3) and (A4), the operator $L(t, x)$ defined by (3.1) has the form

$$\begin{aligned} L(t,x)\psi &= \sum_{i=1}^{m} A_i(t)\psi(-\lambda_i(t)) + \int_{-r}^{0} A(t,\theta)\psi(\theta)\,d\theta + D_3 f(t, x_t, x(t-\tau(t,x_t))) \\ &\quad \times \Big(-\dot{x}(t-\tau(t,x_t))\Big(\sum_{j=1}^{\ell} b_j(t)\psi(-\xi_j(t)) + \int_{-r}^{0} b(t,\theta)\psi(\theta)\,d\theta\Big) + \psi(-\tau(t,x_t))\Big). \end{aligned} \tag{4.1}$$

Our assumptions (A1)–(A4), $(\sigma, \varphi) \in P_2$ and Lemma 3.3 yield that $L(t, x)z_t$ is well-defined for a.e. $t \in [\sigma, \alpha]$ for a function $z : [\sigma - r, \alpha] \to \mathbb{R}^n$, where z restricted to $[\sigma - r, \sigma]$ is in $L^\infty([\sigma - r, \sigma], \mathbb{R}^n)$, and z is continuous on $[\sigma, \alpha]$.

It follows from Theorem 2.1 (ii) and (A4) (i) that the constant L_1 defined by (2.8) can be selected so that

$$\begin{aligned} |D_2 f(t, x_t, x(t-\tau(t,x_t)))|_{\mathcal{L}(C,\mathbb{R}^n)} &\le \sum_{i=1}^{m} |A_i(t)| + \int_{-r}^{0} |A(t,\theta)|\,d\theta \\ &\le L_1, \ t \in [\sigma,\alpha], \ (\sigma,\varphi) \in P_2. \end{aligned} \tag{4.2}$$

Then (2.4), (3.13), (4.1), (2.8) and (4.2) yield

$$|L(t,x)z_t| \le K|z_t|_{L^\infty}, \qquad t \in [\sigma,\alpha], \ (\sigma,\varphi) \in P_2, \tag{4.3}$$

where

$$K := L_1(2 + L_2 N). \tag{4.4}$$

We extend the IVP (3.8)-(3.9) to this case by considering

$$\dot{z}(t) = L(t,x)z_t, \qquad \text{a.e. } t \in [\sigma,\alpha] \tag{4.5}$$

$$z(\sigma) = v, \tag{4.6}$$

$$z(t) = h(t-\sigma), \qquad \text{a.e. } t \in [\sigma - r,\sigma), \tag{4.7}$$

where $v \in \mathbb{R}^n$ and $h \in L^\infty$. By a solution of (4.5)-(4.7) we mean a function $z\colon [\sigma - r, \alpha] \to \mathbb{R}^n$ which is absolutely continuous on $[\sigma, \alpha]$ and satisfies (4.5)-(4.7). It is easy to show that (4.5)-(4.7) has a unique solution $z(t) = z(t, \sigma, \varphi, v, h)$ on $[\sigma - r, \alpha]$ for all $(v, h) \in \mathbb{R}^n \times L^\infty$ and $(\sigma, \varphi) \in P_2$. On $\mathbb{R}^n \times L^\infty$ we use the norm $|(v, h)|_{\mathbb{R}^n \times L^\infty} := |v| + |h|_{L^\infty}$.

Next we state the generalization of Theorem 5.3 of [21] using the parameter set P_2 instead of P_1. First we need to reformulate Lemma 5.2 from [21] for the parameter set P_2.

Lemma 4.1 *Assume f satisfies (A1), (A4), and τ satisfies (A2), (A3). Let P_2 and H_2 be the sets defined by* (3.7) *and* (3.17)*, respectively. Then there exists $N_0 \ge 1$ such that for all $(\sigma, \varphi) \in P_2$ and $(v, h) \in \mathbb{R}^n \times L^\infty$ the corresponding solution $z(t, \sigma, \varphi, v, h)$ of the IVP* (4.5)-(4.7) *satisfies*

$$|z(t,\sigma,\varphi,v,h)| \le N_0(|v| + |h|_{L^\infty}), \quad t \in [\sigma,\alpha] \quad \text{and a.e. } t \in [\sigma - r,\sigma]. \tag{4.8}$$

Moreover, the function

$$\mathbb{R} \times \mathbb{R} \times W^{1,\infty} \supset H_2 \to \mathbb{R}^n, \qquad (t, \sigma, \varphi) \mapsto z(t, \sigma, \varphi, v, h)$$

is continuous for all fixed $(v, h) \in \mathbb{R}^n \times L^\infty$.

Proof The proof of (4.8) is identical to that of the same estimate in [21], so it is omitted here.

In [21] the proof of the continuity of $z(t, \sigma, \varphi, v, h)$ in (t, σ, φ) relied directly on the strict monotonicity of the time lag function and the functions $t \mapsto t - \lambda_i(t)$ and $t \mapsto t - \xi_j(t)$. Since here we assume only piecewise monotonicity of these functions, we give the sketch of the proof, emphasizing only the differences in the argument. Note that the part of the proof we present here is simpler than that given in [21], even though here we assume the weaker condition of piecewise monotonicity instead of the strict monotonicity of the time lag function.

To show the continuity wrt σ and φ, fix $(\sigma, \varphi) \in P_2$ and let $(\sigma_k, \varphi_k) \in P_2$ be a sequence such that $|\sigma_k - \sigma| + |\varphi_k - \varphi|_{W^{1,\infty}} \to 0$ as $k \to \infty$. Let $x^k(t) := x(t, \sigma_k, \varphi_k)$, $x(t) := x(t, \sigma, \varphi)$, $u_k(s) := s - \tau(s, x^k_s)$, $u(s) := s - \tau(s, x_s)$, and for a fixed $(v, h) \in \mathbb{R}^n \times L^\infty$, let $z^k(t) := z(t, \sigma_k, \varphi_k, v, h)$ and $z(t) := z(t, \sigma, \varphi, v, h)$. Then

$$z^k(t) = v + \int_{\sigma_k}^t L(s, x^k) z^k_s \, ds, \qquad t \in [\sigma_k, \alpha],$$

and

$$z(t) = v + \int_{\sigma}^t L(s, x) z_s \, ds, \qquad t \in [\sigma, \alpha].$$

From the above relations it was shown in [21] that

$$|z^k(t) - z(t)| \leq B_k(|v| + |h|_{L^\infty}) + C^h_k + \Big| \int_{\nu_k}^t L(s, x)(z^k_s - z_s) \, ds \Big|, \qquad t \in [\nu_k, \alpha], \tag{4.9}$$

where $\nu_k := \max\{\sigma, \sigma_k\}$, and the sequence $B_k \to 0$ as $k \to \infty$. The constant C^h_k was defined in [21] as

$$C^h_k := L_1 \int_{\nu_k}^\alpha |\bar{h}(u_k(s)) - \bar{h}(u(s))| \, ds,$$

where

$$\bar{h}(s) := \begin{cases} h(s - \sigma), & s \in [\sigma - r, \sigma), \\ 0, & s \in [\sigma, \alpha]. \end{cases}$$

Let $\varepsilon > 0$ be fixed. Then for large enough k we have $\sigma \leq \nu_k < \sigma + \varepsilon$, so for such k Lemma 3.3 implies

$$\begin{aligned} C^h_k &= L_1 \Big(\int_{\nu_k}^{\sigma+\varepsilon} |\bar{h}(u_k(s)) - \bar{h}(u(s))| \, ds + \int_{\sigma+\varepsilon}^\alpha |\bar{h}(u_k(s)) - \bar{h}(u(s))| \, ds \Big) \\ &\leq L_1 \Big(2\varepsilon |h|_{L^\infty} + \int_{\sigma+\varepsilon}^\alpha |\bar{h}(u_k(s)) - \bar{h}(u(s))| \, ds \Big). \end{aligned}$$

Therefore Lemma 3.4 yields that $C^h_k \to 0$ as $k \to \infty$, since ε can be arbitrarily close to 0. Note that B_k contains the term $\int_{\nu_k}^\alpha |\dot{x}(u_k(s)) - \dot{x}(u(s))| \, ds$. Its convergence to 0 follows from an argument similar to the convergence of C^h_k shown above.

