to appear in Journal of Dynamics and Differential Equations

On second-order differentiability with respect to parameters for differential equations with state-dependent delays*

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May 29, 2013

Abstract

In this paper we consider a class of differential equations with state-dependent delays. We show first and second-order differentiability of the solution with respect to parameters in a pointwise sense and also using the C-norm on the state-space, assuming that the state-dependent time lag function is piecewise strictly monotone.

AMS(MOS) subject classification: 34K05
keywords: Delay differential equation, state-dependent delay, differentiability with respect to parameters.

1 Introduction

In this paper we study the SD-DDE

$$\dot{x}(t) = f(t, x_t, x(t - \tau(t, x_t, \xi)), \theta), \qquad t \in [0, T], \tag{1.1}$$

and the corresponding initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{1.2}$$

Let Θ and Ξ be normed linear spaces with norms $|\cdot|_\Theta$ and $|\cdot|_\Xi$, respectively, and suppose $\theta \in \Theta$ and $\xi \in \Xi$. Here we consider the initial function φ, θ and ξ as parameters in the IVP (1.1)-(1.2), and we denote the corresponding solution by $x(t, \varphi, \theta, \xi)$. The main goal of this paper is to discuss the differentiability of $x(t, \varphi, \theta, \xi)$ wrt φ, θ and ξ. By differentiability we mean Fréchet-differentiability throughout the manuscript.

Differentiability of solutions wrt parameters is an important qualitative question, but it also has a natural application in the problem of identification of parameters (see [10]). But even for simple constant delay equations this problem leads to technical difficulties if the parameter is the delay [6, 17]. Similar difficulty arises in SD-DDEs.

*This research was partially supported by the Hungarian National Foundation for Scientific Research Grant No. K101217.

Theorem 3.1 below yields that, under natural assumptions, Lipschitz continuous initial functions generate unique solutions of (1.1). As it is common for delay equations, as the time increases, the solution of (1.1) gets smoother wrt the time: on the interval $[0, r]$ the solution is C^1, on $[r, 2r]$ it is a C^2 function, etc. But for $t \in [0, r]$ the solution segment function x_t is only Lipschitz continuous. Therefore the linearization of the composite function $x(t - \tau(t, x_t, \xi))$ is not straightforward, which is clearly needed at some point of the proof to obtain differentiability wrt parameters.

To illustrate the difficulty of this problem in the case when we cannot assume continuous differentiability of x, we recall a result of Brokate and Colonius [1]. They studied SD-DDEs of the form

$$x'(t) = f\Big(t, x(t - \tau(t, x(t)))\Big), \qquad t \in [a, b],$$

and investigated differentiability of the composition operator

$$A : W^{1,\infty}([a, b]; \mathbb{R}) \supset \bar{X} \to L^p([a, b]; \mathbb{R}), \qquad A(x)(t) := x(t - \tau(t, x(t))).$$

They assumed that τ is twice continuously differentiable satisfying $a \leq t - \tau(t, v) \leq b$ for all $t \in [a, b]$ and $v \in \mathbb{R}$, and considered as domain of A the set

$$\bar{X} := \Big\{x \in W^{1,\infty}([a, b]; \mathbb{R}) \colon \text{There exists } \varepsilon > 0 \text{ s.t. } \frac{d}{dt}\Big(t - \tau(t, x(t))\Big) \geq \varepsilon \text{ for a.e. } t \in [a, b]\Big\}.$$

It was shown in [1] that under these assumptions A is continuously differentiable with the derivative given by

$$(DA(x)u)(t) = -\dot{x}(t - \tau(t, x(t)))D_2\tau(t, x(t))u(t) + u(t - \tau(t, x(t)))$$

for $u \in W^{1,\infty}([a, b], \mathbb{R})$. Both the strong $W^{1,\infty}$-norm on the domain and the weak L^p-norm on the range, together with the choice of the domain seemed to be necessary to obtain the results in [1]. Note that Manitius in [18] used a similar domain and norm when he studied linearization for a class of SD-DDEs.

Differentiability of solutions wrt parameters for SD-DDEs was studied in [2, 9, 12, 16, 21, 22]. In [9] differentiability of the parameter map was established at parameter values where the compatibility condition

$$\varphi \in C^1, \qquad \dot{\varphi}(0-) = f(0, \varphi, \varphi(-\tau(0, \varphi, \xi)), \theta) \tag{1.3}$$

is satisfied. It was proved that the parameter map is differentiable in a pointwise sense, i.e., the map

$$W^{1,\infty} \times \Theta \times \Xi \to \mathbb{R}^n, \qquad (\varphi, \theta, \xi) \mapsto x(t, \varphi, \theta, \xi) \tag{1.4}$$

is differentiable for every fixed t from the domain of the solution. Moreover, it was shown that the map

$$W^{1,\infty} \times \Theta \times \Xi \to C, \qquad (\varphi, \theta, \xi) \mapsto x_t(\cdot, \varphi, \theta, \xi), \tag{1.5}$$

and, under a little more smoothness assumptions, the map

$$W^{1,\infty} \times \Theta \times \Xi \to W^{1,\infty}, \qquad (\varphi, \theta, \xi) \mapsto x_t(\cdot, \varphi, \theta, \xi) \tag{1.6}$$

is also differentiable at fixed parameter values satisfying (1.3). Note that a condition similar to (1.3) was used by Walter in [21] and [22], where he proved the existence of a C^1-smooth solution semiflow for large classes of SD-DDES.

In [16] differentiability of the parameter map was proved without assuming the compatibility condition (1.3). Instead, it was assumed that the time lag function $t \mapsto t - \tau(t, x_t, \xi)$ corresponding to a fixed solution x is strictly monotone increasing, more precisely,

$$\operatorname*{ess\,inf}_{0\le t\le \alpha} \frac{d}{dt}(t - \tau(t, x_t, \xi)) > 0, \tag{1.7}$$

where $\alpha > 0$ is such that the solution exists on $[-r, \alpha]$. Also, instead of a "pointwise" differentiability, the differentiability of the map

$$W^{1,\infty} \times \Theta \times \Xi \to W^{1,p}, \qquad (\varphi, \theta, \xi) \mapsto x_t(\cdot, \varphi, \theta, \xi)$$

was proved in a small neighborhood of the fixed parameter value. Note that here the differentiability was obtained using only a weak norm, the $W^{1,p}$-norm ($1 \le p < \infty$) on the state-space.

Chen, Hu and Wu in [2] extended the above result to proving second-order differentiability of the parameter map using the monotonicity condition (1.7) of the state-dependent time lag function, the $W^{1,p}$-norm ($1 \le p < \infty$) on the state space, and the $W^{2,p}$-norm on the space of initial functions. Note that τ was not given explicitly in [2], it was defined through a coupled differential equation, but it satisfied the monotonicity condition (1.7).

In [12] the IVP

$$\dot{x}(t) = f(t, x_t, x(t - \tau(t, x_t))), \qquad t \in [\sigma, T], \tag{1.8}$$
$$x(t) = \varphi(t - \sigma), \qquad t \in [\sigma - r, \sigma] \tag{1.9}$$

was considered. In this IVP the parameters θ and ξ were omitted for simplicity, but the initial time σ was considered together with the initial function as parameters in the equation. Combining the techniques of [9] and [16], and assuming the appropriate monotonicity condition (1.7), but without assuming the compatibility condition (1.3), the continuous differentiability of the parameter maps

$$W^{1,\infty} \to \mathbb{R}^n, \qquad \varphi \mapsto x(t, \sigma, \varphi)$$

and

$$W^{1,\infty} \to C, \qquad \varphi \mapsto x_t(\cdot, \sigma, \varphi)$$

were proved for a fixed t and σ in a neighborhood of a fixed initial function. Note that with this technique similar result can't be given using the $W^{1,\infty}$-norm on the state-space without using the compatibility condition.

Assuming the compatibility condition (1.3) it was also shown in [12] that the maps

$$[0, \alpha) \to \mathbb{R}^n, \qquad \sigma \mapsto x(t, \sigma, \varphi)$$

and

$$[0, \alpha) \to C, \qquad \sigma \mapsto x_t(\cdot, \sigma, \varphi)$$

are differentiable for all $t \in [\sigma - r, \alpha]$ and $t \in [\sigma, \alpha]$, respectively, and σ, φ in a neighborhood of a fixed parameter (σ, φ), and where $\alpha > 0$ is a certain constant. Assuming that the functions f and τ have a special form in (1.8), i.e., for equations of the form

$$\dot{x}(t) = \bar{f}\Big(t, x(t - \lambda^1(t)), \ldots, x(t - \lambda^m(t)), \int_{-r}^0 A(t,\theta)x(s+\theta)\,ds,$$
$$x\Big(t - \bar{\tau}\Big[t, x(t - \xi^1(t)), \ldots, x(t - \xi^\ell(t)), \int_{-r}^0 B(t,\theta)x(s+\theta)\,ds\Big]\Big)\Big)$$

the differentiability of the map

$$[0,\alpha)\to\mathbb{R}^n,\qquad \sigma\mapsto x(t,\sigma,\varphi)$$

was shown in [12] for $t\in[\sigma,\alpha]$ using the monotonicity assumption (1.7), but without the compatibility condition (1.3). Note that in this case similar result does not hold for the map $\sigma\mapsto x_t(\cdot,\sigma,\varphi)$ using the C-norm, which is not surprising, since it is easy to see [12] that the map $\sigma\mapsto x(t,\sigma,\varphi)$ is differentiable at the point $t=\sigma$ if and only if a compatibility condition similar to (1.3) is satisfied.

We refer the interested reader for related works on dependence of the solutions on parameters in SD-DDEs to [19, 20], and for similar works in neutral SD-DDEs to [11, 13, 23].

The organization of this paper is the following. In Section 2 we summarize some notations and preliminary results that will be used in the manuscript. In Section 3 we formulate a well-posedness result (Theorem 3.1) concerning the IVP (1.1)-(1.2).

In Section 4 using and extending the method introduced in [12], we discuss first order differentiability of the parameter maps associated to the IVP (1.1)-(1.2). In the main result of this section (see Theorem 4.9 below) we show the differentiability of the parameter maps (1.4) and (1.5) without using the compatibility condition (1.3), and also relaxing the monotonicity condition (1.7) to the condition that the time lag function $t\mapsto t-\tau(t,x_t,\xi)$ is "piecewise strictly monotone" in the sense of Definition 2.6. The key assumption in Theorem 4.9 is (apart from the regularity of f and τ) that the initial function belongs to $W^{1,\infty}$. Note that omitting the compatibility condition is essential in the application of this results in [14], where we prove the convergence of the quasilinearization method in the problem of parameter estimation. Also, in this application the existence of the derivative is needed in this strong, pointwise sense, i.e., the differentiability of the map (1.4) is used in [14].

In Section 5 the main result is Theorem 5.17, which proves twice continuous differentiability of the maps

$$W^{2,\infty}\times\Theta\times\Xi\to\mathbb{R}^n,\qquad (\varphi,\theta,\xi)\mapsto x(t,\varphi,\theta,\xi)$$

and

$$W^{2,\infty}\times\Theta\times\Xi\to C,\qquad (\varphi,\theta,\xi)\mapsto x_t(\cdot,\varphi,\theta,\xi)$$

at a parameter value (φ,θ,ξ) satisfying the compatibility condition (1.3) and such that the corresponding time lag function $t\mapsto\tau(t,x_t,\xi)$ is piecewise strictly monotone in the sense of Definition 2.6. Here the assumptions that f and τ are twice continuously differentiable and the initial functions belong to $W^{2,\infty}$ are needed to obtain the second-order differentiability. The only result known in the literature for the existence of a second derivative wrt the parameters in SD-DDEs is the result of Chen, Hu and Wu [2], where the second-order differentiability is proved only using a weak $W^{1,p}$-norm on the state-space. Note that our result shows the existence of the second derivative in a pointwise sense, i.e., at each fixed t, moreover, the technique of the proof is simpler.

We comment that the results of this paper do not imply the existence C^1 or C^2-smooth semiflows of solutions, since in both Theorems 4.9 and 5.17 only C-norm is used in the final segments $x_t(\cdot,\varphi,\theta,\xi)$, but the $W^{1,\infty}$ and $W^{2,\infty}$-norms respectively are needed for the initial functions. So the existence of C^2-smooth solution semiflows is still an open question for SD-DDEs.

2 Notations and preliminaries

Throughout the manuscript $r > 0$ is a fixed constant and $x_t\colon [-r, 0] \to \mathbb{R}^n$, $x_t(\theta) := x(t+\theta)$ is the segment function. To avoid confusion with the notation of the segment function, sequences of functions are denoted using the upper index: x^k.

$\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive and nonnegative integers, respectively. A fixed norm on $\mathbb{R}^n$ and its induced matrix norm on $\mathbb{R}^{n\times n}$ are both denoted by $|\cdot|$. C denotes the Banach space of continuous functions $\psi : [-r, 0] \to \mathbb{R}^n$ equipped with the norm $|\psi|_C = \max\{|\psi(\zeta)| : \zeta \in [-r, 0]\}$. C^1 is the space of continuously differentiable functions $\psi : [-r, 0] \to \mathbb{R}^n$ where the norm is defined by $|\psi|_{C^1} = \max\{|\psi|_C, |\dot\psi|_C\}$. L^∞ is the space of Lebesgue-measurable functions $\psi : [-r, 0] \to \mathbb{R}^n$ which are essentially bounded. The norm on L^∞ is denoted by $|\psi|_{L^\infty} = \operatorname{ess\,sup}\{|\psi(\zeta)|\colon \zeta \in [-r, 0]\}$. $W^{1,p}$ denotes the Banach-space of absolutely continuous functions $\psi\colon [-r, 0] \to \mathbb{R}^n$ of finite norm defined by

$$|\psi|_{W^{1,p}} := \left(\int_{-r}^{0} |\psi(\zeta)|^p + |\dot\psi(\zeta)|^p \, d\zeta\right)^{1/p}, \qquad 1 \le p < \infty,$$

and for $p = \infty$

$$|\psi|_{W^{1,\infty}} := \max\left\{|\psi|_C, |\dot\psi|_{L^\infty}\right\}.$$

We note that $W^{1,\infty}$ is equal to the space of Lipschitz continuous functions from $[-r, 0]$ to $\mathbb{R}^n$. The subset of $W^{1,\infty}$ consisting of those functions which have absolutely continuous first derivative and essentially bounded second derivative is denoted by $W^{2,\infty}$, where the norm is defined by

$$|\psi|_{W^{2,\infty}} := \max\left\{|\psi|_C,\ |\dot\psi|_C,\ |\ddot\psi|_{L^\infty}\right\}.$$

If the domain or the range of the functions is different from $[-r, 0]$ and $\mathbb{R}^n$, respectively, we will use a more detailed notation. E.g., $C(X, Y)$ denotes the space of continuous functions mapping from X to Y. Finally, $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X to Y, where X and Y are normed linear spaces. An open ball in the normed linear space X centered at a point $x \in X$ with radius δ is denoted by $\mathcal{B}_X(x;\, \delta) := \{y \in X\colon |x - y|_X < \delta\}$, where $|\cdot|_X$ is the norm defined on X.

The derivative of a single variable function $v(t)$ wrt t is denoted by $\dot v$. Note that all derivatives we use in this paper are Fréchet derivatives. The partial derivatives of a function $g\colon X_1\times X_2 \to Y$ wrt the first and second variables will be denoted by D_1g and D_2g, respectively. The second-order partial derivative wrt its ith and jth variables $(i, j = 1, 2)$ of the function $g\colon X_1 \times X_2 \to Y$ at the point $(x_1, x_2) \in X_1 \times X_2$ is the bounded bilinear operator $A\langle\cdot,\cdot\rangle\colon X_i \times X_j \to Y$, if

$$\lim_{k\to 0} \sup_{h\neq 0} \frac{|D_ig(x_1 + k\delta_{1j}, x_2 + k\delta_{2j})h - D_ig(x_1, x_2)h - A\langle h, k\rangle|_Y}{|h|_{X_i}|k|_{X_j}} = 0, \qquad h \in X_i,\ k \in X_j,$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$ is the Kronecker-delta. We will use the notation $D_{ij}g(x_1, x_2) = A$. The norm of the bilinear operator $A\langle\cdot,\cdot\rangle\colon X_i \times X_j \to Y$ is defined by

$$|A|_{\mathcal{L}^2(X_i\times X_j,Y)} := \sup\left\{\frac{|A\langle h, k\rangle|_Y}{|h|_{X_i}|k|_{X_j}}\colon\ h \in X_i, h \neq 0,\ k \in X_j, k \neq 0\right\}.$$

In the case when $X_1 = \mathbb{R}$, we simply write $D_1g(x_1, x_2)$ instead of the more precise notation $D_1g(x_1, x_2)1$, i.e., here D_1g denotes the value in Y instead of the linear operator $\mathcal{L}(\mathbb{R}, Y)$. In the case when, let say, $X_2 = \mathbb{R}^n = Y$, then we identify the linear operator $D_2 g(x_1, x_2) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ by an $n \times n$ matrix.

Next we formulate a result which is a simple consequence of the Gronwall's lemma.

Lemma 2.1 (see, e.g., [12]) *Suppose $a > 0$, $b\colon [0,\alpha] \to [0,\infty)$ and $u\colon [-r,\alpha] \to \mathbb{R}^n$ are continuous functions such that $a \geq |u_0|_C$, and*

$$|u(t)| \leq a + \int_0^t b(s)|u_s|_C \, ds, \qquad t \in [0,\alpha]. \tag{2.1}$$

Then

$$|u(t)| \leq |u_t|_C \leq a e^{\int_0^\alpha b(s)\, ds}, \qquad t \in [0,\alpha]. \tag{2.2}$$

We recall the following variant of the Mean Value Theorem.

Lemma 2.2 *Let X_1, X_2, Y be normed linear spaces, $U \subset X_1$ and $V \subset X_2$ convex and open subsets, $g\colon X_1 \times X_2 \supset U \times V \to Y$ continuously differentiable. Then for every $(x,y), (\bar{x},\bar{y}) \in U \times V$*

$$\begin{aligned} |g(x,y) - g(\bar{x},\bar{y})|_Y \quad \leq \quad & \max_{\nu\in[0,1]} |D_1 f(\bar{x} + \nu(x-\bar{x}), \bar{y} + \nu(y-\bar{y}))|_{\mathcal{L}(X_1,Y)} |x-\bar{x}|_{X_1} \\ & + \max_{\nu\in[0,1]} |D_2 f(\bar{x} + \nu(x-\bar{x}), \bar{y} + \nu(y-\bar{y}))|_{\mathcal{L}(X_2,Y)} |y-\bar{y}|_{X_2}. \end{aligned}$$

We recall the following result from [1], which was essential to prove differentiability wrt parameters in SD-DDEs in [2], [12] and [16]. We state the result in a simplified form we need later, it is formulated in a more general form in [1]. Note that the second part of the lemma was stated in [1] under the assumption $|u^k - u|_{W^{1,\infty}([0,\alpha],\mathbb{R})} \to 0$ as $k \to \infty$, but this stronger assumption on the convergence is not needed in the proof. See also the proof of Lemma 4.26 in [8].

Lemma 2.3 ([1]) *Let $g \in L^1([c,d],\mathbb{R}^n)$, $\varepsilon > 0$, and $u \in \mathcal{A}(\varepsilon)$, where*

$$\mathcal{A}(\varepsilon) := \{v \in W^{1,\infty}([a,b],[c,d]) \,:\, \dot{v}(s) \geq \varepsilon \text{ for a.e. } s \in [a,b]\}.$$

Then

$$\int_a^b |g(u(s))|\, ds \leq \frac{1}{\varepsilon} \int_c^d |g(s)|\, ds. \tag{2.3}$$

Moreover, if the sequence $u^k \in \mathcal{A}(\varepsilon)$ is such that $|u^k - u|_{C([a,b],\mathbb{R})} \to 0$ as $k \to \infty$, then

$$\lim_{k\to\infty} \int_a^b \Big| g(u^k(s)) - g(u(s)) \Big|\, ds = 0. \tag{2.4}$$

Remark 2.4 Changing to the new variable $s = -t$ in the integrals in (2.3) and (2.4) give easily that the statements of Lemma 2.3 hold also in the case when conditions $u, u^k \in \mathcal{A}(\varepsilon)$ are replaced by $-u, -u^k \in \mathcal{A}(\varepsilon)$.

In the next lemma we relax the condition $u \in \mathcal{A}(\varepsilon)$ of the previous lemma.