Now we consider the last term of (4.9). We have from (2.4), (4.1) and (2.8) that

$$\begin{aligned}
&\Big|\int_{\nu_k}^t L(s,x)(z_s^k - z_s)\,ds\Big| \\
&\quad\le \sum_{i=1}^m \int_{\nu_k}^t |A_i(s)||z^k(s-\lambda_i(s)) - z(s-\lambda_i(s))|\,ds \\
&\qquad + \int_{\nu_k}^t \int_{-r}^0 |A(s,\theta)||z^k(s+\theta) - z(s+\theta)|\,d\theta\,ds + L_1 \int_{\nu_k}^t |z^k(u(s)) - z(u(s))|\,ds \\
&\qquad + L_1 N \sum_{j=1}^{\ell} \int_{\nu_k}^t |b_j(s)||z^k(s-\xi_j(s)) - z(s-\xi_j(s))|\,ds \\
&\qquad + L_1 N \int_{\nu_k}^t \int_{-r}^0 |b(s,\theta)||z^k(s+\theta) - z(s+\theta)|\,d\theta\,ds, \qquad t\in[\nu_k,\alpha]. \qquad (4.10)
\end{aligned}$$

We introduce the sets

$$\begin{aligned}
U_{k,i}^1 &:= \{s\in[\nu_k,\alpha]\colon\ s-\lambda_i(s) < \sigma - |\sigma_k-\sigma|\}, \quad i=1,\ldots,m,\\
U_{k,i}^2 &:= \{s\in[\nu_k,\alpha]\colon\ \sigma - |\sigma_k-\sigma| \le s-\lambda_i(s) \le \sigma + |\sigma_k-\sigma|\}, \quad i=1,\ldots,m,\\
U_{k,i}^3(t) &:= \{s\in[\nu_k,t]\colon\ \sigma + |\sigma_k-\sigma| < s-\lambda_i(s) \le t\}, \quad i=1,\ldots,m,\\
V_{k,j}^1 &:= \{s\in[\nu_k,\alpha]\colon\ s-\xi_j(s) < \sigma - |\sigma_k-\sigma|\}, \quad j=1,\ldots,\ell,\\
V_{k,j}^2 &:= \{s\in[\nu_k,\alpha]\colon\ \sigma - |\sigma_k-\sigma| \le s-\xi_j(s) \le \sigma + |\sigma_k-\sigma|\}, \quad j=1,\ldots,\ell,\\
V_{k,j}^3(t) &:= \{s\in[\nu_k,t]\colon\ \sigma + |\sigma_k-\sigma| < s-\xi_j(s)\}, \quad j=1,\ldots,\ell,\\
W_k^1 &:= \{s\in[\nu_k,\alpha]\colon\ s-\tau(s,x_s) < \sigma - |\sigma_k-\sigma|\},\\
W_k^2 &:= \{s\in[\nu_k,\alpha]\colon\ \sigma - |\sigma_k-\sigma| \le s-\tau(s,x_s) \le \sigma + |\sigma_k-\sigma|\},\\
W_k^3(t) &:= \{s\in[\nu_k,t]\colon\ \sigma + |\sigma_k-\sigma| < s-\tau(s,x_s)\}.
\end{aligned}$$

Then if $s\in U_{k,i}^1$, then $s-\lambda_i(s)-\sigma_k \le s-\lambda_i(s)-\sigma+|\sigma_k-\sigma| \le 0$. Similarly, if $s\in U_{k,i}^3(t)$, then $s-\lambda_i(s) > \sigma+|\sigma_k-\sigma| \ge \nu_k$. Note that the Lebesgue measure of $U_{k,i}^2$ goes to 0 as $k\to\infty$ because of the assumed piecewise monotonicity of the function $t\mapsto t-\lambda_i(t)$.

Define $w_k(t) := \max\{|z^k(s)-z(s)|\colon\ s\in[\nu_k,t]\}$. Clearly, w_k is monotone increasing. Then the first integral on the right-hand-side of (4.10) can be estimated using (4.8) in the following way

$$
\begin{aligned}
&\int_{\nu_k}^{t} |A_i(s)||z^k(s-\lambda_i(s)) - z(s-\lambda_i(s))|\,ds \\
&\qquad \le \int_{U^1_{k,i}} |A_i(s)||h(s-\lambda_i(s)-\sigma_k) - h(s-\lambda_i(s)-\sigma)|\,ds \\
&\qquad\quad + \int_{U^2_{k,i}} |A_i(s)|(|z^k(s-\lambda_i(s))| + |z(s-\lambda_i(s))|)\,ds \\
&\qquad\quad + \int_{U^3_{k,i}(t)} |A_i(s)| w_k(s-\lambda_i(s))\,ds \\
&\qquad \le \int_{U^1_{k,i}} |A_i(s)||h(s-\lambda_i(s)-\sigma_k) - h(s-\lambda_i(s)-\sigma)|\,ds \\
&\qquad\quad +2N_0(|v| + |h|_{L^\infty}) \int_{U^2_{k,i}} |A_i(s)|\,ds + \int_{\nu_k}^{t} |A_i(s)| w_k(s)\,ds, \quad t \in [\nu_k, \alpha]. \qquad (4.11)
\end{aligned}
$$

We estimate the second integral of (4.10) for $t \ge \nu_k + r$ in the same manner.

$$
\begin{aligned}
&\int_{\nu_k}^{t}\int_{-r}^{0} |A(s,\theta)||z^k(s+\theta) - z(s+\theta)|\,d\theta\,ds \\
&\qquad \le \int_{\nu_k}^{\nu_k+r}\int_{-r}^{\min\{\sigma,\sigma_k\}-s} |A(s,\theta)||h(s+\theta-\sigma_k) - h(s+\theta-\sigma)|\,d\theta\,ds \\
&\qquad\quad + \int_{\nu_k}^{\nu_k+r}\int_{\min\{\sigma,\sigma_k\}-s}^{\nu_k-s} |A(s,\theta)|(|z^k(s+\theta)| + |z(s+\theta)|)\,d\theta\,ds \\
&\qquad\quad + \int_{\nu_k}^{\nu_k+r}\int_{\nu_k-s}^{0} |A(s,\theta)| w_k(s+\theta)\,d\theta\,ds + \int_{\nu_k+r}^{t}\int_{-r}^{0} |A(s,\theta)| w_k(s+\theta)\,d\theta\,ds \\
&\qquad \le \int_{\nu_k}^{\nu_k+r}\int_{-r}^{\min\{\sigma,\sigma_k\}-s} |A(s,\theta)||h(s+\theta-\sigma_k) - h(s+\theta-\sigma)|\,d\theta\,ds \\
&\qquad\quad +2N_0(|v| + |h|_{L^\infty}) \int_{\nu_k}^{\nu_k+r}\int_{\min\{\sigma,\sigma_k\}-s}^{\nu_k-s} |A(s,\theta)|\,d\theta\,ds \\
&\qquad\quad + \int_{\nu_k}^{t} w_k(s) \int_{-r}^{0} |A(s,\theta)|\,d\theta\,ds. \qquad (4.12)
\end{aligned}
$$

Note that the final estimate is true for all $t \in [\nu_k, \alpha]$.

Similar estimates of the other terms of (4.10) together with (3.11), (4.2) and (4.4) give

$$
\Big|\int_{\nu_k}^{t} L(s,x)(z^k_s - z_s)\,ds\Big| \le D^h_k + K\int_{\nu_k}^{t} w_k(s)\,ds, \qquad t \in [\nu_k, \alpha], \qquad (4.13)
$$

where K is defined in (4.4), and

$$\begin{aligned}
D_k^h \;:=\; & \sum_{i=1}^{m}\int_{U_{k,i}^1}|A_i(s)||h(s-\lambda_i(s)-\sigma_k)-h(s-\lambda_i(s)-\sigma)|\,ds \\
& +2N_0(|v|+|h|_{L^\infty})\sum_{i=1}^{m}\int_{U_{k,i}^2}|A_i(s)|\,ds \\
& +\int_{\nu_k}^{\nu_k+r}\int_{-r}^{\min\{\sigma,\sigma_k\}-s}|A(s,\theta)||h(s+\theta-\sigma_k)-h(s+\theta-\sigma)|\,d\theta\,ds \\
& +2N_0(|v|+|h|_{L^\infty})\int_{\nu_k}^{\nu_k+r}\int_{\min\{\sigma,\sigma_k\}-s}^{\nu_k-s}|A(s,\theta)|\,d\theta\,ds \\
& +\int_{W_k^1}|h(s-\lambda_i(s)-\sigma_k)-h(s-\lambda_i(s)-\sigma)|\,ds+\int_{W_k^2}2N_0(|v|+|h|_{L^\infty})\,ds \\
& +L_1N\sum_{j=1}^{\ell}\int_{V_{k,j}^1}|b_j(s)||h(s-\xi_j(s)-\sigma_k)-h(s-\xi_j(s)-\sigma)|\,ds \\
& +L_1N2N_0(|v|+|h|_{L^\infty})\sum_{j=1}^{\ell}\int_{V_{k,j}^2}|b_j(s)|\,ds \\
& +L_1N\int_{\nu_k}^{\nu_k+r}\int_{-r}^{\min\{\sigma,\sigma_k\}-s}|b(s,\theta)||h(s+\theta-\sigma_k)-h(s+\theta-\sigma)|\,d\theta\,ds \\
& +L_1N2N_0(|v|+|h|_{L^\infty})\int_{\nu_k}^{\nu_k+r}\int_{\min\{\sigma,\sigma_k\}-s}^{\nu_k-s}|b(s,\theta)|\,d\theta\,ds.
\end{aligned}$$

The Dominated Convergence Theorem and simple arguments yield that $D_k^h\to 0$ as $k\to\infty$ for each fixed $h\in L^\infty$.

Combining (4.9) and (4.13) we get

$$w_k(t)\le B_k(|v|+|h|_{L^\infty})+C_k^h+D_k^h+K\int_{\nu_k}^{t}w_k(s)\,ds,\qquad t\in[\nu_k,\alpha],$$

therefore the Gronwall's inequality implies

$$|z^k(t)-z(t)|\le w_k(t)\le\Big(B_k(|v|+|h|_{L^\infty})+C_k^h+D_k^h\Big)e^{K\alpha},\qquad t\in[\nu_k,\alpha].$$

This proves the continuity of z wrt σ and φ.

The continuity of $z(t)=z(t,\sigma,\varphi,\cdot,\cdot)$ in t follows from (4.3) and (4.8), since for $\tilde t,t\in[\sigma,\alpha]$ and $(\sigma,\varphi)\in P_2$

$$|z(\tilde t)-z(t)|=\left|\int_t^{\tilde t}L(s,x)z_s\,ds\right|\le KN_0(|v|+|h|_{L^\infty})|\tilde t-t|.$$

This completes the proof. □

For $(t,\sigma,\varphi)\in H_2$ we define the bounded linear operator

$$\mathcal{T}(t,\sigma,\varphi)\colon\ \mathbb{R}^n\times L^\infty\to\mathbb{R}^n,\qquad \mathcal{T}(t,\sigma,\varphi)(v,h):=z(t,\sigma,\varphi,v,h),\tag{4.14}$$

where $z(t,\sigma,\varphi,v,h)$ is the solution of the IVP (4.5)-(4.7).

Theorem 4.2 *Assume f satisfies (A1), (A4), and τ satisfies (A2), (A3). Then the function*

$$\mathbb{R}^2\times W^{1,\infty}\supset H_2\to\mathbb{R}^n,\qquad (t,\sigma,\varphi)\mapsto x(t,\sigma,\varphi)$$

is continuously differentiable wrt σ, and

$$D_2x(t,\sigma,\varphi)=\mathcal{T}(t,\sigma,\varphi)(-f(\sigma,\varphi,\varphi(-\tau(\sigma,\varphi))),-\dot\varphi),\tag{4.15}$$

where $\mathcal{T}(t,\sigma,\varphi)$ is defined by (4.14).

Proof Let $(\sigma,\varphi)\in P_2$ and $t\in[\sigma,\alpha]$. If $t\in(\sigma,\alpha]$, then let $h_k\in\mathbb{R}$ $(k\in\mathbb{N})$ be a sequence such that $h_k\to 0$ as $k\to\infty$, $(\sigma+h_k,\varphi)\in P_2$ and $\sigma+h_k<t$ for $k\in\mathbb{N}$. If $t=\sigma$, then let $h_k<0$ such that $(\sigma+h_k,\varphi)\in P_2$ for $k\in\mathbb{N}$. To simplify notation, let $x^k(t):=x(t,\sigma+h_k,\varphi)$, $x(t):=x(t,\sigma,\varphi)$, $u(s):=s-\tau(s,x_s)$, $u_k(s):=s-\tau(s,x^k_s)$, $v:=-f(\sigma,\varphi,\varphi(-\tau(\sigma,\varphi)))$, $z(t):=\mathcal{T}(t,\sigma,\varphi)(v,-\dot\varphi)$, $\alpha^*_\sigma:=\min\{\sigma+r,\alpha\}$ and $\nu_k:=\max\{\sigma,\sigma+h_k\}$.

Simple calculation shows for $t\in[\nu_k,\alpha]$

$$\begin{aligned}&x^k(t)-x(t)-z(t)h_k\\&\quad= x^k(\nu_k)-x(\nu_k)-z(\nu_k)h_k\\&\qquad+\int_{\nu_k}^t\Big(f(s,x^k_s,x^k(u_k(s)))-f(s,x_s,x(u(s)))-L(s,x)z_sh_k\Big)ds.\end{aligned}$$

Let $q_k(s):=x^k(s)-x(s)-z(s)h_k$, $s\in[\nu_k-r,\alpha]$. Then it was shown in [21] that the above equation leads to

$$\begin{aligned}|q_k(t)|\ &\le\ a^*_k+d_k|h_k|+|q_k(\nu_k)|+\int_{\nu_k}^t\Big(\sum_{i=0}^m|A_i(s)||q_k(s-\lambda_i(s))|\\&\quad+\int_{-r}^0|A(s,\theta)||q_k(s+\theta)|\,d\theta+L_1|q_k(u_k(s))|+L_1N\sum_{j=0}^{\ell}|b_j(s)||q_k(s-\xi_j(s))|\\&\quad+L_1N\int_{-r}^0|b(s,\theta)||q_k(s+\theta)|\,d\theta\Big)ds,\qquad t\in[\nu_k,\alpha],\end{aligned}\tag{4.16}$$

where the sequence a^*_k satisfies $a^*_k/|h_k|\to 0$ as $k\to\infty$ (see [21] for details), and

$$d_k:=\int_{\nu_k}^{\alpha}|z(u_k(s))-z(u(s))|\,ds.$$

Lemma 3.3 and 3.4 yield that $d_k\to 0$ as $k\to\infty$ (see the proof of $C^h_k\to 0$ in Lemma 4.1).

Note that (2.5) and (4.8) yield

$$|q_k(s)|\le|x^k(s)-x(s)|+|z^k(s)||h_k|\le N_1|h_k|,\quad s\in[\sigma-r,\alpha],\tag{4.17}$$

where $N_1 := L + N_0(|v| + |\dot\varphi|_{L^\infty})$.

We define the sets $U^1_{k,i}$, $U^2_{k,i}$, $V^1_{k,j}$, $V^2_{k,j}$ as in the proof of Lemma 4.1, and the sets

$$\begin{aligned}
\widetilde W^1_k &:= \{s \in [\nu_k, \alpha]\colon\ u_k(s) < \sigma - |\sigma_k - \sigma|\},\\
\widetilde W^2_k &:= \{s \in [\nu_k, \alpha]\colon\ \sigma - |\sigma_k - \sigma| \le u_k(s) \le \sigma + |\sigma_k - \sigma|\},
\end{aligned}$$

and define $w_k(t) := \max\{|q_k(s)| :\ s \in [\sigma, t]\}$, $\nu_k := \max\{\sigma, \sigma + h_k\}$. For a.e. $s \in [\nu_k - r, \min\{\sigma + h_k, \sigma\}]$ introduce $\eta_k(s) := \varphi(s - \sigma - h_k) - \varphi(s - \sigma) + \dot\varphi(s - \sigma)h_k$. Then similarly to the estimates used in the proof of Lemma 4.1, we get from (4.16)

$$w_k(t) \le A_k + K\int_{\nu_k}^t w_k(s)\,ds, \qquad t \in [\nu_k, \alpha], \tag{4.18}$$

where w_k is defined in the proof of Lemma 4.1, K is defined in (4.4), and

$$\begin{aligned}
A_k \;:=\;& a^*_k + d_k|h_k| + |q_k(\nu_k)| + \sum_{i=1}^m \int_{U^1_{k,i}} |A_i(s)||\eta_k(s - \lambda_i(s))|\,ds\\
&+ \int_{\nu_k}^{\nu_k + r}\int_{-r}^{\min\{\sigma,\sigma_k\} - s} |A(s,\theta)||\eta_k(s + \theta)|\,d\theta\,ds\\
&+ N_1|h_k|\int_{\nu_k}^{\nu_k + r}\int_{\min\{\sigma,\sigma_k\} - s}^{\nu_k - s} |A(s,\theta)|\,d\theta\,ds + L_1\int_{\widetilde W^1_k} |\eta_k(u_k(s))|\,ds\\
&+ \int_{\widetilde W^2_k} L_1N_1|h_k|\,ds + L_1N\sum_{j=1}^{\ell}\int_{V^1_{k,j}} |b_j(s)||\eta_k(s - \xi_j(s))|\,ds\\
&+ L_1NN_1|h_k|\sum_{j=1}^{\ell}\int_{V^2_{k,j}} |b_j(s)|\,ds\\
&+ \int_{\nu_k}^{\nu_k + r}\int_{-r}^{\min\{\sigma,\sigma_k\} - s} |b(s,\theta)||\eta_k(s + \theta)|\,d\theta\,ds\\
&+ N_1|h_k|\int_{\nu_k}^{\nu_k + r}\int_{\min\{\sigma,\sigma_k\} - s}^{\nu_k - s} |b(s,\theta)|\,d\theta\,ds.
\end{aligned}$$

Hence Gronwall's inequality yields

$$|x^k(t) - x(t) - z(t)h_k| \le w_k(t) \le A_k e^{K\alpha}, \qquad t \in [\nu_k, \alpha]. \tag{4.19}$$

To prove (4.15) it is enough to show that $\frac{A_k}{|h_k|} \to 0$ as $k \to \infty$.