Lemma 2.5 *Suppose $g \in L^\infty([c,d],\mathbb{R})$, and $u\colon [a,b] \to [c,d]$ is an absolutely continuous function, and*

$$\operatorname{ess\,inf}\{\dot u(s)\colon\ s \in [a',b']\} > 0, \qquad \textit{for all}\ \ [a',b'] \subset (a,b). \tag{2.5}$$

Then the composite function $g \circ u \in L^\infty([a,b],\mathbb{R})$, and $|g \circ u|_{L^\infty([a,b],\mathbb{R})} \le |g|_{L^\infty([c,d],\mathbb{R})}$.

Proof First note that since u is absolutely continuous, it is a.e. differentiable on $[a,b]$, and condition (2.5) yields that u is strictly monotone increasing on $[a,b]$. Let $G := \{v \in [c,d] : g(v)$ is not defined or $|g(v)| > |g|_{L^\infty([c,d],\mathbb{R})}\}$. Then $meas(G) = 0$. Let $A := \{t \in [a,b] : g(u(t))$ is not defined or $|g(u(t))| > |g|_{L^\infty([c,d],\mathbb{R})}\}$. Clearly, $A = u^{-1}(G)$. Let $0 < \varepsilon < (b-a)/2$ be fixed. Then let $c' := u(a+\varepsilon)$, $d' := u(b-\varepsilon)$, and let $M := \operatorname{ess\,inf}\{\dot u(s)\colon\ s \in [a+\varepsilon, b-\varepsilon]\}$. Then (2.5) yields $M > 0$. Since G is of measure 0, there exist open intervals (c_i,d_i), $i \in \mathbb{N}$ such that

$$G \subset \bigcup_{i=1}^\infty (c_i,d_i) \quad \text{and} \quad \sum_{i=1}^\infty (d_i - c_i) < \varepsilon M.$$

We have

$$A = u^{-1}(G) = u^{-1}\Big(G \cap [c,c']\Big) \cup u^{-1}\Big(G \cap [c',d']\Big) \cup u^{-1}\Big(G \cap [d',d]\Big),$$

and the monotonicity of u yields $u^{-1}\Big(G \cap [c,c']\Big) \subset [a,a+\varepsilon]$, $u^{-1}\Big(G \cap [d',d]\Big) \subset [b-\varepsilon,b]$, and

$$u^{-1}\Big(G \cap [c',d']\Big) \subset u^{-1}\Big([c',d'] \cap \bigcup_{i=1}^\infty [c_i,d_i]\Big) = \bigcup_{i=1}^\infty u^{-1}\Big([c',d'] \cap [c_i,d_i]\Big) = \bigcup_{i=1}^\infty [a_i,b_i],$$

where $a_i := u^{-1}(\max\{c',c_i\})$ and $b_i := u^{-1}(\min\{d',d_i\})$. The definition of M yields

$$d_i - c_i \ge \min\{d',d_i\} - \max\{c',c_i\} = u(b_i) - u(a_i) = \int_{a_i}^{b_i} \dot u(s)\, ds \ge M(b_i - a_i).$$

Therefore $A \subset [a,a+\varepsilon] \cup [b-\varepsilon,b] \cup \bigcup_{i=1}^\infty [a_i,b_i]$, and the sum of the length of the closed intervals covering A is less than 3ε. Since $\varepsilon > 0$ is arbitrary, we get that A is Lebesgue-measurable and $meas(A) = 0$.

We show that $g \circ u$ is Lebesgue-measurable. Let $\kappa \in \mathbb{R}$, and define $G_\kappa := \{v \in [c,d] : g(v)$ is defined and $g(v) < \kappa\}$. G_κ is a Lebesgue-measurable set, since $g \in L^\infty([c,d],\mathbb{R})$. Therefore there exists a closed set F_κ such that $F_\kappa \subset G_\kappa$ and $meas(G_\kappa \setminus F_\kappa) = 0$. Since u is continuous, $u^{-1}(F_\kappa)$ is a closed set, and therefore, it is Lebesgue-measurable. Moreover, $u^{-1}(G_\kappa) = u^{-1}(F_\kappa) \cup u^{-1}(G_\kappa \setminus F_\kappa)$, and as in the first part of the proof, we get that $u^{-1}(G_\kappa \setminus F_\kappa)$ is measurable, and so is $u^{-1}(G_\kappa)$. □

Clearly, the statement of the previous Lemma is also valid if (2.5) is changed to

$$\operatorname{ess\,sup}\{\dot u(s)\colon\ s \in [a',b']\} < 0, \qquad \text{for all}\ \ [a',b'] \subset (a,b).$$

We will use the following notation.

Definition 2.6 *$\mathcal{PM}([a,b],[c,d])$ denotes the set of absolutely continuous functions $u\colon [a,b] \to [c,d]$ which are piecewise strictly monotone on $[a,b]$ in the sense that there exists a finite mesh $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$ of $[a,b]$ such that for all $i = 0,1,\ldots,m-1$ either*

$$\operatorname{ess\,inf}\{\dot u(s)\colon\ s \in [a',b']\} > 0, \qquad \textit{for all}\ \ [a',b'] \subset (t_i,t_{i+1})$$

or

$$\operatorname{ess\,sup}\{\dot u(s)\colon\ s \in [a',b']\} < 0, \qquad \textit{for all}\ \ [a',b'] \subset (t_i,t_{i+1}).$$

Lemma 2.5 implies the next result immediately.

Lemma 2.7 *Suppose $g \in L^\infty([c,d],\mathbb{R}^n)$, and $u \in \mathcal{PM}([a,b],[c,d])$. Then the composite function $g\circ u \in L^\infty([a,b],\mathbb{R}^n)$ and $|g\circ u|_{L^\infty([a,b],\mathbb{R}^n)} \le |g|_{L^\infty([c,d],\mathbb{R}^n)}$.*

The next lemma generalizes the convergence property (2.4) to the class $\mathcal{PM}$. We comment that to prove the convergence property (2.4) for $u, u^k \in \mathcal{PM}([a,b],[c,d])$, we need the stronger assumption $|u^k - u|_{W^{1,\infty}([a,b],\mathbb{R})} \to 0$ instead of $|u^k - u|_{C([a,b],\mathbb{R})} \to 0$ what is used in Lemma 2.3.

Lemma 2.8 *Suppose $g \in L^\infty([c,d],\mathbb{R}^n)$, and $u, u^k \in \mathcal{PM}([a,b],[c,d])$ $(k\in\mathbb{N})$ satisfying*

$$|u^k - u|_{W^{1,\infty}([a,b],\mathbb{R})} \to 0, \qquad \textit{as } k\to\infty. \tag{2.6}$$

Then

$$\int_a^b |g(u^k(s)) - g(u(s))|\,ds \to 0, \qquad \textit{as } k\to\infty. \tag{2.7}$$

Proof Clearly, it is enough to show (2.7) for the case when g is real valued, i.e., $n=1$.

First note that Lemma 2.7 yields $g\circ u$, $g\circ u^k \in L^\infty([a,b],\mathbb{R})$. We prove (2.7) in three steps.

(i) First suppose that $g \in L^\infty([c,d],\mathbb{R})$ is the characteristic function of an interval $[e,f] \subset [c,d]$, i.e., $g = \chi_{[e,f]}$. Then $|\chi_{[e,f]}(u^k(s)) - \chi_{[e,f]}(u(s))|$ is either 0 or 1, hence

$$meas(\{s\in[a,b]\colon\ \chi_{[e,f]}(u^k(s)) \ne \chi_{[e,f]}(u(s))\}) \le 4|u^k - u|_{C([a,b],\mathbb{R})},$$

and so

$$\int_a^b |\chi_{[e,f]}(u^k(s)) - \chi_{[e,f]}(u(s))|\,ds \le 4|u^k - u|_{C([a,b],\mathbb{R})} \to 0, \qquad \text{as } k\to\infty.$$

(ii) Suppose g is a step function, i.e., $g = \sum_{i=1}^{\ell} c_i\chi_{A_i}$, where A_i are pairwise disjoint intervals with $\cup_{i=1}^{\ell} A_i = [c,d]$. Then

$$\int_a^b |g(u^k(s)) - g(u(s))|\,ds \le \sum_{i=1}^{\ell} |c_i| 4|u^k - u|_{C([a,b],\mathbb{R})} \to 0, \qquad \text{as } k\to\infty.$$

(iii) Let $a = t_0 < t_1 < \cdots < t_m = b$ be the mesh points of u from the Definition 2.6, and let $0 < \varepsilon < \min\{t_{i+1} - t_i\colon\ i = 0,\ldots,m-1\}/2$ be fixed, and introduce $t_i' := t_i + \varepsilon$ for $i = 0,\ldots,m-1$ and $t_i'' := t_i - \varepsilon$ for $i = 1,\ldots,m$, $t_0'' := a$, $t_m' := b$, and let

$$M := \min_{i=0,\ldots,m-1}\ \operatorname*{ess\,inf}_{t\in[t_i',t_{i+1}'']} |\dot u(t)|. \tag{2.8}$$

We have $M > 0$, since $u \in \mathcal{PM}([a,b],[c,d])$.

The set of step functions is dense in $L^1([c,d],\mathbb{R})$ (see, e.g., [4]), so for a fixed $g \in L^\infty([c,d],\mathbb{R})$ and $0 < \delta < \varepsilon M/m$ there exists a step function $h\colon [c,d] \to \mathbb{R}$ such that $|g-h|_{L^1([c,d],\mathbb{R})} < \delta$. Let $h = \sum_{i=1}^{\ell} c_i \chi_{A_i}$, where A_i are pairwise disjoint intervals with $\cup_{i=1}^{\ell} A_i = [c,d]$, and define $h^* := \sum_{i=1}^{\ell} c_i^* \chi_{A_i}$, where

$$c_i^* := \begin{cases} c_i, & \text{if } c_i \le |g|_{L^\infty([c,d],\mathbb{R})}, \\ |g|_{L^\infty([c,d],\mathbb{R})}, & \text{if } c_i > |g|_{L^\infty([c,d],\mathbb{R})}, \\ -|g|_{L^\infty([c,d],\mathbb{R})}, & \text{if } c_i < -|g|_{L^\infty([c,d],\mathbb{R})}. \end{cases}$$

Then it is easy to check that

$$|g(v) - h^*(v)| \le 2|g|_{L^\infty([c,d],\mathbb{R})} \qquad \text{for a.e. } v \in [c,d], \tag{2.9}$$

and

$$\int_c^d |g(v) - h^*(v)|\,dv \le \int_c^d |g(v) - h(v)|\,dv < \delta. \tag{2.10}$$

We have therefore by using relations (2.9), (2.10), $t_i' - t_i'' = 2\varepsilon$, the definition of M in (2.8) and a change of variables $v = u(s)$

$$\begin{aligned}
&\int_a^b |g(u(s)) - h^*(u(s))|\,ds \\
&\quad= \sum_{i=0}^{m} \int_{t_i''}^{t_i'} |g(u(s)) - h^*(u(s))|\,ds + \sum_{i=0}^{m-1} \int_{t_i'}^{t_{i+1}''} |g(u(s)) - h^*(u(s))|\,ds \\
&\quad\le (m+1)2\varepsilon 2|g|_{L^\infty([c,d],\mathbb{R})} + \sum_{i=0}^{m-1} \int_{t_i'}^{t_{i+1}''} |g(u(s)) - h^*(u(s))|\dot u(s)\frac{1}{\dot u(s)}\,ds \\
&\quad\le 4\varepsilon(m+1)|g|_{L^\infty([c,d],\mathbb{R})} + \frac{1}{M}\sum_{i=0}^{m-1} \left| \int_{u(t_i')}^{u(t_{i+1}'')} |g(v) - h^*(v)|\,dv \right| \\
&\quad\le 4\varepsilon(m+1)|g|_{L^\infty([c,d],\mathbb{R})} + \frac{\delta m}{M} \\
&\quad\le 4\varepsilon(m+1)|g|_{L^\infty([c,d],\mathbb{R})} + \varepsilon.
\end{aligned}$$

Assumption (2.6) yields that there exist $k_0 > 0$ such that $|u^k - u|_{W^{1,\infty}([a,b],\,\mathbb{R})} < \frac{M}{2}$ for $k \ge k_0$. Then for $k \ge k_0$ it follows $|\dot u^k(s)| \ge \frac{M}{2}$ for a.e. $s \in [t_i', t_{i+1}'']$ and $i = 0,\dots,m-1$. Therefore similarly to the previous estimate we have for $k \ge k_0$

$$\int_a^b |g(u^k(s)) - h^*(u^k(s))|\,ds \le 4\varepsilon(m+1)|g|_{L^\infty([c,d],\mathbb{R})} + \frac{2\delta m}{M} \le 4\varepsilon(m+1)|g|_{L^\infty([c,d],\mathbb{R})} + 2\varepsilon.$$

Using the above inequalities we get

$$\begin{aligned}
&\int_a^b |g(u^k(s)) - g(u(s))|\,ds \\
&\quad\le \int_a^b |g(u^k(s)) - h^*(u^k(s))|\,ds + \int_a^b |h^*(u^k(s)) - h^*(u(s))|\,ds \\
&\qquad + \int_a^b |g(u(s)) - h^*(u(s))|\,ds \\
&\quad\le 8\varepsilon(m+1)|g|_{L^\infty([c,d],\mathbb{R})} + 3\varepsilon + \int_a^b |h^*(u^k(s)) - h^*(u(s))|\,ds, \qquad k \ge k_0,
\end{aligned}$$

which yields (2.7) using part (ii), since $\varepsilon > 0$ is arbitrary close to 0. $\square$

Lemma 2.9 *Suppose $f^{k,h} \in L^\infty([c,d],\mathbb{R}^n)$ for $k \in \mathbb{N}$ and $h \in H$ for some fixed parameter set H,*

$$\lim_{k\to\infty} \sup_{h\in H} \int_c^d |f^{k,h}(s)|\, ds = 0,$$

and there exists $A \geq 0$ such that $|f^{k,h}(s)| \leq A$ for $k \in \mathbb{N}$, $h \in H$ and a.e. $s \in [c,d]$. Let $u, u^k \in \mathcal{PM}([a,b],[c,d])$ ($k \in \mathbb{N}$) be such that (2.6) holds. Then

$$\lim_{k\to\infty} \sup_{h\in H} \int_a^b |f^{k,h}(u^k(s))|\, ds = 0.$$

Proof Let $a = t_0 < t_1 < \cdots < t_m = b$ be the mesh points of u from the Definition 2.6, and let $0 < \varepsilon < \min\{t_{i+1} - t_i \colon i = 0,\ldots,m-1\}/2$ be fixed, let t_i' and t_i'' be defined as in the proof of Lemma 2.8, and let M be defined by (2.8). Let k_0 be such that $|u^k - u|_{W^{1,\infty}([a,b],\mathbb{R})} \leq M/2$ for $k \geq k_0$. Then for $k \geq k_0$ it follows $|\dot u^k(s)| \geq \frac{M}{2}$ for a.e. $s \in [t_i', t_{i+1}'']$ and $i = 0,\ldots,m-1$. Since $u^k \in \mathcal{PM}([a,b],[c,d])$, it follows from Lemma 2.7 that $|f^{k,h}(u^k(s))| \leq A$ for $k \in \mathbb{N}$, $h \in H$ and a.e. $s \in [a,b]$. Therefore for any $k \in \mathbb{N}$ and $h \in H$ we have

$$\begin{aligned}\int_a^b |f^{k,h}(u^k(s))|\, ds &= \sum_{i=0}^{m} \int_{t_i''}^{t_i'} |f^{k,h}(u^k(s))|\, ds + \sum_{i=0}^{m-1} \int_{t_i'}^{t_{i+1}''} |f^{k,h}(u^k(s))|\, ds \\ &\leq (m+1)A2\varepsilon + \frac{2m}{M} \int_c^d |f^{k,h}(s)|\, ds.\end{aligned}$$

Then

$$\sup_{h\in H} \int_a^b |f^{k,h}(u^k(s))|\, ds \leq (m+1)A2\varepsilon + \sup_{h\in H} \frac{2m}{M} \int_c^d |f^{k,h}(s)|\, ds,$$

which proves the statement, since ε is arbitrarily close to 0. $\square$

3 Well-posedness and continuous dependence on parameters

In this section we discuss the well-posedness of the IVP (1.1)-(1.2) and Lipschitz continuous dependence of the solutions on the parameters φ, θ and ξ. Note that in this manuscript θ does not represent parameters in the delayed terms (e.g. delays) since later we will assume that f is continuously differentible wrt θ. We concentrate on parameters only in the state-dependent delayed term represented by ξ. Note that the results can be easily generalized to the case when there are several state-dependent delayed terms in the equation.

The parameters θ and ξ can be finite dimensional parameters in the formula of f and the delay function τ, respectively, but also we are interested in the case when θ and ξ are infinite dimensional parameters, e.g., coefficient functions. A simple example of (1.1) for this case is when τ and f have the form

$$\tau(t,\psi,\xi) = \bar\tau\Big(t, \psi(-\eta^1(t)),\ldots,\psi(-\eta^\ell(t)), \int_{-r}^0 A(t,\zeta)\psi(\zeta)\, d\zeta, \xi(t)\Big) \tag{3.1}$$

and

$$f(t,\psi,u,\theta) = \bar f\Big(t, \psi(-\nu^1(t)),\dots,\psi(-\nu^m(t)), \int_{-r}^0 B(t,\zeta)\psi(\zeta)\,d\zeta, u, \theta(t)\Big), \tag{3.2}$$

respectively. To keep the notation and the possible applications quite general we investigate Equation (1.1), and assume that the parameters θ and ξ belong to some normed linear spaces Θ and Ξ, respectively. The conditions on the spaces Θ and Ξ we assume later will be satisfied for finite dimensional parameters and for the cases (3.1) and (3.2) too.

We introduce the parameter space

$$\Gamma := W^{1,\infty} \times \Theta \times \Xi$$

equipped with the product norm $|\gamma|_\Gamma := |\varphi|_{W^{1,\infty}} + |\theta|_\Theta + |\xi|_\Xi$ for $\gamma = (\varphi,\theta,\xi) \in \Gamma$, and the set of admissible parameters

$$\Pi := \Big\{(\varphi,\theta,\xi) \in \Gamma\colon\ \varphi \in \Omega_1 \cap W^{1,\infty},\ \varphi(-\tau(0,\varphi)) \in \Omega_2,\ \theta \in \Omega_3,\ \xi \in \Omega_4\Big\} \subset \Gamma.$$

The next theorem shows that for every admissible parameter $\hat\gamma = (\hat\varphi,\hat\theta,\hat\xi) \in \Pi$ there exist a neighborhood P of $\hat\gamma$ and a time $\alpha > 0$ such that the IVP (1.1)-(1.2) has a unique solution on $[-r,\alpha]$ corresponding to all parameters $\gamma = (\varphi,\theta,\xi) \in P$. This solution will be denoted by $x(t,\gamma)$, and its segment function at t is denoted by $x_t(\cdot,\gamma)$.

The well-posedness of several classes of SD-DDEs was studied in many papers (see, e.g., [5, 15, 16, 19, 21, 22]. The next result is a variant of a result from [12] where the initial time is also considered as a parameter, but the parameters θ and ξ were missing in the equation. The proof is similar to that of Theorem 3.1 in [12], (see also the analogous proof of Theorem 3.2 of the neutral case in [13]), therefore it is omitted here. The notations and estimates introduced in the next theorem will be essential in the following sections.

Suppose $\Omega_1 \subset C$, $\Omega_2 \subset \mathbb{R}^n$, $\Omega_3 \subset \Theta$, $\Omega_4 \subset \Xi$ are open subsets of the respective spaces. $T > 0$ is finite or $T = \infty$, in which case $[0,T]$ denotes the interval $[0,\infty)$.

Theorem 3.1 *Assume*

(i) $f : \mathbb{R} \times C \times \mathbb{R}^n \times \Theta \supset [0,T] \times \Omega_1 \times \Omega_2 \times \Omega_3 \to \mathbb{R}^n$ *is continuous and it is locally Lipschitz continuous wrt its second, third and fourth arguments in the following sense: for every finite* $\alpha \in (0,T]$, *for every compact subset* $M_1 \subset \Omega_1$ *of* C, *compact subset* $M_2 \subset \Omega_2$ *of* $\mathbb{R}^n$, *and closed and bounded subset* $M_3 \subset \Omega_3$ *of* Θ *there exists a constant* $L_1 = L_1(\alpha, M_1, M_2, M_3)$ *such that*

$$|f(t,\psi,u,\theta) - f(t,\bar\psi,\bar u,\bar\theta)| \le L_1\Big(|\psi-\bar\psi|_C + |u-\bar u| + |\theta-\bar\theta|_\Theta\Big),$$

for $t \in [0,\alpha]$, $\psi,\bar\psi \in M_1$, $u,\bar u \in M_2$ *and* $\theta,\bar\theta \in M_3$;

(ii) $\tau : \mathbb{R} \times C \times \Xi \supset [0,T] \times \Omega_1 \times \Omega_4 \to [0,r] \subset \mathbb{R}$ *is continuous and it is locally Lipschitz continuous wrt its second and third arguments in the following sense: for every finite* $\alpha \in (0,T]$, *compact subset* $M_1 \subset \Omega_1$ *of* C, *and closed and bounded subset* $M_4 \subset \Omega_4$ *of* Ξ *there exists a constant* $L_2 = L_2(\alpha, M_1, M_4)$ *such that*

$$|\tau(t,\psi,\xi) - \tau(t,\bar\psi,\bar\xi)| \le L_2\Big(|\psi-\bar\psi|_C + |\xi-\bar\xi|_\Xi\Big)$$

for $t \in [0,\alpha]$, $\psi,\bar\psi \in M_1$, $\xi,\bar\xi \in M_4$.

Then there exist a radius $\delta > 0$ and a time $0 < \alpha \leq T$ such that

(i) for all $\gamma = (\varphi, \theta, \xi) \in P := \mathcal{B}_\Gamma(\hat\gamma; \delta)$ the IVP (1.1)-(1.2) *has a unique solution $x(t, \gamma)$ on $[-r, \alpha]$;*

(ii) there exist $M_1^* \subset \Omega_1$ and $M_2^* \subset \Omega_2$ compact and convex subsets of the spaces C and $\mathbb{R}^n$, respectively, such that

$$x_t(\cdot, \gamma) \in M_1^* \quad \text{and} \quad x(t - \tau(t, x_t(\cdot, \gamma), \xi), \gamma) \in M_2^*, \qquad \gamma = (\varphi, \theta, \xi) \in P,\ t \in [0, \alpha]; \tag{3.3}$$

and

(iii) $x_t(\cdot, \gamma) \in W^{1,\infty}$ for $\gamma \in P$ and $t \in [0, \alpha]$, and there exist a bound $N = N(\alpha, \delta)$ and a Lipschitz constant $L_0 = L_0(\alpha, \delta)$ such that

$$|x_t(\cdot, \gamma)|_{W^{1,\infty}} \leq N, \qquad \gamma \in P,\ t \in [0, \alpha], \tag{3.4}$$

and

$$|x_t(\cdot, \gamma) - x_t(\cdot, \bar\gamma)|_{W^{1,\infty}} \leq L_0 |\gamma - \bar\gamma|_\Gamma, \qquad \gamma \in P,\ t \in [0, \alpha]. \tag{3.5}$$

We note that the strong assumptions (i) and (ii) of Theorem 3.1 that f and τ are locally Lipschitz continuous on closed and bounded subsets of the parameter spaces Θ and Ξ are needed to guarantee the existence and uniqueness of the solutions in an open ball of Γ. If Θ and Ξ are locally compact normed linear spaces then this property can be changed to the usual local Lipschitz continuity (on compact sets).

The following result is obvious.

Remark 3.2 Suppose the conditions of Theorem 3.1 hold, P and α are defined by Theorem 3.1, and let $\mathcal{P}$ denote the subset of P consisting of those parameters which satisfy the compatibility condition, i.e.,

$$\mathcal{P} := \Big\{ (\varphi, \theta, \xi) \in P\colon\ \varphi \in C^1, \quad \dot\varphi(0-) = f(0, \varphi, \varphi(-\tau(0, \varphi, \xi)), \theta) \Big\} \subset \Gamma. \tag{3.6}$$

Then for all parameter values $\gamma \in \mathcal{P}$ the corresponding solution $x(t, \gamma)$ is continuously differentiable wrt t for $t \in [-r, \alpha]$.

Motivated by Theorem 3.1 throughout the rest of the paper we will assume that

(A0) there exist an open subset P of Γ, a time $0 < \alpha \leq T$, $M_1^* \subset \Omega_1$ and $M_2^* \subset \Omega_2$ compact and convex subsets of the spaces C and $\mathbb{R}^n$, respectively, such that the solution $x(t, \gamma)$ of the IVP (1.1)-(1.2) exists and it is unique on the interval $[-r, \alpha]$ for all $\gamma \in P$, moreover relations (3.3), (3.4) and (3.5) are satisfied for $\gamma \in P$ and $t \in [0, \alpha]$.

4 First-order differentiability wrt the parameters

In this section we study the differentiability of the solution $x(t,\gamma)$ of the IVP (1.1)-(1.2) wrt γ.

We start this section with an example which shows a case when the solution of a state-dependent delay equation is not differentiable wrt a parameter.

Example 4.1 Consider the scalar initial value problem

$$\dot{x}(t) = x(t - cx(t)), \qquad t \in [0,2], \tag{4.1}$$

$$x(t) = \begin{cases} t+2, & t \in [-2,-1), \\ 1, & t \in [-1,0], \end{cases} \tag{4.2}$$

where $c > 0$.

If $c \in (2/3, 1]$, then the solution of the IVP (4.1)-(4.2) is

$$x(t,c) = x(t) = t+1,$$

since the time lag function is

$$t - cx(t) = t - c(t+1) = (1-c)t - c \in [-1,0], \qquad t \in [0,2].$$

If $c \in (1,2)$, then $0 - cx(0) = -c \in (-2,-1)$, so for small $t > 0$ the IVP (4.1)-(4.2) is equivalent to

$$\dot{x}(t) = t - cx(t) + 2, \qquad x(0) = 1,$$

i.e.,

$$x(t) = \frac{1}{c^2}\left((1-c)^2 e^{-ct} + ct + 2c - 1\right).$$

We have $t - cx(t) \in [-2,-1)$ for $t \geq 0$. Therefore the solution of the IVP (4.1)-(4.2) is

$$x(t,c) = \begin{cases} t+1, & c \in (2/3,1], \\ \frac{1}{c^2}\left((1-c)^2 e^{-ct} + ct + 2c - 1\right), & c \in (1,2). \end{cases}$$

For $c \in (2/3, 1)$ we have

$$D_2 x(t,c) = 0, \qquad t \in [0,2],$$

and for $c \in (1,2)$ and $t \in [0,2]$

$$\begin{aligned} D_2 x(t,c) &= -\frac{2}{c^3}\left((1-c)^2 e^{-ct} + ct + 2c - 1\right) \\ &\quad + \frac{1}{c^2}\left(-2(1-c)e^{-ct} - (1-c)^2 c e^{-ct} + t + 2\right). \end{aligned}$$

For $c \in (1,2)$ and $t \in [0,2]$

$$\begin{aligned} \frac{x(t,c) - x(t,1)}{c-1} &= \frac{1}{c-1}\left[\frac{1}{c^2}\left((1-c)^2 e^{-ct} + ct + 2c - 1\right) - (t+1)\right] \\ &= \frac{1}{c^2}\left[(c-1)e^{-ct} - ct + 1 - c\right] \\ &\to -t, \qquad \text{ha } c \to 1+. \end{aligned}$$

Therefore $x(t,c)$ is not differentiable wrt c at $c = 1$ for any $t > 0$. Note that for $c = 1$ the time lag is $t - cx(t) = -1$ for $t \geq 0$, and the initial function is not differentiable at -1. □

In Theorem 4.9 below we will show differentiability of the solution wrt the parameters under the condition that the time lag function is piecewise monotone. The above example illustrates that this monotonicity assumption is essential in the proofs, since the failure of this condition may result in the loss of differentiability wrt the parameters.

Beside of assumption (A0) for the first-order differentiability results we suppose

(A1) (i) $f : \mathbb{R} \times C \times \mathbb{R}^n \times \Theta \supset [0,T] \times \Omega_1 \times \Omega_2 \times \Omega_3 \to \mathbb{R}^n$ is continuous and it is continuously differentiable wrt its second, third and fourth arguments;

(A2) (i) $\tau : [0,T] \times C \times \Xi \supset [0,T] \times \Omega_1 \times \Omega_4 \to \mathbb{R}$ is continuous and it is continuously differentiable wrt its second and third arguments;

(ii) $\tau(t,\psi,\xi)$ is locally Lipschitz continuous in t, i.e., for every finite $\alpha \in (0,T]$, compact subset $M_1 \subset \Omega_1$ of C, and compact subset $M_4 \subset \Omega_4$ of Ξ there exists a constant $L_2' = L_2'(\alpha, M_1, M_4)$ such that

$$|\tau(t,\psi,\xi) - \tau(\bar{t},\psi,\xi)| \le L_2'|t-\bar{t}|$$

for $t, \bar{t} \in [0,\alpha]$, $\psi \in M_1$, $\xi \in M_4$;

(iii) for every finite $\alpha \in (0,T]$, compact subset $M_1 \subset \Omega_1$ of C, and compact subset $M_4 \subset \Omega_4$ of Ξ there exists $L_3 = L_3(\alpha, M_1, M_4) \ge 0$ such that

$$\Big|\frac{d}{dt}\tau(t,y_t,\xi) - \frac{d}{dt}\tau(t,\bar{y}_t,\bar{\xi})\Big| \le L_3\Big(|y_t - \bar{y}_t|_{W^{1,\infty}} + |\xi - \bar{\xi}|_{\Xi}\Big), \quad \text{a.e. } t \in [0,\alpha],$$

where $\xi, \bar{\xi} \in M_4$, and $y, \bar{y} \in W^{1,\infty}([-r,\alpha],\mathbb{R}^n)$ are such that $y_t, \bar{y}_t \in M_1$ for $t \in [0,\alpha]$.

We note that (A2) (iii) holds under natural assumptions for functions of the form (3.1). Here $\Xi = W^{1,\infty}([0,T],\mathbb{R})$ can be used, and then we have under straightforward assumptions that for a.e. $t \in [0,\alpha]$, $y \in W^{1,\infty}([-r,\alpha],\mathbb{R}^n)$

$$\begin{aligned}
\frac{d}{dt}\tau(t,y_t,\xi) &= D_1\bar{\tau}\Big(t, y(t-\eta^1(t)),\ldots,y(t-\eta^\ell(t)), \int_{-r}^0 A(t,\zeta)y(t+\zeta)\,d\zeta, \xi(t)\Big) \\
&\quad + \sum_{i=1}^{\ell} D_{i+1}\bar{\tau}\Big(t, y(t-\eta^1(t)),\ldots,y(t-\eta^\ell(t)), \int_{-r}^0 A(t,\zeta)y(t+\zeta)\,d\zeta, \xi(t)\Big) \\
&\qquad \times \dot{y}(t-\eta^i(t))(1-\dot{\eta}^i(t)) \\
&\quad + D_{\ell+2}\bar{\tau}\Big(t, y(t-\eta^1(t)),\ldots,y(t-\eta^\ell(t)), \int_{-r}^0 A(t,\zeta)y(t+\zeta)\,d\zeta, \xi(t)\Big) \\
&\qquad \times \int_{-r}^0 [D_1A(t,\zeta)y(t+\zeta) + A(t,\zeta)\dot{y}(t+\zeta)]\,d\zeta \\
&\quad + D_{\ell+3}\bar{\tau}\Big(t, y(t-\eta^1(t)),\ldots,y(t-\eta^\ell(t)), \int_{-r}^0 A(t,\zeta)y(t+\zeta)\,d\zeta, \xi(t)\Big)\dot{\xi}(t).
\end{aligned}$$

So if $\bar{\tau}$ is twice continuously differentiable and η^i are continuously differentiable, then it is easy to argue that (A2) (iii) holds.

The proof of our differentiability results will be based on the following lemmas.

Lemma 4.2 *Let $y \in W^{1,\infty}([-r,\alpha],\mathbb{R}^n)$, $\omega_k \in (0,\infty)$ ($k \in \mathbb{N}$) be a sequence satisfying $\omega_k \to 0$ as $k \to \infty$. Let $u, u^k \in \mathcal{PM}([0,\alpha],[-r,\alpha])$ ($k \in \mathbb{N}$) be such that*

$$|u^k - u|_{W^{1,\infty}([0,\alpha],\mathbb{R})} \le \omega_k, \qquad k \in \mathbb{N}. \tag{4.3}$$

Then

$$\lim_{k\to\infty} \frac{1}{\omega_k} \int_0^\alpha |y(u^k(s)) - y(u(s)) - \dot y(u(s))(u^k(s) - u(s))|\, ds = 0. \tag{4.4}$$

Proof Let $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = \alpha$ be the mesh points of u from the Definition 2.6, and let $0 < \varepsilon < \min\{t_{i+1} - t_i :\ i = 0,\ldots,m-1\}/2$ be fixed, and introduce $t_i' := t_i + \varepsilon$ for $i = 0,\ldots,m-1$, $t_i'' := t_i - \varepsilon$ for $i = 1,\ldots,m$, $t_0'' := 0$, $t_m' := \alpha$, and let

$$M := \min_{i=0,\ldots,m-1} \operatorname*{ess\,inf}_{t\in[t_i',t_{i+1}'']} |\dot u(t)|.$$

We have $M > 0$, since $u \in \mathcal{PM}([0,\alpha],[-r,\alpha])$. Assumption (4.3) yields that there exists $k_0 > 0$ such that $|u^k - u|_{W^{1,\infty}([0,\alpha],\mathbb{R})} < \frac{M}{2}$ for $k \ge k_0$. Then for $k \ge k_0$ it follows $|\dot u^k(s)| \ge \frac{M}{2}$ and $|\dot u(s) + \nu(\dot u^k(s) - \dot u(s))| \ge \frac{M}{2}$ for a.e. $s \in [t_i', t_{i+1}'']$, $i = 0,\ldots,m-1$ and $\nu \in [0,1]$. Let $a_0 := |y|_{W^{1,\infty}}([-r,\alpha],\mathbb{R}^n)$. Then simple manipulations, (4.3) and Fubini's theorem yield

$$\begin{aligned}
&\int_0^\alpha |y(u^k(s)) - y(u(s)) - \dot y(u(s))(u^k(s) - u(s))|\, ds \\
&\quad\le \sum_{i=0}^{m} \int_{t_i''}^{t_i'} \Big(|y(u^k(s)) - y(u(s))| + |\dot y(u(s))||u^k(s) - u(s)| \Big)\, ds \\
&\qquad + \sum_{i=0}^{m-1} \int_{t_i'}^{t_{i+1}''} \Big| \int_{u(s)}^{u^k(s)} \Big(\dot y(v) - \dot y(u(s))\Big) dv \Big|\, ds \\
&\quad\le (m+1)2\varepsilon 2a_0 |u^k - u|_{C([0,\alpha],\mathbb{R})} \\
&\qquad + \sum_{i=0}^{m-1} \int_{t_i'}^{t_{i+1}''} \Big| \int_0^1 \Big[\dot y\Big(u(s) + \nu(u^k(s) - u(s))\Big) - \dot y(u(s))\Big](u^k(s) - u(s))\, d\nu \Big|\, ds \\
&\quad\le \omega_k \Big[(m+1)4a_0\varepsilon + \sum_{i=0}^{m-1} \int_0^1 \int_{t_i'}^{t_{i+1}''} \Big|\dot y\Big(u(s) + \nu(u^k(s) - u(s))\Big) - \dot y(u(s))\Big|\, ds\, d\nu\Big].
\end{aligned}$$

It follows from Lemma 2.3 and Remark 2.4 that for every $\nu \in [0,1]$

$$\lim_{k\to\infty} \int_{t_i'}^{t_{i+1}''} \Big|\dot y\Big(u(s) + \nu(u^k(s) - u(s))\Big) - \dot y(u(s))\Big|\, ds = 0, \qquad i = 0,\ldots,m-1,$$

hence we get by using the Lebesgue's Dominated Convergence Theorem that

$$\limsup_{k\to\infty} \frac{1}{\omega_k} \int_0^\alpha |y(u^k(s)) - y(u(s)) - \dot y(u(s))(u^k(s) - u(s))|\, ds \le (m+1)4a_0\varepsilon.$$

This concludes the proof of (4.4), since $\varepsilon > 0$ can be arbitrary close to 0. □

We introduce the notations

$$\begin{aligned}\omega_f(t,\bar\psi,\bar u,\bar\theta,\psi,u,\theta) &:= f(t,\psi,u,\theta)-f(t,\bar\psi,\bar u,\bar\theta)-D_2f(t,\bar\psi,\bar u,\bar\theta)(\psi-\bar\psi)\\ &\quad -D_3f(t,\bar\psi,\bar u,\bar\theta)(u-\bar u)-D_4f(t,\bar\psi,\bar u,\bar\theta)(\theta-\bar\theta), \qquad (4.5)\\ \omega_\tau(t,\bar\psi,\bar\xi,\psi,\xi) &:= \tau(t,\psi,\xi)-\tau(t,\bar\psi,\bar\xi)-D_2\tau(t,\bar\psi,\bar\xi)(\psi-\bar\psi)\\ &\quad -D_3\tau(t,\bar\psi,\bar\xi)(\xi-\bar\xi) \qquad (4.6)\end{aligned}$$

for $t\in[0,T]$, $\bar\psi,\psi\in\Omega_1$, $\bar u,u\in\Omega_2$, $\bar\theta,\theta\in\Omega_3$, $\bar\xi,\xi\in\Omega_4$.

The following result is an easy generalization of Lemma 4.2 of [12] for the IVP (1.1)-(1.2), therefore we omit its proof here. (See also the related proof of Lemma 5.8 below.)

Lemma 4.3 (see [12]) *Suppose (A0), (A1) (i) and (A2) (i). Let $\gamma=(\varphi,\theta,\xi)\in P$ be fixed, and $h_k=(h_k^\varphi,h_k^\theta,h_k^\xi)\in\Gamma$ ($k\in\mathbb{N}$) be a sequence satisfying $|h_k|_\Gamma\to 0$ as $k\to\infty$, and $\gamma+h_k\in P$ for $k\in\mathbb{N}$. Let $x(t):=x(t,\gamma)$, $x^k(t):=x(t,\gamma+h_k)$, $u(t):=t-\tau(t,x_t,\xi)$ and $u^k(t):=t-\tau(t,x^k,\xi+h_k^\xi)$. Then*

$$\lim_{k\to\infty}\frac{1}{|h_k|_\Gamma}\int_0^\alpha|\omega_f(s,x_s,x(u(s)),\theta,x_s^k,x^k(u^k(s)),\theta+h_k^\theta)|\,ds=0 \qquad (4.7)$$

and

$$\lim_{k\to\infty}\frac{1}{|h_k|_\Gamma}\int_0^\alpha|\omega_\tau(s,x_s,\xi,x_s^k,\xi+h_k^\xi)|\,ds=0. \qquad (4.8)$$

A solution $x(\cdot,\gamma)$ of the IVP (1.1)-(1.2) for $\gamma\in P$ is, in general, only a $W^{1,\infty}$-function on the interval $[-r,0]$, but it is continuously differentiable for $t\ge 0$. In [16] (see also [12]) a parameter set

$$P_1:=\{\gamma=(\varphi,\theta,\xi)\in P\colon\ x(\cdot,\gamma)\in X(\alpha,\xi)\}$$

was considered, where

$$\begin{aligned}X(\alpha,\xi) &:= \Big\{x\in W^{1,\infty}([-r,\alpha],\mathbb{R}^n)\colon\ x_t\in\Omega_1,\ x(t-\tau(t,x_t,\xi))\in\Omega_2 \text{ for } t\in[0,\alpha],\\ &\qquad \text{and}\quad \operatorname{ess\,inf}\Big\{\frac{d}{dt}(t-\tau(t,x_t,\xi))\colon\ \text{a.e. } t\in[0,\alpha^*]\Big\}>0\Big\}\end{aligned}$$

and $\alpha^*:=\min\{r,\alpha\}$. Then Lemma 2.3 yields that the function $t\mapsto\dot x(t-\tau(t,x_t,\xi))$ is well-defined for a.e. $t\in[0,\alpha^*]$, it is integrable on $[0,\alpha^*]$, and it is well-defined and continuous on $[\alpha^*,\alpha]$. Note that it was shown in [16] (see also [12]) that P_1 is an open subset of the parameter set P. In this section we relax this condition. We define the parameter set

$$\begin{aligned}P_2 &:= \{\gamma=(\varphi,\theta,\xi)\in P\colon \text{ the map } [0,\alpha^*]\to\mathbb{R},\ t\mapsto t-\tau(t,x_t(\cdot,\gamma),\xi)\\ &\qquad \text{belongs to } \mathcal{PM}([0,\alpha^*],[-r,\alpha^*])\}\subset\Gamma. \qquad (4.9)\end{aligned}$$

Then we have $P_1\subset P_2\subset P\subset\Gamma$, and Lemma 2.7 yields that for a solution x corresponding to parameter $\gamma\in P_2$ the function $t\mapsto\dot x(t-\tau(t,x_t,\xi))$ is well-defined for a.e. $t\in[0,\alpha^*]$ and it is integrable on $[0,\alpha^*]$. Therefore, as the next discussion will show, the parameter set where the variational equation is defined, and correspondingly the differentiability of the solution wrt the parameters can be obtained is larger than in the previous papers [9, 12, 16].