Suppose first that k is such that $h_k < 0$. Then $\nu_k = \sigma$, and we have

$$\begin{aligned}
\frac{|q_k(\nu_k)|}{|h_k|} &= \frac{1}{|h_k|}|x^k(\sigma) - x(\sigma) - z(\sigma)h_k|\\
&= \frac{1}{|h_k|}\Big|\varphi(0) + \int_{\sigma + h_k}^{\sigma} f(s, x^k_s, x^k(u_k(s)))\,ds - \varphi(0) - vh_k\Big|\\
&= \frac{1}{|h_k|}\Big|\int_{\sigma + h_k}^{\sigma}\Big(f(s, x^k_s, x^k(u_k(s))) - f(\sigma, \varphi, \varphi(u(\sigma)))\Big)\,ds\Big|.
\end{aligned} \tag{4.20}$$

If $h_k > 0$, then $\nu_k = \sigma + h_k$, and

$$\begin{aligned}
\frac{|q_k(\nu_k)|}{|h_k|} &= \frac{1}{h_k}\Big|x^k(\sigma+h_k) - x(\sigma+h_k) - z(\sigma+h_k)h_k\Big| \\
&= \frac{1}{h_k}\Big|\varphi(0) - \varphi(0) - \int_\sigma^{\sigma+h_k} f(s, x_s, x(u(s)))\,ds - vh_k \\
&\qquad - \int_\sigma^{\sigma+h_k} L(s,x)z_s h_k\,ds\Big| \\
&\leq \frac{1}{h_k}\Big|\int_\sigma^{\sigma+h_k}\Big(f(s,x_s,x(u(s))) - f(\sigma,\varphi,\varphi(u(\sigma)))\Big)\,ds\Big| \\
&\quad + h_k K N_0(|v| + |\dot{\varphi}|_{L^\infty}).
\end{aligned} \tag{4.21}$$

Combining (3.10), (4.20) and (4.21) we get

$$\frac{|q_k(\nu_k)|}{|h_k|} \to 0, \qquad \text{as } k \to \infty.$$

Since $\eta_k(s)/h_k \to 0$ for a.e. $s \in [\nu_k - r, \min\{\sigma + h_k, \sigma\}]$, the Dominated Convergence Theorem implies that $A_k/h_k \to 0$ as $k \to \infty$, which concludes the proof of the existence of the derivative $D_2x(t,\sigma,\varphi)$.

The continuity of $D_2x(t,\sigma,\varphi)$ can be argued in the same way as it was shown in [21], but using Lemma 4.1 instead of Lemma 3.1. □

We remark that in Theorem 4.2 the differentiability of $x(t,\sigma,\varphi)$ wrt σ at $t = \sigma$ is considered only as a one-sided derivative. If differentiability of the solution wrt σ is needed for $t \in [\sigma - r, \alpha]$, or the differentiability of the solution segments $x_t(\cdot,\sigma,\varphi)$ is needed, then the compatibility condition $(\sigma,\varphi) \in \mathcal{P}$ is also required. See Theorem 5.1 and Remark 5.4 in [21].

5 Nonlinear variation of constants formula

Consider the SD-DDE

$$\dot{y}(t) = f(t, y_t, y(t - \tau(t, y_t))) + g(t, y_t, y(t - \lambda(t, y_t))), \qquad t \in [\sigma, \alpha] \tag{5.1}$$

and the associated initial condition

$$y(t) = \varphi(t - \sigma), \qquad t \in [\sigma - r, \sigma]. \tag{5.2}$$

We suppose

(A5) $g\colon \mathbb{R} \times C \times \mathbb{R}^n \supset [0,T] \times \Omega_1 \times \Omega_2 \to \mathbb{R}^n$ is continuous, and it is continuously differentiable wrt its second and third arguments;

(A6) $\lambda\colon \mathbb{R} \times C \supset [0,T] \times \Omega_1 \to [0,r]$ is continuously differentiable wrt both arguments.

Suppose $(\hat{\sigma},\hat{\varphi}) \in \Pi$ is such that $\hat{\varphi}(-\lambda(\hat{\sigma},\hat{\varphi})) \in \Omega_2$. It is easy to generalize Theorem 2.1 to the IVP (5.1)-(5.2), and show that there exist σ_0, α and δ such that both the IVP (2.1)-(2.2) and the IVP (5.1)-(5.2) have unique solutions on $[\sigma - r, \alpha]$ for $(\sigma,\varphi) \in P := [\sigma_0, \alpha) \times \mathcal{B}_{W^{1,\infty}}(\hat{\varphi};\, \delta)$.

Moreover, it can also be assumed that (2.4) and (2.5) hold for the solutions of the IVP (5.1)-(5.2) for all $(\sigma, \varphi) \in P_2$.

We define the sets P_2 and H_2 by (3.7) and (3.17), respectively.

First we show the differentiability of the following composite map.

Lemma 5.1 *Assume f satisfies (A1), (A4), τ satisfies (A2), (A3), g satisfies (A5), and λ satisfies (A6). Let $(\sigma, \varphi) \in P$, and let $y(t) = y(t, \sigma, \varphi)$ be the solution of the IVP* (5.1)-(5.2). *Suppose $(s, y_s) \in P_2$ for a.e. $s \in [\sigma, \alpha]$. Then the function $(\sigma, \alpha) \to \mathbb{R}^n$, $s \mapsto x(t, \nu, y_s)$ is differentiable at $s = \nu$ for a.e. $\nu \in (\sigma, \alpha)$ and for all $t \in [\sigma, \alpha]$, and*

$$\frac{d}{ds} x(t, \nu, y_s)|_{s=\nu} = \mathcal{T}(t, \nu, y_\nu)(\dot{y}(\nu), \dot{y}_\nu), \qquad a.e.\ \nu \in (\sigma, \alpha),$$

where $\mathcal{T}$ is defined by (4.14).

Proof Let $\nu \in (\sigma, \alpha)$ be fixed such that $(\nu, y_\nu) \in P_2$, and let h_k $(k \in \mathbb{N})$ be a sequence of non-zero reals with $h_k \to 0$ as $k \to \infty$, and suppose $|h_k| < \min\{\nu - \sigma, \alpha - \nu\}$ for all $k \in \mathbb{N}$. Then Theorem 3.6 yields that there exists a function ω such that

$$x(t, \nu, y_{\nu+h_k}) - x(t, \nu, y_\nu) = D_3 x(t, \nu, y_\nu)(y_{\nu+h_k} - y_\nu) + \omega(t, \nu, y_\nu, y_{\nu+h_k}), \quad t \in [\nu, \alpha], \tag{5.3}$$

where

$$\frac{|\omega(t, \nu, \psi, \tilde{\psi})|}{|\tilde{\psi} - \psi|_{W^{1,\infty}}} \to 0, \qquad \text{as } |\tilde{\psi} - \psi|_{W^{1,\infty}} \to 0.$$

We have from (2.4) that if $|y_{\nu+h_k} - y_\nu|_{W^{1,\infty}} \neq 0$, then

$$\begin{aligned} \frac{|\omega(t, \nu, y_\nu, y_{\nu+h_k})|}{|h_k|} &\leq \frac{|\omega(t, \nu, y_\nu, y_{\nu+h_k})|}{|y_{\nu+h_k} - y_\nu|_{W^{1,\infty}}} \frac{|y_{\nu+h_k} - y_\nu|_{W^{1,\infty}}}{|h_k|} \\ &\leq \frac{|\omega(t, \nu, y_\nu, y_{\nu+h_k})|}{|y_{\nu+h_k} - y_\nu|_{W^{1,\infty}}} N \\ &\to 0 \end{aligned}$$

as $k \to \infty$. Therefore, it is enough to show that

$$\lim_{k\to\infty} D_3 x(t, \nu, y_\nu) \frac{y_{\nu+h_k} - y_\nu}{h_k} = \mathcal{T}(t, \nu, y_\nu)(\dot{y}(\nu), \dot{y}_\nu), \qquad t \in [\nu, \alpha].$$