Let $\gamma = (\varphi, \theta, \xi) \in P_2$ be fixed, and let $x(t) := x(t, \gamma)$. Consider the space $C \times \Theta \times \Xi$ equipped with the product norm $|(h^\varphi, h^\theta, h^\xi)|_{C\times\Theta\times\Xi} := |h^\varphi|_C + |h^\theta|_\Theta + |h^\xi|_\Xi$. Then for a.e. $t \in [0, \alpha]$ we introduce the linear operator $L(t, x)\colon C \times \Theta \times \Xi \to \mathbb{R}^n$ by

$$\begin{aligned} L(t,x)&(h^\varphi, h^\theta, h^\xi) \\ :=\ & D_2 f(t, x_t, x(t-\tau(t,x_t,\xi)), \theta)h^\varphi + D_3 f(t, x_t, x(t-\tau(t,x_t,\xi)), \theta) \\ & \times\Big[-\dot{x}(t-\tau(t,x_t,\xi))\Big(D_2\tau(t,x_t,\xi)h^\varphi + D_3\tau(t,x_t,\xi)h^\xi\Big) + h^\varphi(-\tau(t,x_t,\xi))\Big] \\ & + D_4 f(t, x_t, x(t-\tau(t,x_t,\xi)), \theta)h^\theta \end{aligned} \tag{4.10}$$

for $(h^\varphi, h^\theta, h^\xi) \in C \times \Theta \times \Xi$. We have by (A1) (i) and (A2) (i) and the compactness of M_1^* and M_2^* that the following constants are well-defined

$$L_1(\theta) := \max_{i=2,3,4} \max\{|D_i f(t, \psi, u, \theta)|_{\mathcal{L}(Y_i,\mathbb{R}^n)} \colon t \in [0,\alpha],\ \psi \in M_1^*,\ u \in M_2^*\} \tag{4.11}$$

and

$$L_2(\xi) := \max_{i=2,3} \max\{|D_i \tau(t, \psi, \xi)|_{\mathcal{L}(Z_i,\mathbb{R})} \colon t \in [0,\alpha],\ \psi \in M_1^*\}, \tag{4.12}$$

where $Y_2 = Z_2 = C$, $Y_3 = \mathbb{R}^n$, $Y_4 = \Theta$, $Z_3 = \Xi$. Then (3.4), (4.11) and (4.12) yield

$$\begin{aligned} |L(t,x)(h^\varphi, h^\theta, h^\xi)| &\le L_1(\theta)|h^\varphi|_C + L_1(\theta)\Big[NL_2(\xi)(|h^\varphi|_C + |h^\xi|_\Xi) + |h^\varphi|_C\Big] + L_1(\theta)|h^\theta|_\Theta \\ &\le L_1(\theta)(NL_2(\xi)+3)|(h^\varphi, h^\theta, h^\xi)|_{C\times\Theta\times\Xi}, \qquad \text{a.e. } t \in [0,\alpha]. \end{aligned} \tag{4.13}$$

Therefore

$$|L(t,x)|_{\mathcal{L}(C\times\Theta\times\Xi,\mathbb{R}^n)} \le L_1(\theta)(NL_2(\xi)+3), \qquad \text{a.e. } t \in [0,\alpha].$$

Hence $L(t,x)$ is a bounded linear operator for all t for which $\dot{x}(t-\tau(t,x_t,\xi))$ exists, i.e., for a.e. $t \in [0,\alpha]$.

For $\gamma \in P_2$ we define the variational equation associated to $x = x(\cdot, \gamma)$ as

$$\dot{z}(t) = L(t,x)(z_t, h^\theta, h^\xi) \qquad \text{a.e. } t \in [0,\alpha], \tag{4.14}$$

$$z(t) = h^\varphi(t), \qquad t \in [-r, 0], \tag{4.15}$$

where $h = (h^\varphi, h^\theta, h^\xi) \in C \times \Theta \times \Xi$ is fixed. The IVP (4.14)-(4.15) is a Carathéodory type linear delay equation. By its solution we mean a continuous function $z\colon [-r,\alpha] \to \mathbb{R}^n$, which is absolutely continuous on $[0,\alpha]$, and it satisfies (4.14) for a.e. $t \in [0,\alpha]$ and (4.15) for all $t \in [-r, 0]$. Standard argument ([3], [7]) shows that the IVP (4.14)-(4.15) has a unique solution $z(t) = z(t,\gamma,h)$ for $t \in [-r,\alpha]$, $\gamma \in P_2$ and $h = (h^\varphi, h^\theta, h^\xi) \in C \times \Theta \times \Xi$.

The following result was proved in [12] for the parameter set P_1 (see Lemma 4.4 in [12]), but the proof is identical for the parameter set P_2, as well.

Lemma 4.4 (see [12]) *Assume (A0), (A1) (i) and (A2) (i). Let $\gamma \in P_2$, and $x(t) := x(t,\gamma)$ for $t \in [-r,\alpha]$. Let $h \in C \times \Theta \times \Xi$ and let $z(t,\gamma,h)$ be the corresponding solution of the IVP* (4.14)-(4.15) *on $[-r,\alpha]$. Then*

(i) $z(t,\gamma,\cdot) \in \mathcal{L}(C\times\Theta\times\Xi, \mathbb{R}^n)$, the map $C\times\Theta\times\Xi \to C$, $h \mapsto z_t(\cdot,\gamma,h)$ is in $\mathcal{L}(C\times\Theta\times\Xi, C)$, and there exists $N_1 \ge 0$ such that

$$|z(t,\gamma,h)| \le |z_t(\cdot,\gamma,h)|_C \le N_1|h|_{C\times\Theta\times\Xi}, \qquad t \in [0,\alpha],\ \gamma \in P_2,\ h \in C\times\Theta\times\Xi; \tag{4.16}$$

(ii) there exists $N_2 \geq 0$ such that

$$|z_t(\cdot,\gamma,h)|_{W^{1,\infty}} \leq N_2|h|_\Gamma, \qquad t\in[0,\alpha],\ \gamma\in P_2,\ h\in\Gamma. \tag{4.17}$$

In Lemma 4.7 below we show that the linear operators $z(t,\gamma,\cdot)$ and $z_t(\cdot,\gamma,\cdot)$ are continuous in t and γ, assuming that γ belongs to P_2. First we prove that the time lag function $u(t) := t-\tau(t,x_t,\xi)$ along the solution $x(t)=x(t,\gamma)$ depends continuously on the parameters. Moreover, under the additional smoothness assumption (A2) (ii) and (iii), we show that the map $\Gamma\supset P\ni\gamma\mapsto u(\cdot)\in W^{1,\infty}([0,\alpha],\mathbb{R})$ is continuous. This is the key property in order to apply Lemma 2.8 in the proof of the next lemma.

Lemma 4.5 *Assume (A0), (A1) (i) and (A2) (i), $\gamma=(\varphi,\xi,\theta)\in P$, $h_k=(h_k^\varphi,h_k^\xi,h_k^\theta)\in\Gamma$ is a sequence such that $\gamma+h_k\in P$ for $k\in\mathbb{N}$ and $|h_k|_\Gamma\to 0$ as $k\to\infty$. Let $x(t):=x(t,\gamma)$, $x^k(t):=x(t,\gamma+h_k)$ be the corresponding solutions of the IVP* (1.1)-(1.2)*, and $u^k(s):=t-\tau(t,x_t^k,\xi+h_k^\xi)$ and $u(t):=t-\tau(t,x_t,\xi)$. Then there exists $K_0=K_0(\gamma,h_k)\geq 0$ such that*

$$|u^k(t)-u(t)|\leq K_0|h_k|_\Gamma, \qquad t\in[0,\alpha], \quad k\in\mathbb{N}. \tag{4.18}$$

If, in addition, (A2) (ii) holds, then $u,u^k\in W^{1,\infty}([0,\alpha],\mathbb{R})$, and if (A2) (iii) is also satisfied, then there exists $K_1=K_1(\gamma,h_k)\geq 0$ such that

$$|u^k-u|_{W^{1,\infty}([0,\alpha],\mathbb{R})}\leq K_1|h_k|_\Gamma, \qquad k\in\mathbb{N}. \tag{4.19}$$

Proof Define the compact set $M_4^*:=\{\xi\}\cup\{\xi+h_k^\xi\colon k\in\mathbb{N}\}$. Then by the compatness of M_1^* and M_4^* and assumption (A2) (i) we get that

$$L_2^*:=\max_{i=2,3}\max\{|D_i\tau(t,\psi,\xi)|_{\mathcal{L}(Z_i,\mathbb{R})}\colon t\in[0,\alpha],\ \psi\in M_1^*,\ \xi\in M_4^*\} \tag{4.20}$$

is finite, where $Z_2=C$ and $Z_3=\Xi$. Then the definition of M_1^* and M_4^* and the Mean Value Theorem imply

$$|u^k(t)-u(t)|=|\tau(t,x_t^k,\xi+h_k^\xi)-\tau(t,x_t,\xi)|\leq L_2^*(|x_t^k-x_t|_C+|h_k^\xi|_\Xi), \quad t\in[0,\alpha],$$

so (3.5) yields (4.18) with $K_0:=L_2^*(L_0+1)$.

Now assume (A2) (ii) also holds, and let $L_2'=L_2'(\alpha,M_1^*,M_3^*)$ be the Lipschitz constant from (A2) (ii), and let $L_2'':=\max\{L_2^*,L_2'\}$. For simplicity of the notation let $h_0:=0=(0,0,0)\in\Gamma$, and so $x^0:=x$ and $u^0:=u$. Then (A2) (ii), the Mean Value Theorem and (3.4) imply for $k\in\mathbb{N}_0$ and $t,\bar{t}\in[0,\alpha]$

$$\left|\tau(t,x_t^k,\xi+h_k^\xi)-\tau(\bar{t},x_{\bar{t}}^k,\xi+h_k^\xi)\right|\leq L_2''(|t-\bar{t}|+|x_t^k-x_{\bar{t}}^k|_C)\leq L_2''(1+N)|t-\bar{t}|. \tag{4.21}$$

Hence u^k is Lipschitz continuous, and so it is almost everywhere differentiable on $[0,\alpha]$, and $|\dot{u}^k|_{L^\infty([0,\alpha],\mathbb{R})}\leq L_2''(1+N)$ for $k\in\mathbb{N}_0$. Therefore $u^k\in W^{1,\infty}([0,\alpha],\mathbb{R})$ for $k\in\mathbb{N}_0$.

Let $L_3^*=L_3(\alpha,M_1^*,M_4^*)$ be defined by (A2) (iii). Assumption (A2) (iii) and (3.5) give

$$|\dot{u}^k(t)-\dot{u}(t)|=\left|\frac{d}{dt}\tau(t,x_t^k,\xi+h_k^\xi)-\frac{d}{dt}\tau(t,x_t,\xi)\right|\leq L_3^*(|x_t^k-x_t|_C+|h_k^\xi|_\Xi)\leq L_3^*(L_0+1)|h_k|_\Gamma$$

for a.e. $t\in[0,\alpha]$. Therefore (4.19) holds with $K_1:=\max\{K_0,L_3^*(L_0+1)\}$. $\square$

The next result shows a key estimate will be used in the proofs of Lemma 4.7 and Lemma 5.14.

Lemma 4.6 *Assume (A0), (A1) (i) and (A2) (i). Let* $\gamma \in P_2$, $h_k = (h_k^\varphi, h_k^\theta, h_k^\xi) \in \Gamma$ $(k \in \mathbb{N})$ *be a sequence such that* $|h_k|_\Gamma \to 0$ *as* $k \to \infty$, *and* $\gamma + h_k \in P_2$ *for* $k \in \mathbb{N}$. *Let* $x(s) := x(s, \gamma)$, $x^k(s) := x(s, \gamma + h_k)$, $u(s) := s - \tau(s, x_s, \xi)$, *and* $u^k(s) := s - \tau(s, x_s^k, \xi + h_k^\xi)$. *Then there exist a nonnegative sequence* $c_{0,k} \to 0$ *as* $k \to \infty$ *and a constant* $N_3 = N_3(\gamma, h_k)$ *such that*

$$|L(s, x^k)h - L(s, x)h| \leq c_{0,k}|h|_\Gamma + N_3|\dot{x}(u^k(s)) - \dot{x}(u(s))||h|_\Gamma \tag{4.22}$$

for a.e. $s \in [0, \alpha]$, $k \in \mathbb{N}$ *and* $h \in \Gamma$.

Proof Let $\gamma = (\varphi, \theta, \xi) \in P_2$ be fixed. We have

$$\begin{aligned}
&L(s, x^k)(h^\varphi, h^\theta, h^\xi) - L(s, x)(h^\varphi, h^\theta, h^\xi) \\
&\quad = \Big(D_2 f(s, x_s^k, x^k(u^k(s)), \theta + h_k^\theta) - D_2 f(s, x_s, x(u(s)), \theta)\Big)h^\varphi \\
&\qquad + \Big(D_3 f(s, x_s^k, x^k(u^k(s)), \theta + h_k^\theta) - D_3 f(s, x_s, x(u(s)), \theta)\Big) \\
&\qquad\quad \times \Big(-\dot{x}^k(u^k(s))\Big)\Big(D_2\tau(s, x_s^k, \xi + h_k^\xi)h^\varphi + D_3\tau(s, x_s^k, \xi + h_k^\xi)h^\xi\Big) \\
&\qquad + D_3 f(s, x_s, x(u(s)), \theta)\Big(-\dot{x}^k(u^k(s)) + \dot{x}(u^k(s)))\Big) \\
&\qquad\quad \times \Big(D_2\tau(s, x_s^k, \xi + h_k^\xi)h^\varphi + D_3\tau(s, x_s^k, \xi + h_k^\xi)h^\xi\Big) \\
&\qquad + D_3 f(s, x_s, x(u(s)), \theta)\Big(-\dot{x}(u^k(s)) + \dot{x}(u(s)))\Big) \\
&\qquad\quad \times \Big(D_2\tau(s, x_s^k, \xi + h_k^\xi)h^\varphi + D_3\tau(s, x_s^k, \xi + h_k^\xi)h^\xi\Big) \\
&\qquad + D_3 f(s, x_s, x(u(s)), \theta)\Big(-\dot{x}(u(s))\Big) \\
&\qquad\quad \times \Big[\Big(D_2\tau(s, x_s^k, \xi + h_k^\xi) - D_2\tau(s, x_s, \xi)\Big)h^\varphi \\
&\qquad\qquad + \Big(D_3\tau(s, x_s^k, \xi + h_k^\xi) - D_3\tau(s, x_s, \xi)\Big)h^\xi\Big] \\
&\qquad + \Big(D_3 f(s, x_s^k, x^k(u^k(s)), \theta + h_k^\theta) - D_3 f(s, x_s, x(u(s)), \theta)\Big)h^\varphi(-\tau(s, x_s^k, \xi + h_k^\xi)) \\
&\qquad + D_3 f(s, x_s, x(u(s)), \theta)\Big(h^\varphi(-\tau(s, x_s^k, \xi + h_k^\xi)) - h^\varphi(-\tau(s, x_s, \xi))\Big) \\
&\qquad + \Big(D_4 f(s, x_s^k, x^k(u^k(s)), \theta + h_k^\theta) - D_4 f(s, x_s, x(u(s)), \theta)\Big)h^\theta, \qquad s \in [0, \alpha].
\end{aligned}$$

Relations (3.4), (3.5), (4.18) and the Mean Value Theorem give

$$\begin{aligned}
|x^k(u^k(s)) - x(u(s))| &\leq |x^k(u^k(s)) - x(u^k(s))| + |x(u^k(s)) - x(u(s))| \\
&\leq L_0|h_k|_\Gamma + N|u^k(s) - u(s)| \\
&\leq K_2|h_k|_\Gamma,
\end{aligned} \tag{4.23}$$

with $K_2 := L_0 + NK_0$, where the constants L_0, N and K_0 are defined in (3.5), (3.4) and (4.18), respectively, and

$$|x_s^k - x_s|_C + |x^k(u^k(s)) - x(u(s))| + |h_k^\theta|_\Theta \leq K_3|h_k|_\Gamma, \tag{4.24}$$

with $K_3 := L_0 + K_2 + 1$, and

$$|x_s^k - x_s|_C + |h_k^\xi|_\Xi \le (L_0+1)|h_k|_\Gamma. \tag{4.25}$$

Define the sets $M_3^* := \{\theta\} \cup \{\theta + h_k^\theta \colon k \in \mathbb{N}\}$ and $M_4^* := \{\xi\} \cup \{\xi + h_k^\xi \colon k \in \mathbb{N}\}$, and the modulus of continuity of $D_i f$ $(i = 2,3,4)$ and $D_i\tau$ $(i=2,3)$ by

$$\begin{aligned}
\Omega_f(\varepsilon) \;:=\; & \max_{i=2,3,4} \sup\Big\{ |D_i f(t,\bar\psi,\bar u,\bar\theta) - D_i f(t,\tilde\psi,\tilde u,\tilde\theta)|_{\mathcal{L}(Y_i,\mathbb{R}^n)} \colon \\
& \qquad |\bar\psi - \tilde\psi|_C + |\bar u - \tilde u| + |\bar\theta - \tilde\theta|_\Theta \le \varepsilon, \quad t\in[0,\alpha],\ \bar\psi,\tilde\psi \in M_1^*, \\
& \qquad \bar u, \tilde u \in M_2^*,\ \bar\theta,\tilde\theta \in M_3^* \Big\}, \qquad (4.26)\\
\Omega_\tau(\varepsilon) \;:=\; & \max_{i=2,3} \sup\Big\{ |D_i\tau(t,\bar\psi,\bar\xi) - D_i\tau(t,\bar\psi,\bar\xi)|_{\mathcal{L}(Z_i,\mathbb{R})} \colon |\bar\psi - \bar\psi|_C + |\bar\xi - \bar\xi|_\Xi \le \varepsilon, \\
& \qquad t\in[0,\alpha],\ \bar\psi,\bar\psi \in M_1^*,\ \bar\xi,\bar\xi \in M_4^* \Big\}, \qquad (4.27)
\end{aligned}$$

where $Y_2 := C$, $Y_3 := \mathbb{R}^n$, $Y_4 := \Theta$, $Z_2 := C$ and $Z_3 := \Xi$. The compactness of the sets $[0,\alpha] \times M_1^* \times M_2^* \times M_3^*$ and $[0,\alpha]\times M_1^* \times M_4^*$ in the respective spaces and the continuity of the partial derivatives of f and τ yield that $\Omega_f(\varepsilon) \to 0$ and $\Omega_\tau(\varepsilon)\to 0$ as $\varepsilon \to 0$. Define

$$L_1^* := \max_{i=2,3,4} \max\{ |D_i f(t,\psi,u,\theta)|_{\mathcal{L}(Y_i,\mathbb{R}^n)} \colon t\in[0,\alpha],\ \psi\in M_1^*,\ u \in M_2^*,\ \theta \in M_3^*\}, \tag{4.28}$$

and let L_2^* be defined by (4.20). Combining the definitions of Ω_f, Ω_τ, L_1^* and L_2^* with (3.4), (3.5), (4.18), (4.24), (4.25) and (4.25) we get

$$\begin{aligned}
& |L(s,x^k)(h^\varphi,h^\theta,h^\xi) - L(s,x)(h^\varphi,h^\theta,h^\xi)| \\
& \quad \le \Omega_f\Big(K_3|h_k|_\Gamma\Big)|h^\varphi|_C + \Omega_f\Big(K_3|h_k|_\Gamma\Big) N L_2^*(|h^\varphi|_C + |h^\xi|_\Xi) \\
& \qquad + L_1^* L_0 |h_k|_\Gamma L_2^*(|h^\varphi|_C + |h^\xi|_\Xi) + L_1^*\Big|\dot x(u^k(s)) - \dot x(u(s))\Big| L_2^*(|h^\varphi|_C + |h^\xi|_\Xi) \\
& \qquad + L_1^* N \Omega_\tau\Big((L_0+1)|h_k|_\Gamma\Big)\big)(|h^\varphi|_C + |h^\xi|_\Xi) + \Omega_f\Big(K_3|h_k|_\Gamma\Big)|h^\varphi|_C \\
& \qquad + L_1^* |\dot h^\varphi|_{L^\infty} K_0 |h_k|_\Gamma + \Omega_f\Big(K_3|h_k|_\Gamma\Big)|h^\theta|_\Theta, \qquad s\in[0,\alpha],
\end{aligned}$$

which yields (4.22) with $c_{0,k} := (NL_2^*+3)\Omega_f\Big(K_3|h_k|_\Gamma\Big) + L_0 L_1^* L_2^* |h_k|_\Gamma + L_1^* N \Omega_\tau\Big((L_0+1)|h_k|_\Gamma\Big) + L_1^* K_0 |h_k|_\Gamma$ and $N_3 := L_1^* L_2^*$. ☐

Now we show that $z(t,\gamma,\cdot)$ and $z_t(\cdot,\gamma,\cdot)$ are continuous in t and γ.