Introduce $z^k(t) := D_3 x(t, \nu, y_\nu) \frac{y_{\nu+h_k} - y_\nu}{h_k}$, $z(t) := \mathcal{T}(t, \nu, y_\nu)(\dot{y}(\nu), \dot{y}_\nu)$, and let $x := x(\cdot, \nu, y_\nu)$. Since z^k and z are solutions of the IVP (3.8)-(3.9) and (4.5)–(4.7), respectively, we have for $t \in [\sigma, \alpha]$

$$\begin{aligned} z^k(t) &= \frac{y(\nu + h_k) - y(\nu)}{h_k} + \int_\nu^t L(s, x) z^k_s \, ds \\ z(t) &= \dot{y}(\nu) + \int_\nu^t L(s, x) z_s \, ds, \end{aligned}$$

therefore

$$|z^k(t) - z(t)| \leq |\mu_k(\nu)| + \int_\nu^t |L(s, x)(z^k_s - z_s)| \, ds,$$

where

$$\mu_k(s) := \frac{y(s+h_k)-y(s)}{h_k} - \dot{y}(s), \qquad \text{a.e. } s\in[\nu-r,\nu].$$

Similarly to the proof of Lemma 4.1 we get the estimate

$$w_k(t) \le |\mu_k(\nu)| + a_k + K\int_\nu^t w_k(s)\,ds, \qquad t\in[\nu,\alpha], \tag{5.4}$$

where $w_k(t) := \max\{|z^k(s)-z(s)|\colon \nu\le s\le t\}$ for $t\in[\nu,\alpha]$, K is defined by (4.4),

$$\begin{aligned} a_k \;:=\; & \sum_{i=1}^m \int_{U_i} |A_i(s)||\mu_k(s-\lambda_i(s))|\,ds + \int_\nu^{\min\{\nu+r,\alpha\}}\int_{-r}^{\nu-s} |A(s,\theta)||\mu_k(s+\theta)|\,d\theta\,ds \\ & + L_1\Big[N\Big(\sum_{j=1}^{\ell}\int_{V_j} |b_j(s)||\mu_k(s-\xi_j(s))|\,ds \\ & + \int_\nu^{\min\{\nu+r,\alpha\}}\int_{-r}^{\nu-s} |b(s,\theta)||\mu_k(s+\theta)|\,d\theta\,ds\Big) + \int_W |\mu_k(s-\tau(s,x_s))|\,ds\Big], \end{aligned}$$

and

$$\begin{aligned} U_i &:= \{s\in[\nu,\alpha]\colon s-\lambda_i(s)<\nu\}, \quad i=1,\dots,m,\\ V_j &:= \{s\in[\nu,\alpha]\colon s-\xi_j(s)<\nu\}, \quad j=1,\dots,\ell,\\ W &:= \{s\in[\nu,\alpha]\colon s-\tau(s,x_s)<\nu\}. \end{aligned}$$

We have $a_k\to 0$ as $k\to\infty$, since $\mu_k(s)\to 0$ for a.e. $s\in[\nu-r,\nu]$.

Gronwall's inequality yields from (5.4) that

$$|z^k(t)-z(t)| \le w_k(t) \le (\mu_k(\nu)+a_k)e^{K\alpha}, \qquad t\in[\nu,\alpha].$$

This concludes the proof, since $\mu_k(\nu)\to 0$ and $a_k\to\infty$ as $k\to\infty$. □

Now we are ready to formulate and prove the nonlinear variation of constants formula for (5.1).

Theorem 5.2 *Assume f satisfies (A1), (A4), τ satisfies (A2), (A3), g satisfies (A5), and λ satisfies (A6). Let $(\sigma,\varphi)\in P$, let $y(t)=y(t,\sigma,\varphi)$ be the solution of the IVP* (5.1)-(5.2), *and let $x(t)=x(t,\sigma,\varphi)$ be the solution of the IVP* (2.1)-(2.2) *for $t\in[\sigma,\alpha]$. Suppose $(s,y_s)\in P_2$ for a.e. $s\in[\sigma,\alpha]$. Then*

$$y(t) = x(t) + \int_\sigma^t \mathcal{T}(t,s,y_s)\Big(g(s,y_s,y(s-\lambda(s,y_s))),\mathbf{0}\Big)\,ds, \qquad t\in[\sigma,\alpha], \tag{5.5}$$

where T is defined by (4.14), *and* $\mathbf{0}$ *is the identically zero function in C.*

Proof Let $s\in(\sigma,\alpha)$ be such that $(s,y_s)\in P_2$. Then for such s Theorem 4.2 and Lemma 5.1 yield

$$\begin{aligned} \frac{d}{ds}x(t,s,y_s) &= D_2x(t,s,y_s) + \frac{d}{d\nu}x(t,s,y_\nu)\Big|_{\nu=s} \\ &= \mathcal{T}(t,s,y_s)(-f(s,y_s,y(s-\tau(s,y_s))),-\dot{y}_s) + \mathcal{T}(t,s,y_s)(\dot{y}(s),\dot{y}_s). \end{aligned}$$

Since $\dot{y}(s) = f(s, y_s, y(s - \tau(s, y_s))) + g(s, y_s, y(s - \lambda(s, y_s)))$, we get

$$\frac{d}{ds}x(t, s, y_s) = \mathcal{T}(t, s, y_s)\Big(g(s, y_s, y(s - \lambda(s, y_s))), \mathbf{0}\Big).$$

Hence, using that $y_\sigma = \varphi$, it follows

$$\begin{aligned} y(t) - x(t) &= x(t, t, y_t) - x(t, \sigma, y_\sigma) \\ &= \int_\sigma^t \frac{d}{ds}x(t, s, y_s)\, ds \\ &= \int_\sigma^t \mathcal{T}(t, s, y_s)\Big(g(s, y_s, y(s - \lambda(s, y_s))), \mathbf{0}\Big)\, ds, \qquad t \in [\sigma, \alpha]. \end{aligned}$$

□

6 Stability of perturbed scalar SD-DDEs

Stability of several classes of SD-DDEs was studied, e.g., in [12], [16], [17], [20], [25], [30]. In these papers it was proved that the asymptotic stabiity of the trivial solution of an associated linear state-independent delay equation imples that of the fixed solution of the SD-DDE.

In this section we consider the scalar version of (2.1), where f and τ are time-independent, f depends only on the state-dependent term, and also the delay τ depends only on $x(t)$ instead of past values of the solution. Let $\sigma \geq 0$, and we consider the following special form of (2.1):

$$\dot{x}(t) = f(x(t - \tau(x(t)))), \qquad t \geq \sigma, \tag{6.1}$$

and the corresponding initial condition is

$$x(t) = \varphi(t - \sigma), \qquad t \in [\sigma - r, \sigma]. \tag{6.2}$$

We consider the perturbed system

$$\dot{y}(t) = f(y(t - \tau(y(t)))) + g(t, y_t, y(t - \lambda(t, y_t))), \qquad t \geq \sigma, \tag{6.3}$$

with the initial condition

$$y(t) = \varphi(t - \sigma), \qquad t \in [\sigma - r, \sigma]. \tag{6.4}$$

Our goal in this section is to give sufficient conditions which imply that the exponential stability of the trivial solution of (6.1) is preserved under certain perturbations given in (6.3).

We summarize our conditions on the parameters of (6.1) and (6.3).

(H1) $f \in C^2(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, $\tau \in C^2(\mathbb{R}, [0, r])$, τ is piecewise monotone in the sense of Definition 3.2 on any finite interval, and τ is not linear on any interval;

(H2) $g\colon [0, \infty) \times C \times \mathbb{R} \to \mathbb{R}$ is continuous and it is continuously differentiable wrt its second and third arguments, $g(t, \mathbf{0}, 0) = 0$ for $t \geq \sigma$, and $\lambda\colon [0, \infty) \times C \to [0, r]$ is continuous and it is continuously differentiable wrt to its second argument;

(H3) $\varphi \in W^{1,\infty}$.

Under conditions (H1)–(H3) a simple generalization of Theorem 2.1 yields that both the IVP (6.1)-(6.2) and the IVP (6.3)-(6.4) have unique solutions on an interval $[\sigma, \alpha]$ for some $\alpha > 0$. We also suppose the solutions of both IVPs exist on $[\sigma, \infty)$.

It is easy to see that, e.g., the delay function $\tau(u) = \frac{ru^2}{1+u^2}$ satisfies all conditions assumed for τ in (H1).

Definition 6.1 *We say that the trivial solution of the IVP* (6.1)-(6.2) *is* exponentially stable, *if there exist constants $\delta > 0$, $\mu_1 > 0$ and $K_1 \geq 1$ such that*

$$|x(t, \sigma, \varphi)| \leq K_1 e^{-\mu_1(t-\sigma)} |\varphi|_C, \qquad t \geq \sigma \tag{6.5}$$

for $\varphi \in W^{1,\infty}$ satisfying $|\varphi|_C \leq \delta$.

Remark 6.2 *We note that Theorem 4.2 of [17] and a well-known stability condition for a single delay equation (see, e.g., [18]) yields that if f and τ satisfy (H1), and*

$$0 < -f'(0)\tau(0) < \frac{\pi}{2}, \tag{6.6}$$

then the tivial solution of (6.1) *is exponentially stable.*

Lemma 6.3 *Assume (H1). Then the parameter set P_2 defined in* (3.7) *satisfies $P_2 = [0, \infty) \times W^{1,\infty}$.*

Proof Suppose for some $s \in (\sigma, \alpha)$ and $\psi \in W^{1,\infty}$ we have $(s, \psi) \notin P_2$. Then there exists an interval $[t_1, t_2] \subset [s, s+r]$ and a constant $c \in [s-r, s+r]$ such that the solution $x(t) = x(t, s, \psi)$ of the IVP (6.1)-(6.2) satisfies

$$t - \tau(x(t)) = c, \qquad t \in [t_1, t_2].$$

Then (6.1) yields

$$\dot{x}(t) = f(x(t - \tau(x(t)))) = f(x(c)), \qquad t \in [t_1, t_2],$$

hence x is linear on $[t_1, t_2]$. Then, using the assumed piecewise monotonicity of τ, there exist a subset $[u_1, u_2] \subset [t_1, t_2]$ such that on the range $\{x(t)\colon\ t \in [u_1, u_2]\}$ τ^{-1} is well-defined. Then $x(t) = \tau^{-1}(t - c)$ for $t \in [u_1, u_2]$, which contradicts to the assumption that τ is not linear, so $\tau^{-1}(t-c)$ is not linear too on any interval. This means that $t - \tau(x(t))$ is piecewise monotone in the sense ot Definition 3.2, i.e., $P_2 = [0, \infty) \times W^{1,\infty}$. □

According to the previous lemma, all conditions of Theorem 5.2 are satisfied for this case. Therefore (5.5) holds for all $(\sigma, \varphi) \in [0, \infty) \times W^{1,\infty}$. Suppose $x(t)$ is a fixed solution of (6.1). In this case the IVP (4.5)-(4.7) corresponding to this x has the form

$$\dot{z}(t) = f'(x(t - \tau(x(t))))\Big(-\dot{x}(t - \tau(x(t)))\tau'(x(t))z(t) + z(t - \tau(x(t)))\Big), \ t \geq \sigma, \tag{6.7}$$

$$z(\sigma) = v, \tag{6.8}$$

$$z(t) = h(t - \sigma), \qquad t \in [\sigma - r, \sigma]. \tag{6.9}$$

Using the solution $z(t) = z(t, \sigma, \varphi, v, h)$ of the IVP (6.7)-(6.9) we define $\mathcal{T}$ by (4.14).

The asymptotic properties of perturbed nonlinear differential equations using the nonlinear variation of constants formula was studied in several papers [4], [5], [7], [13], [14], [34]. Motivated by [5] and [13], we introduce the following definition.

Definition 6.4 *We say that the trivial solution of the IVP* (6.1)-(6.2) *is* exponentially stable in variation, *if there exist constants* $\mu_0 > 0$ *and* $K_0 \geq 1$ *such that*

$$|\mathcal{T}(t,s,\mathbf{0})(v,\mathbf{0})| \leq K_0 e^{-\mu_0(t-s)}|v|, \qquad \sigma \leq s \leq t, \ v \in \mathbb{R}. \tag{6.10}$$

The variational equation (6.7) corresponding to the trivial solution $\bar{x}(t) = 0$ of (6.1) has the form

$$\dot{z}(t) = -f'(0)z(t-\tau(0)), \qquad t \geq \sigma. \tag{6.11}$$

Consider its fundamental solution, i.e., the solution of the IVP

$$\begin{aligned} \dot{v}(t) &= -f'(0)z(t-\tau(0)), \qquad t \geq 0 &(6.12)\\ v(0) &= 1, &(6.13)\\ v(t) &= 0, \qquad t \in [-r,0). &(6.14) \end{aligned}$$

Then $\mathcal{T}(t,s,\mathbf{0})(1,\mathbf{0}) = v(t-s)$.

Remark 6.5 *It is known (see, e.g., [18]) that* (6.6) *implies that the trivial solution of* (6.11) *is exponentially stable, and so* (6.10) *holds, i.e, the trivial solution of the IVP* (6.1)-(6.2) *is exponentially stable in variation.*

Lemma 6.6 *Suppose (H1)-(H2). Let* $\sigma \geq 0$ *be fixed, and suppose* (6.6) *holds. Then there exist constants* $\delta > 0$, $K_2 \geq 1$ *and* $\mu_2 > 0$ *such that*

$$|(\mathcal{T}(t,s,y_s) - \mathcal{T}(t,s,\mathbf{0}))(v,\mathbf{0})| \leq K_2 e^{-\mu_2(t-s)}|v|, \qquad \sigma \leq s \leq t, \ v \in \mathbb{R}, \tag{6.15}$$

for any function $y\colon [\sigma - r, \infty) \to \mathbb{R}$ *satisfying* $y_s \in W^{1,\infty}$ *and* $|y_s|_C < \delta$ *for* $s \in [\sigma, \infty)$.

Proof Let $s \geq \sigma$ and $v \in \mathbb{R}$ be fixed, and let $x(t) := x(t,s,y_s)$, $z(t) := z(t,s,y_s,v,\mathbf{0})$ and $\bar{z}(t) := \bar{z}(t,s,\mathbf{0},v,\mathbf{0})$ be the solutions of the corresponding IVP (6.7)-(6.9). Furthermore, we introduce the short notations $\eta(t) := \tau(x(t))$, $\bar{\eta} := \tau(0)$, $a(t) := -f'(x(t-\tau(x(t))))\dot{x}(t-\tau(x(t)))\tau'(x(t))$, $b(t) := f'(x(t-\tau(x(t))))$, $\bar{b} := f'(0)$. Then it follows from (6.7) that

$$\dot{z}(t) - \dot{\bar{z}}(t) = \bar{b}[z(t-\bar{\eta}) - \bar{z}(t-\bar{\eta})] + F(t), \quad t \geq s,$$

where

$$\begin{aligned} F(t) \ := \ & a(t)(z(t)-\bar{z}(t)) + a(t)\bar{z}(t) + (b(t)-\bar{b})[z(t-\eta(t)) - \bar{z}(t-\eta(t))] \\ & +(b(t)-\bar{b})\bar{z}(t-\eta(t)) + \bar{b}[(z(t-\eta(t)) - \bar{z}(t-\eta(t))) - (z(t-\bar{\eta}) - \bar{z}(t-\bar{\eta}))] \\ & +\bar{b}[\bar{z}(t-\eta(t)) - \bar{z}(t-\bar{\eta})]. \end{aligned}$$

Let $v(t)$ be the fundamental solution of (6.11), i.e., the solution of the IVP (6.12)-(6.14). It follows from our assumptions and Remark 6.5 that

$$|v(t)| \leq K_0 e^{-\mu_0 t}, \qquad t \geq 0. \tag{6.16}$$

Since $z(t) = \bar{z}(t) = 0$ for $t \in [s-\tau, s]$, the function $\Omega(t) := z(t) - \bar{z}(t)$ satisfies

$$\Omega(t) = \int_s^t v(t-u)F(u)\,du, \qquad t \geq s. \tag{6.17}$$

On the other hand, using the linear variation of constants formula, we get

$$\Omega(t) = \Psi(t) + \int_{s+2r}^{t} v(t-u)F(u)\,du, \qquad t \geq s+2r, \tag{6.18}$$

where Ψ is the solution of the linear IVP

$$\begin{aligned} \dot{\Psi}(t) &= \bar{b}\Psi(t-\bar{\eta}), \qquad t \geq s+2r, \\ \Psi(t) &= z(t) - \bar{z}(t), \qquad t \in [s+r, s+2r]. \end{aligned}$$

From (6.16) it follows that there exists $M_0 \geq 1$ such that

$$|\Psi(t)| \leq M_0 e^{-\mu_0(t-s-2r)} \max_{s+r\leq\theta\leq s+2r} |z(\theta) - \bar{z}(\theta)|, \qquad t \geq s+2r. \tag{6.19}$$

Let δ_1 be the constant from the definition of the exponential stability, i.e., x satisfies (6.5) with $\delta = \delta_1$. Suppose $y\colon [\sigma - r, \infty) \to \mathbb{R}$ is such that $y_s \in W^{1,\infty}$ and $|y_s|_C < \delta_1$ for $s \in [\sigma, \infty)$. Using (6.5), we get that $|x(t)| \leq K_1\delta_1$ for $t \in [s-r, \infty)$ and $|y_s|_C < \delta_1$. Then using $f \in C^2(\mathbb{R}, \mathbb{R})$ and $\tau \in C^1([0,\infty), [0,r])$, there exist constants $L_{f'}$ and L_τ such that

$$|b(t) - \bar{b}| = |f'(x(t-\tau(x(t)))) - f'(0)| \leq L_{f'}|x(t-\tau(x(t)))| \leq L_{f'}K_1|y_s|_C, \quad t \geq s \geq \sigma,$$

where in the above estimates we also used estimate (6.5). Therefore there exists a constant $K_b \geq 0$ such that

$$|b(t) - \bar{b}| \leq K_b|y_s|_C, \qquad t \geq s \geq \sigma. \tag{6.20}$$

Similar argument shows that there exist constants $K_a \geq 0$ and $K_\tau \geq 0$ such that

$$|a(t)| \leq K_a|y_s|_C \quad \text{and} \quad |\eta(t) - \bar{\eta}| \leq K_\tau|y_s|_C \quad \text{for } t \geq s \geq \sigma. \tag{6.21}$$