Lemma 4.7 *Assume (A0), (A1) (i), (A2) (i)–(iii). Let $\gamma \in P_2$, and $x(t) := x(t,\gamma)$ for $t \in [-r,\alpha]$. Let $h \in C \times \Omega \times \Xi$ and let $z(t,\gamma,h)$ be the corresponding solution of the IVP* (4.14)-(4.15) *on $[-r,\alpha]$. Then the maps*

$$\mathbb{R}\times\Gamma \supset [0,\alpha]\times P_2 \to \mathcal{L}(\Gamma,\mathbb{R}^n), \quad (t,\gamma) \mapsto z(t,\gamma,\cdot)$$

and

$$\mathbb{R}\times\Gamma \supset [0,\alpha]\times P_2 \to \mathcal{L}(\Gamma,C), \quad (t,\gamma)\mapsto z_t(\cdot,\gamma,\cdot)$$

are continuous.

Proof Let $\gamma \in P_2$ be fixed, and let $h_k = (h_k^\varphi, h_k^\theta, h_k^\xi) \in \Gamma$ ($k \in \mathbb{N}$) be a sequence such that $|h_k|_\Gamma \to 0$ as $k \to \infty$ and $\gamma + h_k \in P_2$ for $k \in \mathbb{N}$. For a fixed $h = (h^\varphi, h^\theta, h^\xi) \in \Gamma$ we define the short notations $x^k(t) := x(t, \gamma + h_k)$, $x(t) := x(t, \gamma)$, $u^k(t) := t - \tau(t, x_t^k, \xi + h_k^\xi)$, $u(t) := t - \tau(t, x_t, \xi)$, $z^{k,h}(t) := z(t, \gamma + h_k, h)$ and $z^h(t) := z(t, \gamma, h)$. The functions $z^{k,h}$ and z^h satisfy

$$\begin{aligned} z^{k,h}(t) &= h^\varphi(0) + \int_0^t L(s, x^k)(z_s^{k,h}, h^\theta, h^\xi)\, ds, \qquad t \in [0, \alpha], \\ z^h(t) &= h^\varphi(0) + \int_0^t L(s, x)(z_s^h, h^\theta, h^\xi)\, ds, \qquad t \in [0, \alpha], \end{aligned}$$

and therefore for $t \in [0, \alpha]$

$$|z^{k,h}(t) - z^h(t)| \le \int_0^t \Big| \Big(L(s, x^k) - L(s, x) \Big)(z_s^h, h^\theta, h^\xi) + L(s, x^k)(z_s^{k,h} - z_s^h, 0, 0) \Big|\, ds. \tag{4.29}$$

We have by (4.17) that

$$|(z_s^h, h^\theta, h^\xi)|_\Gamma \le N_2 |h|_\Gamma + |h^\theta|_\Theta + |h^\xi|_\Xi \le (N_2 + 1)|h|_\Gamma. \tag{4.30}$$

Let L_1^* and L_2^* be defined by (4.28) and (4.20), respectively. Then (4.11) and (4.12) yield $L_1(\theta + h_k^\theta) \le L_1^*$ and $L_2(\xi + h_k^\xi) \le L_2^*$ for $k \in \mathbb{N}$, so (4.13) yields

$$|L(t, x)h| \le N_4 |h|_{C \times \Theta \times \Xi}, \quad |L(t, x^k)h| \le N_4 |h|_{C \times \Theta \times \Xi}, \qquad \text{for } t \in [0, \alpha],\ h \in \Gamma,\ k \in \mathbb{N}, \tag{4.31}$$

where $N_4 := L_1^*(N L_2^* + 3)$. Then (4.31), (4.22), (4.29) and (4.30) imply

$$|z^{k,h}(t) - z^h(t)| \le c_{1,k}|h|_\Gamma + N_4 \int_0^t |z_s^{k,h} - z_s^h|_C\, ds, \qquad t \in [0, \alpha], \tag{4.32}$$

where

$$c_{1,k} := \alpha c_{0,k}(N_2 + 1) + N_3(N_2 + 1) \int_0^\alpha |\dot{x}(u^k(s)) - \dot{x}(u(s))|\, ds.$$

Relation (4.19) and Lemma 2.8 yield that

$$\lim_{k \to \infty} \int_0^\alpha |\dot{x}(u^k(s)) - \dot{x}(u(s))|\, ds = 0. \tag{4.33}$$

Hence $c_{1,k} \to 0$ as $k \to \infty$.

Lemma 2.1 is applicable for (4.32) since $|z_0^{k,h} - z_0^h|_C = 0$, and it gives

$$|z^{k,h}(t) - z^h(t)| \le |z_t^{k,h} - z_t^h|_C \le c_{1,k} N_5 |h|_\Gamma, \qquad t \in [0, \alpha], \tag{4.34}$$

where $N_5 := e^{N_4 \alpha}$. Therefore we get for $t \in [0, \alpha]$

$$|z(t, \gamma + h_k, \cdot) - z(t, \gamma, \cdot)|_{\mathcal{L}(W^{1,\infty}, \mathbb{R}^n)} \le |z_t(\cdot, \gamma + h_k, \cdot) - z_t(\cdot, \gamma, \cdot)|_{\mathcal{L}(W^{1,\infty}, C)} \le c_{1,k} N_5 \tag{4.35}$$

for all $k \in \mathbb{N}$. This proves the continuity of the maps wrt γ.

Let $t \in [0, \alpha]$ be fixed, and let ν_k be a sequence of real numbers such that $t + \nu_k \in [0, \alpha]$ for $k \in \mathbb{N}$ and $\nu_k \to 0$ as $k \to \infty$. Then (4.17) and the Mean Value Theorem yield

$$|z_{t+\nu_k}(\cdot, \gamma + h_k, h) - z_t(\cdot, \gamma + h_k, h)|_C \le N_2|\nu_k||h|_\Gamma, \qquad k \ge k_0, \quad h \in \Gamma.$$

Combining this relation with (4.35) we get

$$\begin{aligned}
&|z(t + \nu_k, \gamma + h_k, h) - z(t, \gamma, h)| \\
&\quad \le \ |z_{t+\nu_k}(\cdot, \gamma + h_k, h) - z_t(\cdot, \gamma, h)|_C \\
&\quad \le \ |z_{t+\nu_k}(\cdot, \gamma + h_k, h) - z_t(\cdot, \gamma + h_k, h)|_C + |z_t(\cdot, \gamma + h_k, h) - z_t(\cdot, \gamma, h)|_C \\
&\quad \le \ (N_2|\nu_k| + c_{1,k}N_5)|h|_\Gamma, \qquad h \in \Gamma,
\end{aligned}$$

which completes the proof, since $|\nu_k| + c_{1,k} \to 0$ as $k \to \infty$. □

Remark 4.8 Note that if in the statement of Lemma 4.7 the parameter set P_2 is replaced by the smaller set P_1, then assumptions (A2) (ii) and (iii) are not needed to prove the statement, since in this case (4.18) and Lemma 2.3 can be used to show that $c_{1,k} \to 0$ as $k \to \infty$.

Now we are ready to prove the Fréchet-differentiability of the function $x(t, \gamma)$ wrt γ. We will denote this derivative by $D_2x(t, \gamma)$.

Theorem 4.9 *Assume (A0), f satisfies (A1) (i), τ satisfies (A2) (i)–(iii), and let P_2 be defined by* (4.9)*. Then the functions*

$$\mathbb{R} \times \Gamma \supset [0, \alpha] \times P \to \mathbb{R}^n, \qquad (t, \gamma) \mapsto x(t, \gamma)$$

and

$$\mathbb{R} \times \Gamma \supset [0, \alpha] \times P \to C, \qquad (t, \gamma) \mapsto x_t(\cdot, \gamma)$$

are both differentiable wrt γ for every $\gamma \in P_2$, and

$$D_2x(t, \gamma)h = z(t, \gamma, h), \qquad h \in \Gamma,\ t \in [0, \alpha],\ \gamma \in P_2, \tag{4.36}$$

and

$$D_2x_t(\cdot, \gamma)h = z_t(\cdot, \gamma, h), \qquad h \in \Gamma,\ t \in [0, \alpha],\ \gamma \in P_2, \tag{4.37}$$

where $z(t, \gamma, h)$ is the solution of the IVP (4.14)-(4.15) *for $t \in [0, \alpha]$, $\gamma \in P_2$ and $h \in \Gamma$. Moreover, the functions*

$$\mathbb{R} \times \Gamma \supset [0, \alpha] \times P_2 \to \mathcal{L}(\Gamma, \mathbb{R}^n), \qquad (t, \gamma) \mapsto D_2x(t, \gamma)$$

and

$$\mathbb{R} \times \Gamma \supset [0, \alpha] \times P_2 \to \mathcal{L}(\Gamma, C), \qquad (t, \gamma) \mapsto D_2x_t(\cdot, \gamma)$$

are continuous.

Proof Let $\gamma = (\varphi, \theta, \xi) \in P_2$ be fixed, and let $h_k = (h_k^\varphi, h_k^\theta, h_k^\xi) \in \Gamma$ ($k \in \mathbb{N}$) be a sequence with $|h_k|_\Gamma \to 0$ as $k \to \infty$ and $\gamma + h_k \in P$ for $k \in \mathbb{N}$. To simplify notation, let $x^k(t) := x(t, \gamma + h_k)$, $x(t) := x(t, \gamma)$, $u(s) := s - \tau(s, x_s, \xi)$, $u^k(s) := s - \tau(s, x_s^k, \xi + h_k^\xi)$ and $z^{h_k}(t) := z(t, \gamma, h_k)$. Then

$$\begin{aligned} x^k(t) &= \varphi(0) + h_k^\varphi(0) + \int_0^t f(s, x_s^k, x^k(u^k(s)), \theta + h_k^\theta)\, ds, \qquad t \in [0, \alpha], \\ x(t) &= \varphi(0) + \int_0^t f(s, x_s, x(u(s)), \theta)\, ds, \qquad t \in [0, \alpha], \end{aligned}$$

and

$$z^{h_k}(t) = h_k^\varphi(0) + \int_0^t L(s, x)(z_s^{h_k}, h_k^\theta, h_k^\xi)\, ds, \qquad t \in [0, \alpha].$$

We have

$$\begin{aligned} x^k(t) - x(t) - z^{h_k}(t) &= \int_0^t \Big(f(s, x_s^k, x^k(u^k(s)), \theta + h_k^\theta) - f(s, x_s, x(u(s)), \theta) \\ &\qquad - L(s, x)(z_s^{h_k}, h_k^\theta, h_k^\xi) \Big) ds. \end{aligned} \tag{4.38}$$

The definitions of ω_f and $L(s, x)$ (see (4.5) and (4.10), respectively) yield for $s \in [0, \alpha]$

$$\begin{aligned} & f(s, x_s^k, x^k(u^k(s)), \theta + h_k^\theta) - f(s, x_s, x(u(s)), \theta) - L(s, x)(z_s^{h_k}, h_k^\theta, h_k^\xi) \\ &= D_2 f(s, x_s, x(u(s)), \theta)(x_s^k - x_s - z_s^{h_k}) + D_3 f(s, x_s, x(u(s)), \theta)\Big(x^k(u^k(s)) - x(u(s))\Big) \\ &\quad + D_3 f(s, x_s, x(u(s)), \theta)\Big(\dot{x}(u(s))\Big(D_2\tau(s, x_s, \xi) z_s^{h_k} + D_3\tau(s, x_s, \xi) h_k^\xi\Big) - z^{h_k}(u(s))\Big) \\ &\quad + \omega_f(s, x_s, x(u(s), \theta, x_s^k, x^k(u^k(s)), \theta + h_k^\theta). \end{aligned} \tag{4.39}$$

Relation (4.6) and simple manipulations give

$$\begin{aligned} & x^k(u^k(s)) - x(u(s)) + \dot{x}(u(s))\Big(D_2\tau(s, x_s, \xi) z_s^{h_k} + D_3\tau(s, x_s, \xi) h_k^\xi\Big) - z^{h_k}(u(s)) \\ &= x^k(u^k(s)) - x(u^k(s)) - z^{h_k}(u^k(s)) + x(u^k(s)) - x(u(s)) - \dot{x}(u(s))(u^k(s) - u(s)) \\ &\quad - \dot{x}(u(s))\omega_\tau(s, x_s, \xi, x_s^k, \xi + h_k^\xi) - \dot{x}(u(s)) D_2\tau(s, x_s, \xi)(x_s^k - x_s - z_s^{h_k}) \\ &\quad + z^{h_k}(u^k(s)) - z^{h_k}(u(s)). \end{aligned} \tag{4.40}$$

Relation (4.17) and (4.18) imply

$$|z^{h_k}(u^k(s)) - z^{h_k}(u(s))| \le N_2 |h_k|_\Gamma |u^k(s) - u(s)| \le N_2 K_0 |h_k|_\Gamma^2. \tag{4.41}$$

Let L_1^* and L_2^* be defined by (4.28) and (4.20), respectively. Then using (3.4), (4.38), (4.39), (4.40) and (4.41) we get

$$\begin{aligned} & |x^k(t) - x(t) - z^{h_k}(t)| \\ &\le \int_0^t \Big[L_1^* \Big(|x_s^k - x_s - z_s^{h_k}|_C + |x^k(u^k(s)) - x(u^k(s)) - z^{h_k}(u^k(s))| \\ &\qquad + |x(u^k(s)) - x(u(s)) - \dot{x}(u(s))(u^k(s) - u(s))| \\ &\qquad + N|\omega_\tau(s, x_s, \xi, x_s^k, \xi + h_k^\xi)| + N L_2^* |x_s^k - x_s - z_s^{h_k}|_C + N_2 K_0 |h_k|_\Gamma^2 \Big) \\ &\qquad + |\omega_f(s, x_s, x(u(s)), \theta, x_s^k, x^k(u^k(s)), \theta + h_k^\theta)| \Big] ds, \qquad t \in [0, \alpha]. \end{aligned} \tag{4.42}$$

Then

$$|x^k(t) - x(t) - z^{h_k}(t)| \le a_k + b_k + c_k + d_k + N_6 \int_0^t |x_s^k - x_s - z_s^{h_k}|_C \, ds, \qquad t \in [0, \alpha], \quad (4.43)$$

where $N_6 := L_1^*(NL_2^* + 2)$, and

$$a_k := \int_0^\alpha |\omega_f(s, x_s, x(u(s)), \theta, x_s^k, x^k(u^k(s)), \theta + h_k^\theta)| \, ds, \quad (4.44)$$

$$b_k := L_1^* N \int_0^\alpha |\omega_\tau(s, x_s, \xi, s, x_s^k, \xi + h_k^\xi)| \, ds, \quad (4.45)$$

$$c_k := L_1^* \int_0^\alpha |x(u^k(s)) - x(u(s)) - \dot{x}(u(s))(u^k(s) - u(s))| \, ds, \quad (4.46)$$

and

$$d_k := \alpha L_1^* N_2 K_0 |h_k|_\Gamma^2. \quad (4.47)$$

Since $|x_0^k - x_0 - z_0|_C = 0$, Lemma 2.1 is applicable for (4.43), and it yields

$$|x^k(t) - x(t) - z^{h_k}(t)| \le |x_t^k - x_t - z_t|_C \le (a_k + b_k + c_k + d_k) e^{N_6 \alpha}, \qquad t \in [0, \alpha], \quad (4.48)$$

and hence

$$\frac{|x^k(t) - x(t) - z^{h_k}(t)|}{|h_k|_\Gamma} \le \frac{|x_t^k - x_t - z_t^{h_k}|_C}{|h_k|_\Gamma} \le \frac{a_k + b_k + c_k + d_k}{|h_k|_\Gamma} e^{N_6 \alpha}, \quad t \in [0, \alpha], \quad (4.49)$$

which proves both (4.36) and (4.37), since Lemmas 4.2, 4.3 and (4.47) show that

$$\lim_{k \to \infty} \frac{a_k + b_k + c_k + d_k}{|h_k|_\Gamma} = 0. \quad (4.50)$$

The continuity of $D_2 x(t, \gamma)$ follows from Lemma 4.7. □

Remark 4.10 We comment that if in the statement of Theorem 4.9 the set P_2 is replaced by P_1, the statements are valid without assumptions (A2) (ii) and (iii). To see this we refer to Remark 4.8, and in the proof of Theorem 4.9 we use Lemma 4.1 of [12] to show that $c_k/|h_k|_\Gamma \to 0$ as $k \to \infty$. We also note that continuous differentiability of x wrt the parameters holds in a neighborhood of γ, since P_1 is open in P. See Theorem 4.7 in [12] for a related result.

5 Second-order differentiability wrt the parameters

To obtain second-order differentiability wrt the parameters we need more smoothness of the initial functions. Therefore we introduce the parameter set

$$\Gamma_2 := W^{2,\infty} \times \Theta \times \Xi$$

equipped with the norm $|h|_{\Gamma_2} := |h^\varphi|_{W^{2,\infty}} + |h^\theta|_\Theta + |h^\xi|_\Xi$. We will show in Theorem 5.17 below that the parameter map

$$\Gamma_2 \supset (P_2 \cap \Gamma_2 \cap \mathcal{P}) \to \mathbb{R}^n, \qquad \gamma \to x(t, \gamma)$$

is twice continuously differentiable, where $\mathcal{P}$ and P_2 are defined in (3.6) and (4.9), respectively.

In addition to (A0), (A1) (i), (A2) (i)–(iii) for the second-order differentiability result we assume that f and τ satisfy

(A1) (ii) $f(t, \psi, u, \theta)$ is locally Lipschitz continuous wrt t, i.e., for every finite $\alpha \in (0, T]$, for every compact subset $M_1 \subset \Omega_1$ of C, compact subset $M_2 \subset \Omega_2$ of $\mathbb{R}^n$, and compact subset $M_3 \subset \Omega_3$ of Θ there exists a constant $L_1' = L_1'(\alpha, M_1, M_2, M_3)$ such that

$$|f(t, \psi, u, \theta) - f(\bar{t}, \psi, u, \theta)| \leq L_1' |t - \bar{t}|$$

for $t, \bar{t} \in [0, \alpha]$, $\psi \in M_1$, $u \in M_2$ and $\theta \in M_3$;

(iii) $D_2 f$, $D_3 f$ and $D_4 f$ are continuously differentiable wrt their second, third and fourth arguments on $[0, T] \times \Omega_1 \times \Omega_2 \times \Omega_3$;

(iv) $D_2 f$, $D_3 f$ and $D_4 f$ are locally Lipschitz continuous wrt t, i.e., for every finite $\alpha \in (0, T]$, for every compact subset $M_1 \subset \Omega_1$ of C, compact subset $M_2 \subset \Omega_2$ of $\mathbb{R}^n$, and compact subset $M_3 \subset \Omega_3$ of Θ there exists $L_4' = L_4'(\alpha, M_1, M_2, M_3)$ such that

$$|D_i f(t, \psi, u, \theta) - D_i f(\bar{t}, \psi, u, \theta)|_{\mathcal{L}(Y_i, \mathbb{R}^n)} \leq L_4' |t - \bar{t}|$$

for $i = 2, 3, 4$, $t, \bar{t} \in [0, \alpha]$, $\psi \in M_1$, $u \in M_2$ and $\theta \in M_3$, where $Y_2 := C$, $Y_3 := \mathbb{R}^n$ and $Y_4 := \Theta$;

(A2) (iv) $D_2 \tau$ and $D_3 \tau$ are continuously differentiable wrt their second and third arguments on $[0, T] \times \Omega_1 \times \Omega_4$;

(v) $D_2 \tau$ and $D_3 \tau$ are locally Lipschitz continuous wrt t, i.e., for every finite $\alpha \in (0, T]$, compact subset $M_1 \subset \Omega_1$ of C, and compact subset $M_4 \subset \Omega_4$ of Ξ there exists a constant $L_5' = L_5'(\alpha, M_1, M_4)$ such that

$$|D_i \tau(t, \psi, \xi) - D_i \tau(\bar{t}, \psi, \xi)|_{\mathcal{L}(Z_i, \mathbb{R})} \leq L_5' |t - \bar{t}|$$

for $i = 2, 3$, $t, \bar{t} \in [0, \alpha]$, $\psi, \bar{\psi} \in M_1$, $\xi, \bar{\xi} \in M_4$, where $Z_2 := C$ and $Z_3 := \Xi$;

(vi) for every finite $\alpha \in (0, T]$, for every compact subset $M_1 \subset \Omega_1$ of C, compact subset $M_2 \subset \Omega_2$ of $\mathbb{R}^n$, compact subsets $M_3 \subset \Omega_3$ of Θ and $M_4 \subset \Omega_4$ of Ξ, for every $\bar{\gamma} = (\bar{\varphi}, \bar{\theta}, \bar{\xi}) \in \Gamma$ satisfying $\bar{\theta} \in M_3$ and $\bar{\xi} \in M_4$, for every $\bar{y} \in W^{2,\infty}([-r, \alpha], \mathbb{R}^n)$ satisfying $\bar{y}_t \in M_1$ and $\bar{y}(t - \tau(t, \bar{y}_t, \bar{\xi})) \in M_2$ for $t \in [0, \alpha]$ there exists $L_6 = L_6(\alpha, M_1, M_2, M_3, M_4, \bar{\gamma}, \bar{y})$ such that

$$\begin{aligned} \Big| \frac{d}{dt} f(t, y_t, y(t - \tau(t, y_t, \xi)), \theta) - \frac{d}{dt} f(t, \bar{y}_t, \bar{y}(t - \tau(t, \bar{y}_t, \bar{\xi})), \bar{\theta}) \Big| \\ \leq \quad L_6 \Big(|y_t - \bar{y}_t|_{W^{1,\infty}} + |\xi - \bar{\xi}|_\Xi + |\theta - \bar{\theta}|_\Xi \Big), \quad \text{a.e. } t \in [0, \alpha], \end{aligned}$$

where $\theta \in M_3$, $\xi \in M_4$, and $y \in W^{1,\infty}([-r, \alpha], \mathbb{R}^n)$ are such that $y_t \in M_1$ and $y(t - \tau(t, y_t, \xi)) \in M_2$ for $t \in [0, \alpha]$.

If f and τ are twice continuously differentiable wrt all variables then conditions (A1) (i)-(iv) and (A2) (i),(ii),(iv) and (v) are satisfied. Differentiability wrt t is not needed, only the weaker assumption, Lipschitz continuity is needed for our proofs. We note that assumption (A2) (iii) and (vi) are satisfied for τ and f of the form (3.1) and (3.2) under natural assumptions.

We use throughout this section the following notations

(H) $\gamma = (\varphi, \theta, \xi) \in P_2 \cap \Gamma_2$, $h = (h^\varphi, h^\theta, h^\xi) \in \Gamma$, $h_k = (h^\varphi_k, h^\theta_k, h^\xi_k) \in \Gamma$ $(k \in \mathbb{N})$ are so that $|h_k|_\Gamma \to 0$ as $k \to \infty$, $\gamma + h_k \in P_2$ for $k \in \mathbb{N}$, and $|h_k|_\Gamma \neq 0$ for $k \in \mathbb{N}$. Let $M^*_3 := \{\theta\} \cup \{\theta + h^\theta_k\colon\ k \in \mathbb{N}\}$ and $M^*_4 := \{\xi\} \cup \{\xi + h^\xi_k\colon\ k \in \mathbb{N}\}$. Let $x^k(t) := x(t, \gamma + h_k)$ and $x(t) := x(t, \gamma)$ be the solutions of the IVP (1.1)-(1.2), $z^{k,h}(t) := D_2x(t, \gamma + h_k)h$ and $z^h(t) := D_2x(t, \gamma)h$ be the solutions of the IVP (4.14)-(4.15).

The simplifying notations for $t \in [0, \alpha]$ and $k \in \mathbb{N}$

$$
\begin{aligned}
u(t) &:= t - \tau(t, x_t, \xi),\\
u^k(t) &:= t - \tau(t, x^k_t, \xi + h^\xi_k),\\
\mathbf{v}(t) &:= (t, x_t, x(u(t)), \theta),\\
\mathbf{v}^k(t) &:= (t, x^k_t, x^k(u^k(t)), \theta),\\
A(t, h^\varphi, h^\xi) &:= D_2\tau(t, x_t, \xi)h^\varphi + D_3\tau(t, x_t, \xi)h^\xi,\\
A^k(t, h^\varphi, h^\xi) &:= D_2\tau(t, x^k_t, \xi + h^\xi_k)h^\varphi + D_3\tau(t, x^k_t, \xi + h^\xi_k)h^\xi,\\
E(t, h^\varphi, h^\xi) &:= -\dot{x}(u(t))A(t, h^\varphi, h^\xi) + h^\varphi(-\tau(t, x_t, \xi)), \quad \text{a.e. } t \in [0, \alpha],\\
E^k(t, h^\varphi, h^\xi) &:= -\dot{x}^k(u^k(t))A^k(t, h^\varphi, h^\xi) + h^\varphi(-\tau(t, x^k_t, \xi + h^\xi_k)), \quad \text{a.e. } t \in [0, \alpha],\\
F(t, h^\varphi, h^\xi) &:= -\ddot{x}(u(t))A(t, h^\varphi, h^\xi) + \dot{h}^\varphi(-\tau(t, x_t, \xi)), \quad \text{a.e. } t \in [0, \alpha],\\
F^k(t, h^\varphi, h^\xi) &:= -\ddot{x}^k(u^k(t))A^k(t, h^\varphi, h^\xi) + \dot{h}^\varphi(-\tau(t, x^k_t, \xi + h^\xi_k)), \quad \text{a.e. } t \in [0, \alpha]
\end{aligned}
$$

will be used throughout this section. For simplicity of the notation we define $h_0 := 0 = (0, 0, 0) \in \Gamma$, and accordingly, $x^0 := x$, $u^0 := u$, $z^{0,h} := z^h$, $A^0 := A$, $E^0 := E$. Note that in all the above abbreviations the dependence on γ is omitted from the notation but it should be kept in mind. With these notations the operator $L(t, x)$ defined by (4.10) can be written shortly as

$$L(t, x)h = D_2f(\mathbf{v}(t))h^\varphi + D_3f(\mathbf{v}(t))E(t, h^\varphi, h^\xi) + D_4f(\mathbf{v}(t))h^\theta. \tag{5.1}$$

The proof of second-order differentiability (Theorem 5.17) is broken up to several lemmas. First we show that if the compatibility condition $\gamma \in \mathcal{P}$ holds then the solution of the IVP (1.1)-(1.2) is a $W^{2,\infty}$-smooth function.

Lemma 5.1 *Assume (A0), (A1) (i), (ii), (A2) (i)–(ii) and $\gamma = (\varphi, \theta, \xi) \in P \cap \Gamma_2$. Then there exists $K_4 = K_4(\gamma) \geq 0$ such that the solution $x(t) = x(t, \gamma)$ of the IVP (1.1)-(1.2) satisfies*

$$|\dot{x}(t) - \dot{x}(\bar{t})| \leq K_4|t - \bar{t}| \qquad \textit{for } t, \bar{t} \in [-r, 0) \quad \textit{and} \quad t, \bar{t} \in (0, \alpha]. \tag{5.2}$$

Moreover, if in addition $\gamma \in \mathcal{P}$, then $x \in W^{2,\infty}([-r, \alpha], \mathbb{R}^n)$, and

$$|\dot{x}(t) - \dot{x}(\bar{t})| \leq K_4|t - \bar{t}| \qquad \text{for } t, \bar{t} \in [-r, \alpha]. \tag{5.3}$$

Proof The Mean Value Theorem and the definition of the $W^{2,\infty}$-norm yield

$$|\dot{x}(t) - \dot{x}(\bar{t})| = |\dot{\varphi}(t) - \dot{\varphi}(\bar{t})| \le |\varphi|_{W^{2,\infty}}|t - \bar{t}|, \qquad t, \bar{t} \in [-r, 0).$$

Let $L_1(\theta)$ and $L_2(\xi)$ be defined by (4.11) and (4.12), respectively, and $L_1' = L_1'(\alpha, M_1^*, M_2^*, \{\theta\})$ and $L_2' = L_2'(\alpha, M_1^*, \{\xi\})$ be the Lipschitz constants from (A1) (ii) and (A2) (ii), respectively. Define $L_1'' := \max(L_1(\theta), L_1')$ and $L_2'' := \max(L_2(\xi), L_2')$. For $t, \bar{t} \in (0, \alpha]$ it follows from the definitions of L_1'' and L_2'' and the Mean Value Theorem, (3.4) and (4.21) with $k = 0$

$$\begin{aligned} |\dot{x}(t) - \dot{x}(\bar{t})| &= |f(t, x_t, x(u(t)), \theta) - f(\bar{t}, x_{\bar{t}}, x(u(\bar{t})), \theta)| \\ &\le L_1''\Big(|t - \bar{t}| + |x_t - x_{\bar{t}}|_C + |x(u(t)) - x(u(\bar{t}))|\Big) \\ &\le L_1''\Big(1 + N + NL_2''(1 + N)\Big)|t - \bar{t}|. \end{aligned}$$

Hence (5.2) is satisfied with $K_4 := \max\{|\varphi|_{W^{2,\infty}}, L_1''[1 + N + NL_2''(1 + N)]\}$.

If $\gamma \in \mathcal{P}$, then $\dot{x}$ is continuous, and (5.2) yields that it is Lipschitz continuous on $[-r, \alpha]$ with the Lipschitz constant K_4, so, in particular, $x \in W^{2,\infty}([-r, \alpha], \mathbb{R}^n)$. □

Lemma 5.2 *Assume (A0), (A1) (i), (A2) (i)–(iii) and (H). Then*

$$\lim_{k\to\infty} \frac{1}{|h_k|_\Gamma} \int_0^\alpha |\dot{x}^k(s) - \dot{x}(s) - \dot{z}^{h_k}(s)|\, ds = 0, \tag{5.4}$$

and

$$\lim_{k\to\infty} \frac{1}{|h_k|_\Gamma} \int_0^\alpha |\dot{x}^k(u^k(s)) - \dot{x}(u^k(s)) - \dot{z}^{h_k}(u^k(s))|\, ds = 0. \tag{5.5}$$

Proof Using (4.38), (4.42), (4.43) and (4.48) we get

$$\begin{aligned} &\int_0^\alpha |\dot{x}^k(s) - \dot{x}(s) - \dot{z}^{h_k}(s)|\, ds \\ &\le \int_0^\alpha \Big[L_1^*\Big(|x_s^k - x_s - z_s^{h_k}|_C + |x^k(u^k(s)) - x(u^k(s)) - z^{h_k}(u^k(s))| \\ &\qquad + |x(u^k(s)) - x(u(s)) - \dot{x}(u(s))(u^k(s) - u(s))| \\ &\qquad + N|\omega_\tau(s, x_s, \xi, x_s^k, \xi + h_k^\xi)| + NL_2^*|x_s^k - x_s - z_s^{h_k}|_C + N_2K_0|h_k|_\Gamma^2\Big) \\ &\qquad + |\omega_f(s, x_s, x(u(s)), \theta, x_s^k, x^k(u^k(s)), \theta + h_k^\theta)|\Big]\, ds \\ &\le a_k + b_k + c_k + d_k + N_6 \int_0^\alpha |x_s^k - x_s - z_s^{h_k}|_C\, ds \\ &\le (a_k + b_k + c_k + d_k)(1 + N_6\alpha e^{N_6\alpha}), \end{aligned}$$

where a_k, b_k, c_k and d_k are defined by (4.44)–(4.47), respectively. Then (5.4) is obtained from (4.50).

Relation (5.5) follows from (5.4), $x^k(s) - x(s) - z^{h_k}(s) = 0$ for $s \in [-r, 0]$, $|\dot{x}^k(s) - \dot{x}(s) - \dot{z}^{h_k}(s)| \le (L_0 + N_2)|h_k|_\Gamma$ for $s \in [-r, 0]$, and Lemmas 2.9 and 4.5. □

Next we show that $\dot{z}^h(t)$ is Lipschitz continuous in t for any $h \in \Gamma_2$. For this property we assume Lipschitz continuity of f, $D_i f$ $(i = 2, 3, 4)$ and τ, $D_i\tau$ $(i = 2, 3)$ in t. Note that the restriction of h to Γ_2 instead of Γ is essential.

Lemma 5.3 *Assume (A0), (A1) (i)-(iv), (A2) (i)–(v), (H) and $\gamma \in P_2 \cap \Gamma_2 \cap \mathcal{P}$. Then there exists $N_7 = N_7(\gamma) \geq 0$ such that*

$$|\dot{z}^h(s) - \dot{z}^h(\bar{s})| \leq N_7 |h|_{\Gamma_2} |s - \bar{s}|, \quad \text{for} \quad s, \bar{s} \in [-r, 0) \quad \text{and} \quad s, \bar{s} \in (0, \alpha], \quad h \in \Gamma_2. \tag{5.6}$$

Proof For $h \in \Gamma_2$, i.e., $h^\varphi \in W^{2,\infty}$, the function $\dot{h}^\varphi$ is continuous, and for $s, \bar{s} \in [-r, 0)$

$$|\dot{z}^h(s) - \dot{z}^h(\bar{s})| = |\dot{h}^\varphi(s) - \dot{h}^\varphi(\bar{s})| \leq |h^\varphi|_{W^{2,\infty}} |s - \bar{s}| \leq |h|_{\Gamma_2} |s - \bar{s}|.$$

Let $L_1(\theta)$ and $L_2(\xi)$ be defined by (4.11) and (4.12), respectively, let $L_2' = L_2'(\alpha, M_1^*, \{\xi\})$ be the Lipschitz constant from (A2) (ii), and let $L_2'' := \max\{L_2(\theta), L_2'\}$. Since $\gamma \in \mathcal{P}$, $L(s, x)$ is defined and continuous for all $s \in [0, \alpha]$, so $\dot{z}^h$ is continuous on $(0, \alpha]$. For $s, \bar{s} \in (0, \alpha]$ (4.13), (4.14) and (5.1) imply

$$\begin{aligned} |\dot{z}^h(s) - \dot{z}^h(\bar{s})| &= |L(s, x)(z_s^h, h^\theta, h^\xi) - L(\bar{s}, x)(z_{\bar{s}}^h, h^\theta, h^\xi)| \\ &\leq |[L(s, x) - L(\bar{s}, x)](z_s^h, h^\theta, h^\xi)| + |L(\bar{s}, x)(z_s^h - z_{\bar{s}}^h, 0, 0)| \\ &\leq |[D_2 f(\mathbf{v}(s)) - D_2 f(\mathbf{v}(\bar{s}))] z_s^h| + |[D_3 f(\mathbf{v}(s)) - D_3 f(\mathbf{v}(\bar{s}))] E(s, z_s^h, h^\xi)| \\ &\quad + |D_3 f(\mathbf{v}(\bar{s}))[E(s, z_s^h, h^\xi) - E(\bar{s}, z_{\bar{s}}^h, h^\xi)]| \\ &\quad + |[D_4 f(\mathbf{v}(s)) - D_4 f(\mathbf{v}(\bar{s}))] h^\theta| + L_1(\theta)(N L_2(\theta) + 3) |z_s^h - z_{\bar{s}}^h|_C. \end{aligned} \tag{5.7}$$

We have by (3.4) and (4.21) with $k = 0$ for $s, \bar{s} \in [0, \alpha]$

$$|\mathbf{v}(s) - \mathbf{v}(\bar{s})| := |s - \bar{s}| + |x_s - x_{\bar{s}}|_C + |x(u(s)) - x(u(\bar{s}))| \leq K_5 |s - \bar{s}| \tag{5.8}$$

with $K_5 := (1 + N + N L_2''(1 + N))$, and

$$|(s, x_s, \xi) - (\bar{s}, x_{\bar{s}}, \xi)| := |s - \bar{s}| + |x_s - x_{\bar{s}}|_C \leq (1 + N)|s - \bar{s}|. \tag{5.9}$$

The definition of A, (4.12) and (4.16) give

$$|A(s, z_s^h, h^\xi)| \leq |D_2\tau(s, x_s, \xi) z_s^h| + |D_3\tau(s, x_s, \xi) h^\xi| \leq K_6 |h|_\Gamma, \quad s \in [0, \alpha], \; h \in \Gamma \tag{5.10}$$

with $K_6 := L_2(\xi)(N_1 + 1)$. Let

$$L_4(\theta) := \max_{i,j=2,3,4} \max\{|D_{ij} f(t, \psi, u, \theta)|_{\mathcal{L}^2(Y_i \times Y_j, \mathbb{R}^n)} \colon t \in [0, \alpha], \; \psi \in M_1^*, \; u \in M_2^*\}, \tag{5.11}$$

where $Y_2 := C$, $Y_3 := \mathbb{R}^n$, $Y_4 := \Theta$, and

$$L_5(\xi) := \max_{i,j=2,3} \max\{|D_{ij}\tau(t, \psi, \xi)|_{\mathcal{L}^2(Z_i \times Z_j, \mathbb{R})} \colon t \in [0, \alpha], \; \psi \in M_1^*\}, \tag{5.12}$$

where $Z_2 := C$, $Z_3 := \Xi$. Let $L_4' := L_4'(\alpha, M_1^*, M_2^*, \{\theta\})$ and $L_5' := L_5'(\alpha, M_1^*, \{\xi\})$ be defined by (A1) (iv) and (A2) (v), respectively, and define

$$L_4'' := \max\{L_4(\theta), L_4'\}, \qquad L_5'' := \max\{L_5(\xi), L_5'\}. \tag{5.13}$$

Then the Mean Value Theorem and (5.8) yield for $i = 2, 3, 4$

$$|D_i f(\mathbf{v}(s)) - D_i f(\mathbf{v}(\bar{s}))|_{\mathcal{L}(Y_i,\mathbb{R}^n)} \le L_4'' K_5 |s - \bar{s}|, \qquad s, \bar{s} \in [0, \alpha]. \tag{5.14}$$

Similarly, the Mean Value Theorem, (4.16), (4.17) and (5.9) give

$$\begin{aligned} |A(s, z_s^h, h^\xi) - A(\bar{s}, z_{\bar{s}}^h, h^\xi)| &\le |[D_2\tau(s, x_s, \xi) - D_2\tau(\bar{s}, x_{\bar{s}}, \xi)]z_s^h| + |D_2\tau(\bar{s}, x_{\bar{s}}, \xi)[z_s^h - z_{\bar{s}}^h]| \\ &\quad + |[D_3\tau(s, x_s, \xi) - D_3\tau(\bar{s}, x_{\bar{s}}, \xi)]h^\xi| \\ &\le K_7 |s - \bar{s}| |h|_\Gamma, \qquad s, \bar{s} \in [0, \alpha] \end{aligned} \tag{5.15}$$

with $K_7 := L_5''(1 + N)N_1 + L_2(\xi)N_2 + L_5''(1 + N)$. Relations (3.4), (4.16) and (5.10) yield

$$\begin{aligned} |E(s, z_s^h, h^\xi)| &\le |\dot{x}(u(s))||A(s, z_s^h, h^\xi)| + |z^h(u(s))| \\ &\le K_8 |h|_\Gamma, \qquad s \in [0, \alpha],\ h \in \Gamma,\ \gamma \in P_2 \end{aligned} \tag{5.16}$$

with $K_8 := NK_6 + N_1$, and using (3.4), (4.21) with $k = 0$, (4.17), (5.3), (5.10) and (5.15)

$$\begin{aligned} &|E(s, z_s^h, h^\xi) - E(\bar{s}, z_{\bar{s}}^h, h^\xi)| \\ &\quad \le |[\dot{x}(u(s)) - \dot{x}(u(\bar{s}))]A(s, z_s^h, h^\xi)| + |\dot{x}(u(\bar{s}))[A(s, z_s^h, h^\xi) - A(\bar{s}, z_{\bar{s}}^h, h^\xi)]| \\ &\qquad + |z^h(u(s)) - z^h(u(\bar{s}))| \\ &\quad \le K_9 |s - \bar{s}| |h|_\Gamma, \qquad s, \bar{s} \in [0, \alpha] \end{aligned} \tag{5.17}$$

with $K_9 = K_9(\gamma) := K_4 L_2''(1+N)K_6 + NK_7 + N_2 L_2''(1+N)$. Then combining (5.7) with (4.11), (4.16), (5.14), (5.16) and (5.17) yields

$$|\dot{z}^h(s) - \dot{z}^h(\bar{s})| \le (L_4'' K_5 N_1 + L_4'' K_5 K_8 + L_1(\theta)K_9 + L_4'' K_5 + L_1(\theta)(NL_2(\theta) + 3)N_2)|s - \bar{s}||h|_\Gamma$$

for $s, \bar{s} \in [0, \alpha]$ and $h \in \Gamma$. Hence $N_7 := \max\{1, L_4'' K_5 N_1 + L_4'' K_5 K_8 + L_1^* K_9 + L_4'' K_5 + L_1(\theta)(NL_2(\theta) + 3)N_2\}$ satisfies (5.6). □

The next two results will be used in the proof of Lemma 5.7.