Then, combining the definition of F together with (6.20), (6.21) and (6.10), we get

$$\begin{aligned} |F(t)| \leq\; & K_a|y_s|_C|\Omega(t)| + K_a|y_s|_C K_0 e^{-\mu_0(t-s)}|v| \\ & + K_b|y_s|_C|\Omega(t-\eta(t))| + K_b|y_s|_C K_0 e^{-\mu_0(t-\eta(t)-s)}|v| \\ & + \bar{b}|\Omega(t-\eta(t)) - \Omega(t-\bar{\eta})| + \bar{b}|\bar{z}(t-\eta(t)) - \bar{z}(t-\bar{\eta})|, \quad t \geq s. \end{aligned} \tag{6.22}$$

Therefore, using $|\bar{z}(t-\eta(t))| \leq K_0 e^{-\mu_0(t-\eta(t)-s)}|v|$, (6.10) and $|y_s|_C < \delta_1$, there exist M_1 and M_2 such that

$$|F(t)| \leq M_1 \max_{t-r\leq u\leq t} |\Omega(u)| + M_2 e^{-\mu_0(t-s)}|v|, \qquad t \geq s. \tag{6.23}$$

Substituting this into (6.17) we get

$$\begin{aligned} |\Omega(t)| &\leq \int_s^t K_0 e^{-\mu_0(t-u)}\Big(M_1 \max_{s\leq\theta\leq u} |\Omega(\theta)| + M_2 e^{-\mu_0(u-s)}|v|\Big)\,du \\ &\leq K_0 M_2 2r|v| + \int_s^t K_0 M_1 \max_{s\leq\theta\leq u} |\Omega(\theta)|\,du, \qquad t \in [s, s+2r]. \end{aligned}$$

So Gronwall's inequality yields

$$\max_{s\leq\theta\leq s+2r} |\Omega(\theta)| \leq M_3|v| \tag{6.24}$$

with $M_3 := K_0 M_2 2r e^{K_0 M_1 2r}$.

For $t \geq s + 2r$ we have

$$
\begin{aligned}
|\bar{z}(t-\eta(t)) - \bar{z}(t-\bar{\eta})| &= \left| \int_{t-\bar{\eta}}^{t-\eta(t)} \bar{b}\bar{z}(u-\bar{\eta})\,du \right| \\
&\leq \left| \int_{t-\bar{\eta}}^{t-\eta(t)} \bar{b}K_0 e^{-\mu_0(u-\bar{\eta}-s)}|v|\,du \right| \\
&\leq \left| \int_{t-\bar{\eta}}^{t-\eta(t)} \bar{b}K_0 e^{-\mu_0(t-2r-s)}|v|\,du \right| \\
&\leq K_3 e^{-\mu_0(t-s)}|v||y_s|_C \qquad (6.25)
\end{aligned}
$$

with an appropriate constant $K_3 \geq 0$. Similarly, using (6.23), we obtain

$$
\begin{aligned}
&|\Omega(t-\eta(t)) - \Omega(t-\bar{\eta})| \\
&\quad = \left| \int_{t-\bar{\eta}}^{t-\eta(t)} \bar{b}\Omega(u-\bar{\eta}) + F(s)\,du \right| \\
&\quad \leq \Big((\bar{b} + M_1) \max_{t-2r\leq\theta\leq t} |\Omega(\theta)| + M_2 e^{-\mu_0(t-r-s)}|v| \Big) |\eta(t) - \bar{\eta}| \\
&\quad \leq |y_s|_C \Big(K_4 \max_{t-2r\leq\theta\leq t} |\Omega(\theta)| + K_5 e^{-\mu_0(t-s)}|v| \Big), \qquad t \geq s + 2r \qquad (6.26)
\end{aligned}
$$

with some $K_4 \geq 0$ and $K_5 \geq 0$. Hence (6.22), (6.25) and (6.26) imply

$$
|F(t)| \leq |y_s|_C \Big(K_6 \max_{t-2r\leq\theta\leq t} |\Omega(\theta)| + K_7 e^{-\mu_0(t-s)}|v| \Big), \qquad t \geq s + 2r
$$

with some appropriate constants $K_6 \geq 0$ and $K_7 \geq 0$. Then, combining this relation with (6.18), (6.19), (6.24), we get

$$
\begin{aligned}
|\Omega(t)| &\leq M_0 e^{-\mu_0(t-s-2r)} M_3 |v| \\
&\quad + \int_s^t K_0 e^{-\mu_0(t-u)} |y_s|_C \Big(K_6 \max_{u-2r\leq\theta\leq u} |\Omega(\theta)| + K_7 e^{-\mu_0(u-s)}|v| \Big)\,du \\
&\leq M_0 e^{-\mu_0(t-s-2r)} M_3 |v| + K_0 |y_s|_C K_7 |v| e^{-\mu_0(t-s)}(t-s) \\
&\quad + K_0 K_6 |y_s|_C \int_s^t e^{-\mu_0(t-u)} \max_{u-2r\leq\theta\leq u} |\Omega(\theta)|\,du, \quad t \geq s.
\end{aligned}
$$

Let $0 < \mu_2 < \mu_0$ be fixed. Then $\sup_{t\geq s} e^{-(\mu_0-\mu_2)(t-s)}(t-s) < \infty$, so there exists $M_4 \geq e^{\mu_2 2r} M_3$

such that

$$\begin{aligned}
&e^{\mu_2(t-s)}|\Omega(t)| \\
&\leq M_4|v| + K_0K_6|y_s|_C e^{-(\mu_0-\mu_2)t-\mu_2 s} \\
&\quad \times \int_s^t e^{\mu_0 u} \max_{u-2r\leq\theta\leq u}\Big(e^{-\mu_2(\theta-s)}e^{\mu_2(\theta-s)}|\Omega(\theta)|\Big)\,du \\
&\leq M_4|v| + K_0K_6|y_s|_C e^{-(\mu_0-\mu_2)t-\mu_2 s} \\
&\quad \times \int_s^t e^{(\mu_0-\mu_2)u}e^{\mu_2(2r+s)} \max_{u-2r\leq\theta\leq u} e^{\mu_2(\theta-s)}|\Omega(\theta)|\,du \\
&\leq M_4|v| + K_0K_6|y_s|_C e^{-(\mu_0-\mu_2)t} \max_{s\leq\theta\leq t} e^{\mu_2(\theta-s)}|\Omega(\theta)| \int_s^t e^{(\mu_0-\mu_2)u}e^{2r\mu_2}\,du \\
&\leq M_4|v| + \frac{K_0K_6|y_s|_C e^{2r\mu_2}}{\mu_0-\mu_2} \max_{s\leq\theta\leq t} e^{\mu_2(\theta-s)}|\Omega(\theta)|, \qquad t\geq s+2r. \qquad (6.27)
\end{aligned}$$

Let $0<\delta\leq\delta_1$ be such that

$$M_5 := \frac{K_0K_6e^{2r\mu_2}}{\mu_0-\mu_2}\delta < 1.$$

Since $M_4 \geq e^{\mu_2 2r}M_3$, (6.27) yields for $|y_s|_C<\delta$ that

$$\max_{s\leq\theta\leq t} e^{\mu_2(\theta-s)}|\Omega(\theta)| \leq M_4|v| + M_5\max_{s\leq\theta\leq t} e^{\mu_2(\theta-s)}|\Omega(\theta)|, \qquad t\geq s,$$

hence

$$e^{\mu_2(t-s)}|\Omega(t)| \leq \max_{s\leq\theta\leq t} e^{\mu_2(\theta-s)}|\Omega(\theta)| \leq \frac{M_4}{1-M_5}|v|, \qquad t\geq s,$$

which completes the proof of (6.15). □

Our main theorem shows that the exponential stability of the trivial solution of (6.1) is preserved for that of (6.3) under suitable conditions of the perturbation.