Lemma 5.4 *Assume (A0), (A1) (i)-(iv), (A2) (i)–(v), (H) and* $\gamma \in P_2 \cap \Gamma_2 \cap \mathcal{P}$. *Then*

$$\lim_{k\to\infty} \frac{1}{|h_k|_{\Gamma_2}} \int_0^\alpha |\dot{z}^{h_k}(u^k(s)) - \dot{z}^{h_k}(u(s))|\, ds = 0. \tag{5.18}$$

Proof Since $\gamma \in P_2$ and $u(0) \le 0$, it follows that u has finitely many zeros on $[0, \alpha]$. Let $0 \le s_1 < s_2 < \cdots < s_\ell \le \alpha$ be the mesh points where $u(s_i) = 0$, $0 < \varepsilon < \min\{s_{i+1} - s_i : i = 1, \ldots, \ell - 1\}/2$ be fixed, and introduce $s_i' := \min\{s_i + \varepsilon, \alpha\}$ and $s_i'' := \max\{s_i - \varepsilon, 0\}$ for $i = 1, \ldots, \ell$, $s_0' := 0$, $s_{\ell+1}'' := \alpha$, and let

$$M := \min_{i=1,\ldots,\ell-1} \min_{s\in[s_i', s_{i+1}'']} |u(s)|.$$

We have $M > 0$. Relation (4.18) yields that there exist $k_0 > 0$ such that $|u^k - u|_{C([0,\alpha],\mathbb{R})} < \frac{M}{2}$ for $k \ge k_0$. Then for $k \ge k_0$ it follows $|u^k(s)| \ge \frac{M}{2}$ for $s \in [s_i', s_{i+1}'']$ and $i = 0, \ldots, \ell$. Note that $h_k \in \Gamma_2$ and $\gamma \in \mathcal{P}$ yield $\dot{z}^{h_k}$ is continuous on $[-r, 0)$ and $(0, \alpha]$, and (4.17) implies $|\dot{z}^{h_k}(s)| \le N_2 |h_k|_\Gamma \le N_2 |h_k|_{\Gamma_2}$ for $s \ne 0$ and $k \in \mathbb{N}$. Therefore $|\dot{z}^{h_k}(u^k(s))| \le N_2 |h_k|_{\Gamma_2}$ for a.e.

$s \in [0,\alpha]$, since, by assumption (H), $\gamma + h_k \in P_2$, hence $u^k \in \mathcal{PM}([0,\alpha],[-r,\alpha])$. Then (4.17), (4.18) and (5.6) yield for $k \geq k_0$

$$\begin{aligned}
&\int_0^\alpha |\dot z^{h_k}(u^k(s)) - \dot z^{h_k}(u(s))|\,ds \\
&\quad\leq \sum_{i=1}^{\ell} \int_{s_i''}^{s_i'} [|\dot z^{h_k}(u^k(s))| + |\dot z^{h_k}(u(s))|]\,ds + \sum_{i=0}^{\ell} \int_{s_i'}^{s_{i+1}''} |\dot z^{h_k}(u^k(s)) - \dot z^{h_k}(u(s))|\,ds \\
&\quad\leq 4\ell\varepsilon N_2 |h_k|_{\Gamma_2} + (\ell+1)\alpha N_7 K_0 |h_k|_{\Gamma_2} |h_k|_\Gamma.
\end{aligned}$$

This concludes the proof of (5.18), since $\varepsilon > 0$ can be arbitrary close to 0. □

Lemma 5.5 *Assume (A0), (A1) (i)-(iv), (A2) (i)–(v), (H) and $\gamma \in P_2 \cap \Gamma_2 \cap \mathcal{P}$. Then*

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0\\ h\in\Gamma_2}} \frac{1}{|h|_{\Gamma_2}|h_k|_\Gamma} \int_0^\alpha |z^h(u^k(s)) - z^h(u(s)) - \dot z^h(u(s))(u^k(s)-u(s))|\,ds = 0. \tag{5.19}$$

Proof Let s_i, s_i', s_i'', ℓ, ε, M and k_0 be defined as in the proof of Lemma 5.4. Then $|u(s) + \nu(u^k(s)-u(s))| > \frac{M}{2}$, and $u(s)$ and $u(s)+\nu(u^k(s)-u(s))$ are both either positive or negative for $s \in [s_i', s_{i+1}'']$, $\nu \in [0,1]$, $k \geq k_0$ and $i = 0,\ldots,\ell$, and therefore (4.18) and (5.6) yield

$$|\dot z^h(u(s)+\nu(u^k(s)-u(s))) - \dot z^h(u(s))| \leq N_7 |h|_{\Gamma_2} |u^k(s)-u(s)| \leq N_7 K_0 |h|_{\Gamma_2} |h_k|_\Gamma.$$

Hence, using Fubini's Theorem, (4.18) and (4.17) we have

$$\begin{aligned}
&\int_0^\alpha |z^h(u^k(s)) - z^h(u(s)) - \dot z^h(u(s))(u^k(s)-u(s))|\,ds \\
&\quad\leq \sum_{i=1}^{\ell} \int_{s_i''}^{s_i'} \Big(|z^h(u^k(s)) - z^h(u(s))| + |\dot z^h(u(s))||u^k(s)-u(s))|\Big)ds \\
&\qquad + \sum_{i=0}^{\ell} \int_{s_i'}^{s_{i+1}''} |z^h(u^k(s)) - z^h(u(s)) - \dot z^h(u(s))(u^k(s)-u(s))|\,ds \\
&\quad\leq 4\varepsilon\ell N_2 K_0 |h|_\Gamma |h_k|_\Gamma \\
&\qquad + \sum_{i=0}^{\ell} \int_{s_i'}^{s_{i+1}''} \left| \int_0^1 [\dot z^h(u(s)+\nu(u^k(s)-u(s))) - \dot z^h(u(s))][u^k(s)-u(s)]\,d\nu \right| ds \\
&\quad\leq 4\varepsilon\ell N_2 K_0 |h|_\Gamma |h_k|_\Gamma \\
&\qquad + K_0 |h_k|_\Gamma \sum_{i=0}^{\ell} \int_0^1 \int_{s_i'}^{s_{i+1}''} |\dot z^h(u(s)+\nu(u^k(s)-u(s))) - \dot z^h(u(s))|ds\,d\nu \\
&\quad\leq 4\varepsilon\ell N_2 K_0 |h|_{\Gamma_2} |h_k|_\Gamma + K_0^2(\ell+1)\alpha N_7 |h|_{\Gamma_2} |h_k|_\Gamma^2.
\end{aligned}$$

This completes the proof of (5.19), since $\varepsilon > 0$ is arbitrary close to 0. □

The following result will be needed in the proof of Lemma 5.10 below.

Lemma 5.6 *Assume (A0), (A1) (i), (A2) (i)–(iii) and (H). Then*

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0\\ h\in\Gamma}} \frac{1}{|h|_\Gamma} \int_0^\alpha |\dot{z}^{k,h}(s) - \dot{z}^h(s)|\, ds = 0, \tag{5.20}$$

and

$$\lim_{k\to\infty} \sup_{|h|_\Gamma\neq 0} \frac{1}{|h|_\Gamma |h_k|_\Gamma} \int_0^\alpha |z^{k,h}(u^k(s)) - z^h(u^k(s)) - [z^{k,h}(u(s)) - z^h(u(s))]|\, ds = 0. \tag{5.21}$$

Proof For a.e. $s\in[0,\alpha]$ combining (4.14), (4.22), (4.30), (4.31) and (4.34) we get

$$\begin{aligned}
&|\dot{z}^{k,h}(s) - \dot{z}^h(s)|\\
&\quad\leq |L(s,x^k)(z_s^{k,h} - z_s^h, 0, 0)| + |(L(s,x^k) - L(s,x))(z_s^h, h^\theta, h^\xi)|\\
&\quad\leq N_4 c_{1,k} N_5 |h|_\Gamma + c_{0,k}(N_2+1)|h|_\Gamma + N_3(N_2+1)|\dot{x}(u^k(s)) - \dot{x}(u(s))||h|_\Gamma.
\end{aligned} \tag{5.22}$$

Hence Lemmas 2.8 and 4.5 yield (5.20).

Define the functions

$$f^{k,h}(s) := \frac{|\dot{z}^{k,h}(s) - \dot{z}^h(s)|}{|h|_\Gamma},$$

and the set $H := \{h\in\Gamma:\ h\neq 0\}$. Note that (4.14), (4.16) and (4.31) yield $|\dot{z}^{k,h}(s)| = |L(s,x^k)z_s^{k,h}| \leq N_4 N_1 |h|_\Gamma$ for $k\in\mathbb{N}_0$ and $s\in[0,\alpha]$, so $|f^{k,h}(s)|\leq 2N_4N_1$ for a.e. $s\in[-r,\alpha]$, $k\in\mathbb{N}$ and $h\in H$. Then it follows from (5.20), $z^{k,h}(s) - z^h(s) = 0$ for $s\in[-r,0]$, and Lemmas 2.9 and 4.5 that for any fixed $\nu\in[0,1]$

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0\\ h\in\Gamma}} \frac{1}{|h|_\Gamma} \int_0^\alpha \Big|\dot{z}^{k,h}\Big(u(s)+\nu(u^k(s)-u(s))\Big) - \dot{z}^h\Big(u(s)+\nu(u^k(s)-u(s))\Big)\Big|\, ds = 0. \tag{5.23}$$

Relation (4.18) and Fubini's Theorem yield

$$\begin{aligned}
&\int_0^\alpha |z^{k,h}(u^k(s)) - z^h(u^k(s)) - [z^{k,h}(u(s)) - z^h(u(s))]|\, ds\\
&= \int_0^\alpha \Big|\int_0^1 \Big[\dot{z}^{k,h}\Big(u(s)+\nu(u^k(s)-u(s))\Big) - \dot{z}^h\Big(u(s)+\nu(u^k(s)-u(s))\Big)\Big]\\
&\qquad \times [u^k(s)-u(s)]\, d\nu\Big|\, ds\\
&\leq K_0|h_k|_\Gamma \int_0^1\int_0^\alpha \Big|\dot{z}^{k,h}\Big(u(s)+\nu(u^k(s)-u(s))\Big) - \dot{z}^h\Big(u(s)+\nu(u^k(s)-u(s))\Big)\Big|\, ds\, d\nu.
\end{aligned}$$

Therefore (5.23) and the Dominated Convergence Theorem imply (5.21). □

Introduce the notation

$$p^k(t) := x^k(t) - x(t) - z^{h_k}(t).$$

Then, under the assumptions of Theorem 4.9, (4.49) and (4.50) give

$$\lim_{k\to\infty} \max_{s\in[-r,\alpha]} \frac{|p^k(s)|}{|h_k|_\Gamma} = 0. \tag{5.24}$$

To linearize Equation (4.14) around a fixed solution z we will need the following results, where we discuss the linearization of $u^k(s) - u(s)$, $x^k(u^k(s)) - x(u(s))$ and $\dot{x}^k(u^k(s)) - \dot{x}(u(s))$. Note that for the last formula we need the compatibility assumption.

Lemma 5.7 *Assume (A0), (A1) (i)–(iv), (A2) (i)–(v), (H) and* $\gamma \in P_2 \cap \Gamma_2 \cap \mathcal{P}$. *Then*

(i)

$$u^k(s) - u(s) + A(s, z_s^{h_k}, h_k^\xi) = g_0^k(s), \qquad s \in [0,\alpha], \tag{5.25}$$

where

$$g_0^k(s) := -\omega_\tau(s, x_s, \xi, x_s^k, \xi + h_k^\xi) - D_2\tau(s, x_s, \xi) p_s^k$$

satisfies

$$\lim_{k\to\infty} \frac{1}{|h_k|_\Gamma} \int_0^\alpha |g_0^k(s)|\, ds = 0; \tag{5.26}$$

(ii)

$$x^k(u^k(s)) - x(u(s)) - E(s, z_s^{h_k}, h_k^\xi) = g_1^k(s), \qquad s \in [0,\alpha], \tag{5.27}$$

where

$$\begin{aligned} g_1^k(s) \quad := \quad & p^k(u^k(s)) + x(u^k(s)) - x(u(s)) - \dot{x}(u(s))(u^k(s) - u(s)) + \dot{x}(u(s)) g_0^k(s) \\ & + z^{h_k}(u^k(s)) - z^{h_k}(u(s)) \end{aligned}$$

satisfies

$$\lim_{k\to\infty} \frac{1}{|h_k|_\Gamma} \int_0^\alpha |g_1^k(s)|\, ds = 0; \tag{5.28}$$

and

(iii) if $h_k \in \Gamma_2$ *for* $k \in \mathbb{N}$, *then*

$$\dot{x}^k(u^k(s)) - \dot{x}(u(s)) - F(s, z_s^{h_k}, h_k^\xi) = g_2^k(s), \qquad s \in [0,\alpha], \tag{5.29}$$

where

$$\begin{aligned} g_2^k(s) \quad := \quad & \dot{x}^k(u^k(s)) - \dot{x}(u^k(s)) - \dot{z}^{h_k}(u^k(s)) + \dot{z}^{h_k}(u^k(s)) - \dot{z}^{h_k}(u(s)) \\ & + \dot{x}(u^k(s)) - \dot{x}(u(s)) - \ddot{x}(u(s))(u^k(s) - u(s)) \\ & - \ddot{x}(u(s))\omega_\tau(s, x_s, \xi, x_s^k, \xi + h_k^\xi) - \ddot{x}(u(s)) D_2\tau(s, x_s, \xi) p_s^k \end{aligned}$$

satisfies

$$\lim_{k\to\infty} \frac{1}{|h_k|_{\Gamma_2}} \int_0^\alpha |g_2^k(s)|\, ds = 0. \tag{5.30}$$

Proof The definition of ω_τ and A imply

$$
\begin{aligned}
&u^k(s) - u(s) + A(s, z_s^{h_k}, h_k^\xi) \\
&\quad = -[\tau(s, x_s^k, \xi + h_k^\xi) - \tau(s, x_s, \xi) - D_2\tau(s, x_s, \xi)(x_s^k - x_s) - D_2\tau(s, x_s, \xi)h_k^\xi] \\
&\qquad -D_2\tau(s, x_s, \xi)(x_s^k - x_s - z_s^{h_k}), \qquad s \in [0, \alpha],
\end{aligned}
$$

which shows (5.25). Let L_2^* be defined by (4.20). (5.26) follows from $|D_2\tau(s, x_s, \xi)|_{\mathcal{L}(C,\mathbb{R})} \le L_2^*$ for $s \in [0, \alpha]$, (4.8) and (5.24).

Relation (4.40) and the definition of g_1^k yield (5.27). We have by (3.4) and (4.41)

$$
\begin{aligned}
\int_0^\alpha |g_1^k(s)|\, ds \;\le\; & \alpha \max_{s\in[-r,\alpha]} |p^k(s)| + \int_0^\alpha |x(u^k(s)) - x(u(s)) - \dot{x}(u(s))(u^k(s) - u(s))|\, ds \\
& + N\int_0^\alpha |g_0^k(s)|\, ds + \alpha N_2 K_0 |h_k|_\Gamma^2.
\end{aligned}
$$

Therefore (5.24), (5.26), and Lemmas 4.2 and 4.5 yield (5.28).

Simple computation and the definition of g_2^k imply (5.29) immediately. Note that $\gamma \in \mathcal{P}$ yields that $\dot{x}$ is continuous on $[-r, \alpha]$, and $\varphi \in W^{2,\infty}$ and Lemma 5.1 imply that $x \in W^{2,\infty}([-r, \alpha], \mathbb{R}^n)$. Then (4.19) and Lemma 4.2 with $y = \dot{x}$ yield

$$
\lim_{k\to\infty} \frac{1}{|h_k|_\Gamma} \int_0^\alpha |\dot{x}(u^k(s)) - \dot{x}(u(s)) - \ddot{x}(u(s))(u^k(s) - u(s))|\, ds = 0. \tag{5.31}
$$

We have by (5.3) and Lemma 2.7 that $|\ddot{x}(u(s))| \le K_4$ for a.e. $s \in [0, \alpha]$, therefore

$$
\begin{aligned}
\int_0^\alpha |g_2^k(s)|\, ds \;\le\; & \int_0^\alpha |\dot{x}^k(u^k(s)) - \dot{x}(u^k(s)) - \dot{z}^{h_k}(u^k(s))|\, ds \\
& + \int_0^\alpha |\dot{z}^{h_k}(u^k(s)) - \dot{z}^{h_k}(u(s))|\, ds \\
& + \int_0^\alpha |\dot{x}(u^k(s)) - \dot{x}(u(s)) - \ddot{x}(u(s))(u^k(s) - u(s))|\, ds \\
& + K_4 \int_0^\alpha |\omega_\tau(s, x_s, \xi, x_s^k, \xi + h_k^\xi)|\, ds + \alpha K_4 L_2^* \max_{s\in[0,\alpha]} |p_s^k|_C.
\end{aligned}
$$

Hence (4.8), (5.5), (5.18), (5.24) and (5.31) imply (5.30). □

We define the notations

$$
\begin{aligned}
&\omega_{D_2\tau}(s, \bar{\varphi}, \bar{\xi}, \varphi, \xi, \psi) \\
&\quad := D_2\tau(s, \varphi, \xi)\psi - D_2\tau(s, \bar{\varphi}, \bar{\xi})\psi - D_{22}\tau(s, \bar{\varphi}, \bar{\xi})\langle\psi, \varphi - \bar{\varphi}\rangle - D_{23}\tau(s, \bar{\varphi}, \bar{\xi})\langle\psi, \xi - \bar{\xi}\rangle \\
&\omega_{D_3\tau}(s, \bar{\varphi}, \bar{\xi}, \varphi, \xi, \chi) \\
&\quad := D_3\tau(s, \varphi, \xi)\chi - D_3\tau(s, \bar{\varphi}, \bar{\xi})\chi - D_{32}\tau(s, \bar{\varphi}, \bar{\xi})\langle\chi, \varphi - \bar{\varphi}\rangle - D_{33}\tau(s, \bar{\varphi}, \bar{\xi})\langle\chi, \xi - \bar{\xi}\rangle
\end{aligned}
$$

for $s \in [0, \alpha]$, $\bar{\varphi}, \varphi \in \Omega_1$, $\bar{\xi}, \xi \in \Omega_4$, $\psi \in C$ and $\chi \in \Xi$.