Theorem 6.7 *Suppose (H1)-(H3) and* (6.6) *hold. We assume that for every $\rho>0$ there exists $\gamma_\rho\colon [\sigma,\infty)\to[0,\infty)$ such that $\gamma_\rho(t)\to 0$ monotone decreasingly as $t\to\infty$, and*

$$|g(t,\psi,u)-g(t,\bar\psi,\bar u)| \leq \gamma_\rho(t)(|\psi-\bar\psi|_C + |u-\bar u|), \qquad t\geq\sigma \qquad (6.28)$$

for $\psi,\bar\psi\in W^{1,\infty}$, $u,\bar u\in\mathbb{R}$ satisfying $|\psi|_C,|\bar\psi|_C\leq\rho$ and $|u|,|\bar u|<\rho$. Then the trivial solution of (6.3) *is also exponentially stable.*

Proof It follows from our assumptions that $g(t,\mathbf{0},0)=0$, and so $y(t,\sigma,\mathbf{0})=0$ for all $t\geq\sigma$.

Let $\delta>0$ and $\varphi\in W^{1,\infty}$ be fixed, $x(t):=x(t,\sigma,\varphi)$, $y(t):=y(t,\sigma,\varphi)$, $v\in\mathbb{R}$, $z(t):=z(t,\sigma,\varphi,v,\mathbf{0})$, $\bar z(t):=z(t,\sigma,\mathbf{0},v,\mathbf{0})$. It follows from (5.5) that

$$y(t) = x(t) + \int_\sigma^t \mathcal{T}(t,s,y_s)(g(s,y_s,y(s-\lambda(s,y_s))),\mathbf{0})\,ds, \quad t\geq\sigma.$$

Hence we have

$$\begin{aligned}
y(t) &= x(t) + \int_\sigma^t [\mathcal{T}(t,s,y_s)-\mathcal{T}(t,s,\mathbf{0})](g(s,y_s,y(s-\lambda(s,y_s)))-g(s,\mathbf{0},0),\mathbf{0})\,ds \\
&\quad + \int_\sigma^t \mathcal{T}(t,s,\mathbf{0})\Big(g(s,y_s,y(s-\lambda(s,y_s)))-g(s,\mathbf{0},0),\mathbf{0}\Big)\,ds, \qquad t\geq\sigma.
\end{aligned}$$

Let δ_1 and δ_2 be defined by Definition 6.1 and Lemma 6.6, respectively, i.e., (6.5) and (6.15) hold with $\delta = \delta_1$ and $\delta = \delta_2$, respectively. Let $\delta_3 := \min\{\delta_1, \delta_2\}$. Suppose $|\varphi|_C < \delta_3$. Then the continuity of y yields that there exists $\alpha > \sigma$ such that $|y_s|_C < \delta_3$ for $s \in [\sigma, \alpha)$. Then relations (6.10), (6.5) and (6.15) imply

$$\begin{aligned}|y(t)| &\leq |x(t)| + \int_\sigma^t K_2 e^{-\mu_2(t-s)} |g(s, y_s, y(s-\lambda(s,y_s))) - g(s, \mathbf{0}, 0)|\, ds \\ &\quad + \int_\sigma^t K_0 e^{-\mu_0(t-s)} \Big| g(s, y_s, y(s-\lambda(s,y_s))) - g(s, \mathbf{0}, 0) \Big|\, ds \\ &\leq K_1 e^{-\mu_1(t-\sigma)} |\varphi|_C + K_2 \int_\sigma^t e^{-\mu_2(t-s)} \gamma_{\delta_1}(s) \Big(|y_s|_C + |y(s-\lambda(s,y_s))| \Big)\, ds \\ &\quad + K_0 \int_\sigma^t e^{-\mu_0(t-s)} \gamma_{\delta_1}(s) \Big(|y_s|_C + |y(s-\lambda(s,y_s))| \Big)\, ds, \qquad t \in [\sigma, \alpha).\end{aligned}$$

We have from the proof of Lemma 6.6 that $\mu_2 < \mu_0$. Let $0 < \mu_3 < \min\{\mu_1, \mu_2\}$ be fixed. Then

$$\begin{aligned}e^{\mu_3(t-\sigma)}|y(t)| &\leq K_1 |\varphi|_C + (K_0 + K_2) e^{-(\mu_2-\mu_3)t - \mu_3\sigma} \\ &\quad \times \int_\sigma^t e^{\mu_2 s} \gamma_{\delta_1}(s) \Big(|y_s|_C + |y(s-\lambda(s,y_s))| \Big)\, ds \\ &\leq K_1 |\varphi|_C + 2(K_0 + K_2) e^{-(\mu_2-\mu_3)t} \\ &\quad \times \int_\sigma^t e^{(\mu_2-\mu_3)s + \mu_3 r} \gamma_{\delta_1}(s) \max_{s-r \leq u \leq s} e^{\mu_3(u-\sigma)} |y(u)|\, ds\end{aligned}$$

for $t \in [\sigma, \alpha)$. Since $e^{\mu_3(t-\sigma)}|y(t)| \leq |\varphi|_C \leq K_1 |\varphi|_C$ for $t \in [\sigma - r, \sigma]$, it follows that

$$\begin{aligned}&\max_{\sigma-r \leq u \leq t} e^{\mu_3(u-\sigma)} |y(u)| \\ &\quad \leq K_1 |\varphi|_C + M_6 e^{-(\mu_2-\mu_3)t} \int_\sigma^t e^{(\mu_2-\mu_3)s} \gamma_{\delta_1}(s) \max_{\sigma-r \leq u \leq s} e^{\mu_3(u-\sigma)} |y(u)|\, ds\end{aligned} \tag{6.29}$$

for $t \in [\sigma, \alpha)$ with $M_6 := 2(K_0 + K_2) e^{\mu_3 r}$. In particular, we have for $t \in [\sigma, \alpha)$

$$\max_{\sigma-r \leq u \leq t} e^{\mu_3(u-\sigma)} |y(u)| \leq K_1 |\varphi|_C + M_6 \gamma_{\delta_1}(0) \int_\sigma^t \max_{\sigma-r \leq u \leq s} e^{\mu_3(u-\sigma)} |y(u)|\, ds.$$

Then Gronwall's inequality yields

$$\max_{\sigma-r \leq u \leq t} e^{\mu_3(u-\sigma)} |y(u)| \leq K_1 |\varphi|_C e^{M_6 \gamma_{\delta_1}(0) t}, \qquad t \geq \sigma. \tag{6.30}$$

Let $t_1 > \sigma$ be fixed such that

$$\frac{M_6 \gamma_{\delta_1}(t_1)}{\mu_2 - \mu_3} < 1. \tag{6.31}$$

Then (6.30) gives

$$e^{-\mu_3 r} \max_{\sigma-r \leq u \leq t_1} |y(u)| \leq \max_{\sigma-r \leq u \leq t_1} e^{\mu_3(u-\sigma)} |y(u)| \leq K_1 |\varphi|_C e^{M_6 \gamma_{\delta_1}(0) t_1}. \tag{6.32}$$

So if $0 < \delta_4 \leq \delta_3$ is such that

$$K_1 \delta_4 e^{M_6 \gamma_{\delta_1}(0) t_1 + \mu_3 r} < \delta_3,$$

then $|y(t)| < \delta_3$ for $t \in [\sigma, t_1]$, so α can be selected so that $\alpha > t_1$. Therefore (6.29) implies

$$\begin{aligned}
&\max_{\sigma-r\le u\le t} e^{\mu_3(u-\sigma)}|y(u)| \\
&\qquad \le \; K_1|\varphi|_C + M_6 e^{-(\mu_2-\mu_3)t}\int_\sigma^{t_1} e^{(\mu_2-\mu_3)s}\gamma_{\delta_1}(s)\max_{\sigma-r\le u\le s} e^{\mu_3(u-\sigma)}|y(u)|\,ds \\
&\qquad\qquad + M_6 e^{-(\mu_2-\mu_3)t}\int_{t_1}^{t} e^{(\mu_2-\mu_3)s}\gamma_{\delta_1}(s)\max_{\sigma-r\le u\le s} e^{\mu_3(u-\sigma)}|y(u)|\,ds \\
&\qquad \le \; M_7|\varphi|_C + \frac{M_6\gamma_{\delta_1}(t_1)}{\mu_2-\mu_3}\max_{\sigma-r\le u\le t} e^{\mu_3(u-\sigma)}|y(u)|, \qquad t\in[\sigma,\alpha), \qquad (6.33)
\end{aligned}$$

with some $M_7 \ge 0$. Then (6.33) and (6.31) yield

$$e^{\mu_3(t-\sigma)}|y(t)| \le M|\varphi|_C,$$

or equivalently,

$$|y(t)| \le Me^{-\mu_3(t-\sigma)}|\varphi|_C \qquad (6.34)$$

for $t \in [\sigma, \alpha)$ with some $M \ge 1$. Let $0 < \delta \le \delta_4$ be such that $M\delta < \delta_3$. Then it is easy to see that (6.34) holds for all $t \ge \sigma$, which completes the proof. □

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