Lemma 5.8 *Assume (A0), (A2) (i)–(v) and (H). Then*

$$
\lim_{k\to\infty} \sup_{\substack{h\neq 0\\ h\in\Gamma}} \frac{1}{|h|_\Gamma |h_k|_\Gamma} \int_0^\alpha |\omega_{D_2\tau}(s, x_s, \xi, x_s^k, \xi + h_k^\xi, z_s^{k,h})|\, ds = 0, \tag{5.32}
$$

and

$$\lim_{k\to\infty}\sup_{\substack{h\neq 0\\ h\in\Gamma}}\frac{1}{|h|_\Gamma|h_k|_\Gamma}\int_0^\alpha|\omega_{D_3\tau}(s,x_s,\xi,x_s^k,\xi+h_k^\xi,h^\xi)|\,ds=0. \tag{5.33}$$

Proof Let L_5^* be defined by (5.12). Then the Mean Value Theorem, (3.5), (4.16) and (4.25) yield for $s\in[0,\alpha]$

$$\begin{aligned}|D_2\tau(s,x_s^k,\xi+h_k^\xi)z_s^{k,h}-D_2\tau(s,x_s,\xi)z_s^{k,h}| &\le L_5^*(L_0+1)N_1|h_k|_\Gamma|h|_\Gamma,\\ |D_{22}\tau(s,x_s,\xi)\langle z_s^{k,h},x_s^k-x_s\rangle| &\le L_5^*N_1L_0|h|_\Gamma|h_k|_\Gamma,\\ |D_{23}\tau(s,x_s,\xi)\langle z_s^{k,h},h_k^\xi\rangle| &\le L_5^*N_1|h|_\Gamma|h_k|_\Gamma,\end{aligned}$$

and hence,

$$|\omega_{D_2\tau}(s,x_s,\xi,x_s^k,\xi+h_k^\xi,z_s^{k,h})|\le 2L_5^*(L_0+1)N_1|h_k|_\Gamma|h|_\Gamma,\qquad s\in[0,\alpha].$$

On the other hand, for $s\in[0,\alpha]$, $k\in\mathbb{N}$ and $0\neq h\in\Gamma$ such that $|x_s^k-x_s|_C+|h_k^\xi|_\Gamma\neq 0$ and $|z_s^{k,h}|_C\neq 0$, assumption (A2) (iv), (3.5) and (4.16) yield

$$\begin{aligned}&\sup_{|h|_\Gamma\neq 0}\frac{|\omega_{D_2\tau}(s,x_s,\xi,x_s^k,\xi+h_k^\xi,z_s^{k,h})|}{|h|_\Gamma|h_k|_\Gamma}\\ &=\sup_{|h|_\Gamma\neq 0}\frac{|\omega_{D_2\tau}(s,x_s,\xi,x_s^k,\xi+h_k^\xi,z_s^{k,h})|}{(|x_s^k-x_s|_C+|h_k^\xi|_\Gamma)|z_s^{k,h}|_C}\cdot\frac{(|x_s^k-x_s|_C+|h_k^\xi|_\Gamma)|z_s^{k,h}|_C}{|h|_\Gamma|h_k|_\Gamma}\\ &\le (L_0+1)N_1\sup_{|h|_\Gamma\neq 0}\frac{|\omega_{D_2\tau}(s,x_s,\xi,x_s^k,\xi+h_k^\xi,z_s^{k,h})|}{(|x_s^k-x_s|_C+|h_k^\xi|_\Gamma)|z_s^{k,h}|_C}\\ &\to 0,\qquad k\to\infty.\end{aligned}$$

Note that for s,k and h such that $|x_s^k-x_s|_C+|h_k^\xi|_\Gamma=0$ or $|z_s^{k,h}|_C=0$, $|\omega_{D_2\tau}(s,x_s,\xi,x_s^k,\xi+h_k^\xi,z_s^{k,h})|=0$. Therefore the Dominated Convergence Theorem implies (5.32).

The proof of (5.33) is similar. □

For a.e. $s\in[0,\alpha]$, $h,y\in\Gamma$ we introduce the bilinear operators by

$$\begin{aligned}G(s)\langle(h^\varphi,h^\xi),(y^\varphi,y^\xi)\rangle &:= D_{22}\tau(s,x_s,\xi)\langle h^\varphi,y^\varphi\rangle+D_{23}\tau(s,x_s,\xi)\langle h^\varphi,y^\xi\rangle\\ &\quad+D_{32}\tau(s,x_s,\xi)\langle h^\xi,y^\varphi\rangle+D_{33}\tau(s,x_s,\xi)\langle h^\xi,y^\xi\rangle,\\ H(s)\langle(h^\varphi,h^\xi),(y^\varphi,y^\xi)\rangle &:= -A(s,h^\varphi,h^\xi)F(s,y^\varphi,y^\xi)-\dot{x}(u(s))G(s)\langle(h^\varphi,h^\xi),(y^\varphi,y^\xi)\rangle\\ &\quad-\dot{h}^\varphi(-\tau(s,x_s,\xi))A(s,y^\varphi,y^\xi),\end{aligned}$$

and

$$\begin{aligned}B(s)\langle h,y\rangle &:= D_{22}f(\mathbf{v}(s))\langle h^\varphi,y^\varphi\rangle+D_{23}f(\mathbf{v}(s))\langle h^\varphi,E(s,y^\varphi,y^\xi)\rangle+D_{24}f(\mathbf{v}(s))\langle h^\varphi,y^\theta\rangle\\ &\quad+D_{32}f(\mathbf{v}(s))\langle E(s,h^\varphi,h^\xi),y^\varphi\rangle+D_{33}f(\mathbf{v}(s))\langle E(s,h^\varphi,h^\xi),E(s,y^\varphi,y^\xi)\rangle\\ &\quad+D_{34}f(\mathbf{v}(s))\langle E(s,h^\varphi,h^\xi),y^\theta\rangle+D_{42}f(\mathbf{v}(s))\langle h^\theta,y^\varphi\rangle\\ &\quad+D_{43}f(\mathbf{v}(s))\langle h^\theta,E(s,y^\varphi,y^\xi)\rangle+D_{44}f(\mathbf{v}(s))\langle h^\theta,y^\theta\rangle\\ &\quad+D_3f(\mathbf{v}(s))H(s)\langle(h^\varphi,h^\xi),(y^\varphi,y^\xi)\rangle.\end{aligned}$$

Note that G, H and B correspond to γ, but this dependence is omitted for simplicity in the notation.

For $\gamma \in P_2 \cap \Gamma_2$ consider the corresponding solution x of the IVP (1.1)-(1.2), and let z^h and z^y be the solutions of the IVP (4.14)-(4.15) corresponding to a fixed $h, y \in \Gamma$. We consider the IVP

$$\begin{aligned} \dot{w}(t) &= L(t,x)(w_t,0,0) + B(t)\langle (z_t^h, h^\theta, h^\xi), (z_t^y, y^\theta, y^\xi)\rangle, \quad \text{a.e. } t \in [0,\alpha], && (5.34) \\ w(t) &= 0, \qquad t \in [-r,0]. && (5.35) \end{aligned}$$

The IVP (5.34)-(5.35) is a Carathéodory type inhomogeneous linear delay system with time-dependent but state-independent delays. It is easy to see that under assumptions (A0), (A1) (i), (iii), (A2) (i),(iv) the IVP (5.34)-(5.35) has a unique solution on $[-r, \alpha]$, which will be denoted by $w^{h,y}(t) := w(t, \gamma, h, y)$. It is easy to see that $\Gamma \times \Gamma \to \mathbb{R}^n$, $(h, y) \mapsto w(t, \gamma, h, y)$ is a bilinear map for a fixed $t \in [0, \alpha]$ and $\gamma \in P_2 \cap \Gamma_2$. In Lemma 5.13 below we will show that this bilinear map is bounded, and our main Theorem 5.17 will show that it is the second-order partial derivative of $x(t, \gamma)$ wrt γ.

The next results show the covergence of A^k to A in different senses.

Lemma 5.9 *Assume (A0), (A1) (i), (A2) (i)–(v) and (H). Then there exists $K_{10} = K_{10}(\gamma, h_k) \geq 0$ such that*

$$|A^k(s, z_s^{j,h}, h^\xi) - A(s, z_s^{j,h}, h^\xi)| \leq K_{10}|h|_\Gamma |h_k|_\Gamma, \qquad s \in [0,\alpha],\ k \in \mathbb{N},\ j \in \mathbb{N}_0, \tag{5.36}$$

and there exists a sequence $c_{2,k} \geq 0$ satisfying $c_{2,k} \to 0$ as $k \to \infty$ such that

$$|A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^h, h^\xi)| \leq c_{2,k}|h|_\Gamma, \qquad s \in [0,\alpha],\ k \in \mathbb{N}. \tag{5.37}$$

Proof Let L_2^* be defined by (4.20), and L_5^* be defined by

$$L_5^* := \max_{i,j=2,3} \max\{|D_{ij}\tau(t,\psi,\xi)|_{\mathcal{L}^2(Z_i \times Z_j, \mathbb{R})} \colon t \in [0,\alpha],\ \psi \in M_1^*,\ \xi \in M_4^*\}, \tag{5.38}$$

where $Z_2 = C$, $Z_3 = \Xi$. To show (5.37) we use (3.5), (4.16), (4.25) and the Mean Value Theorem to get

$$\begin{aligned} &|A^k(s, z_s^{j,h}, h^\xi) - A(s, z_s^{j,h}, h^\xi)| \\ &\quad \leq |D_2\tau(s, x_s^k, \xi + h_k^\xi) z_s^{j,h} - D_2\tau(s, x_s, \xi) z_s^{j,h}| + |D_3\tau(s, x_s^k, \xi + h_k^\xi) h^\xi - D_3\tau(s, x_s, \xi) h^\xi| \\ &\quad \leq L_5^*(L_0+1)|h_k|_\Gamma N_1 |h|_\Gamma + L_5^*(L_0+1)|h_k|_\Gamma |h|_\Gamma, \qquad s \in [0,\alpha],\ k \in \mathbb{N},\ j \in \mathbb{N}_0, \end{aligned}$$

which yields (5.36). Using (4.34), (5.36), (5.38) and the Mean Value Theorem we get

$$\begin{aligned} &|A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^h, h^\xi)| \\ &\quad \leq |A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^{k,h}, h^\xi)| + |A(s, z_s^{k,h}, h^\xi) - A(s, z_s^h, h^\xi)| \\ &\quad \leq K_{10}|h|_\Gamma |h_k|_\Gamma + |D_2\tau(s, x_s, \xi)(z_s^{k,h} - z_s^h)| \\ &\quad \leq K_{10}|h_k|_\Gamma |h|_\Gamma + L_2^* c_{1,k} N_5 |h|_\Gamma, \qquad s \in [0,\alpha],\ k \in \mathbb{N}, \end{aligned}$$

therefore (5.37) holds. □

The next Lemma gives the linearization of $A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^h, h^\xi)$ and $E^k(s, z_s^{k,h}, h^\xi) - E(s, z_s^h, h^\xi)$. We need the further notation

$$q^{k,h}(s) \quad := \quad z^{k,h}(s) - z^h(s) - w^{h,h_k}(s), \quad s \in [-r, \alpha].$$

Lemma 5.10 *Assume (A0), (A1) (i)–(iv), (A2) (i)–(v), (H) and* $\gamma \in P_2 \cap \Gamma_2 \cap \mathcal{P}$. *Then*

$$\begin{aligned} & A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^h, h^\xi) - G(s)\langle (z_s^h, h^\xi), (z_s^{h_k}, h_k^\xi)\rangle - A(s, w_s^{h,h_k}, 0) \\ &= \quad A(s, q_s^{k,h}, 0) + g_3^{k,h}(s), \qquad s \in [0, \alpha],\ h \in \Gamma,\ k \in \mathbb{N}, \end{aligned} \tag{5.39}$$

where

$$\begin{aligned} g_3^{k,h}(s) \quad := \quad & D_{22}\tau(s, x_s, \xi)\langle z_s^{k,h} - z_s^h, x_s^k - x_s\rangle + D_{22}\tau(s, x_s, \xi)\langle z_s^h, p_s^k\rangle \\ & + D_{23}\tau(s, x_s, \xi)\langle z_s^{k,h} - z_s^h, h_k^\xi\rangle + D_{32}\tau(s, x_s, \xi)\langle h^\xi, p_s^k\rangle \\ & + \omega_{D_2\tau}(s, x_s, \xi, x_s^k, \xi + h_k^\xi, z_s^{k,h}) + \omega_{D_3\tau}(s, x_s, \xi, x_s^k, \xi + h_k^\xi, h^\xi) \end{aligned}$$

satisfies

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0 \\ h\in\Gamma}} \frac{1}{|h|_\Gamma |h_k|_\Gamma} \int_0^\alpha |g_3^{k,h}(s)|\, ds = 0; \tag{5.40}$$

and if $h_k \in \Gamma_2$ *for* $k \in \mathbb{N}$, *then*

$$\begin{aligned} & E^k(s, z_s^{k,h}, h^\xi) - E(s, z_s^h, h^\xi) - H(s)\langle (z_s^h, h^\xi), (z_s^{h_k}, h_k^\xi)\rangle - E(s, w_s^{h,h_k}, 0) \\ &= \quad E(s, q_s^{k,h}, 0) + g_4^{k,h}(s), \qquad \text{a.e. } s \in [0, \alpha],\ h \in \Gamma,\ k \in \mathbb{N} \end{aligned} \tag{5.41}$$

with

$$\begin{aligned} g_4^{k,h}(s) \quad := \quad & -[\dot{x}^k(u^k(s)) - \dot{x}(u(s))][A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^{k,h}, h^\xi)] - g_2^k(s) A(s, z_s^{k,h}, h^\xi) \\ & - \dot{x}(u(s)) g_3^{k,h}(s) + z^{k,h}(u^k(s)) - z^h(u^k(s)) - [z^{k,h}(u(s)) - z^h(u(s))] \\ & + z^h(u^k(s)) - z^h(u(s)) - \dot{z}^h(u(s))(u^k(s) - u(s)) + \dot{z}^h(u(s)) g_0^k(s) \end{aligned}$$

satisfying

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0 \\ h\in\Gamma_2}} \frac{1}{|h|_{\Gamma_2} |h_k|_{\Gamma_2}} \int_0^\alpha |g_4^{k,h}(s)|\, ds = 0. \tag{5.42}$$

Proof The definitions of $A^k, A, G, g_3^{k,h}, \omega_{D_2\tau}, \omega_{D_3\tau}$ and relation

$$A(s, z_s^{k,h}, h^\xi) - A(s, z_s^h, h^\xi) - A(s, w_s^{h,h_k}, 0) = A(s, z_s^{k,h} - z_s^h - w_s^{h,h_k}, 0)$$

yield

$$\begin{aligned} & A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^h, h^\xi) - G(s)\langle (z_s^h, h^\xi), (z_s^{h_k}, h_k^\xi)\rangle - A(s, w_s^{h,h_k}, 0) \\ &= \quad A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^{k,h}, h^\xi) - G(s)\langle (z_s^h, h^\xi), (z_s^{h_k}, h_k^\xi)\rangle + A(s, q_s^{k,h}, 0) \\ &= \quad D_2\tau(s, x_s^k, \xi + h_k^\xi) z_s^{k,h} - D_2\tau(s, x_s, \xi) z_s^{k,h} - D_{22}\tau(s, x_s, \xi)\langle z_s^{k,h}, x_s^k - x_s\rangle \\ & \quad - D_{23}\tau(s, x_s, \xi)\langle z_s^{k,h}, h_k^\xi\rangle + D_{22}\tau(s, x_s, \xi)\langle z_s^{k,h} - z_s^h, x_s^k - x_s\rangle \\ & \quad + D_{22}\tau(s, x_s, \xi)\langle z_s^h, p_s^k\rangle + D_{23}\tau(s, x_s, \xi)\langle z_s^{k,h} - z_s^h, h_k^\xi\rangle \\ & \quad + D_3\tau(t, x_s^k, \xi + h_k^\xi) h^\xi - D_3\tau(s, x_s, \xi) h^\xi - D_{32}\tau(s, x_s, \xi)\langle h^\xi, x_s^k - x_s\rangle \\ & \quad - D_{33}\tau(s, x_s, \xi)\langle h^\xi, h_k^\xi\rangle + D_{32}\tau(s, x_s, \xi)\langle h^\xi, p_s^k\rangle + A(s, q_s^{k,h}, 0) \\ &= \quad A(s, q_s^{k,h}, 0) + g_3^{k,h}(s). \end{aligned}$$

Let L_5^* be defined by (5.38). Then we have by (3.5), (4.16) and (4.34)

$$\begin{aligned}\int_0^\alpha |g_3^{k,h}(s)|\,ds \quad &\le \quad \alpha L_5^* c_{1,k} N_5 |h|_\Gamma L_0 |h_k|_\Gamma + \alpha L_5^* N_1 |h|_\Gamma \max_{s\in[0,\alpha]} |p_s^k|_C + \alpha L_5^* c_{1,k} N_5 |h|_\Gamma |h_k|_\Gamma \\ &\quad + \alpha L_5^* |h|_\Gamma \max_{s\in[0,\alpha]} |p_s^k|_C + \int_0^\alpha |\omega_{D_2\tau}(s, x_s, \xi, x_s^k, \xi + h_k^\xi, z_s^{k,h})|\,ds \\ &\quad + \int_0^\alpha |\omega_{D_3\tau}(s, x_s, \xi, x_s^k, \xi + h_k^\xi, h^\xi)|\,ds.\end{aligned}$$

Hence $c_{1,k} \to 0$ as $k \to \infty$, (5.24), (5.32) and (5.33) imply (5.40).

Relation

$$E(s, z_s^{k,h}, h^\xi) - E(s, z_s^h, h^\xi) - E(s, w_s^{h,h_k}, 0) = E(s, z_s^{k,h} - z_s^h - w_s^{h,h_k}, 0)$$

and the definition of E, E^k and H give

$$\begin{aligned} & E^k(s, z_s^{k,h}, h^\xi) - E(s, z_s^h, h^\xi) - H(s)\langle (z_s^h, h^\xi), (z_s^{h_k}, h_k^\xi)\rangle - E(s, w_s^{h,h_k}, 0) \\ &= \quad E^k(s, z_s^{k,h}, h^\xi) - E(s, z_s^{k,h}, h^\xi) - H(s)\langle (z_s^h, h^\xi), (z_s^{h_k}, h_k^\xi)\rangle + E(s, q_s^{k,h}, 0) \\ &= \quad -\dot{x}^k(u^k(s)) A^k(s, z_s^{k,h}, h^\xi) + \dot{x}(u(s)) A(s, z_s^{k,h}, h^\xi) + z^{k,h}(u^k(s)) - z^{k,h}(u(s)) \\ &\quad\quad + A(s, z_s^h, h^\xi) F(s, z_s^{h_k}, h_k^\xi) + \dot{x}(u(s)) G(s)\langle (z_s^h, h^\xi), (z_s^{h_k}, h_k^\xi)\rangle \\ &\quad\quad + \dot{z}^h(u(s)) A(s, z_s^{h_k}, h_k^\xi) + E(s, q_s^{k,h}, 0) \\ &= \quad -[\dot{x}^k(u^k(s)) - \dot{x}(u(s))][A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^{k,h}, h^\xi)] \\ &\quad\quad -[\dot{x}^k(u^k(s)) - \dot{x}(u(s)) - F(s, z_s^{h_k}, h_k^\xi)] A(s, z_s^{k,h}, h^\xi) \\ &\quad\quad -\dot{x}(u(s)) \Big[A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^{k,h}, h^\xi) - G(s)\langle (z_s^h, h^\xi), (z_s^{h_k}, h_k^\xi)\rangle\Big] \\ &\quad\quad + z^{k,h}(u^k(s)) - z^h(u^k(s)) - [z^{k,h}(u(s)) - z^h(u(s))] \\ &\quad\quad + z^h(u^k(s)) - z^h(u(s)) - \dot{z}^h(u(s))(u^k(s) - u(s)) \\ &\quad\quad + \dot{z}^h(u(s)) \Big(u^k(s) - u(s) + A(s, z_s^{h_k}, h_k^\xi)\Big) + E(s, q_s^{k,h}, 0),\end{aligned}$$

which, together with (5.29) and (5.39), yields (5.41).

To prove (5.42) first note that by (3.5), (4.18) and (5.3)

$$\begin{aligned} |\dot{x}^k(u^k(s)) - \dot{x}(u(s))| \quad &\le \quad |\dot{x}^k(u^k(s)) - \dot{x}(u^k(s))| + |\dot{x}(u^k(s)) - \dot{x}(u(s))| \\ &\le \quad L_0 |h_k|_\Gamma + K_4 K_0 |h_k|_\Gamma. \end{aligned} \tag{5.43}$$

Hence (5.36) and (5.43) give

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0 \\ h\in\Gamma}} \frac{1}{|h|_\Gamma |h_k|_\Gamma} \int_0^\alpha |\dot{x}^k(u^k(s)) - \dot{x}(u(s))| |A^k(s, z_s^{k,h}, h^\xi) - A(s, z_s^{k,h}, h^\xi)|\,ds = 0.$$

Relations (3.4), (5.10), (5.30) and (5.40) imply for $h_k \in \Gamma_2$ for $k \in \mathbb{N}$

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0 \\ h\in\Gamma}} \frac{1}{|h|_\Gamma |h_k|_{\Gamma_2}} \int_0^\alpha |g_2^k(s) A(s, z_s^{k,h}, h^\xi)|\,ds \le \lim_{k\to\infty} \frac{K_6}{|h_k|_{\Gamma_2}} \int_0^\alpha |g_2^k(s)|\,ds = 0$$

and

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0 \\ h\in\Gamma}} \frac{1}{|h|_\Gamma |h_k|_\Gamma} \int_0^\alpha |\dot{x}(u(s)) g_3^{k,h}(s)|\,ds \le \lim_{k\to\infty} \frac{N}{|h|_\Gamma |h_k|_\Gamma} \int_0^\alpha |g_3^{k,h}(s)|\,ds = 0.$$

The above limits and (5.19), (5.21), $|\dot{z}^h(u(s))| \le N_2 |h|_{\Gamma_2}$ and (5.26) yield (5.42).

□

The next estimates will be used in the proof of Lemmas 5.12 and 5.16.

Lemma 5.11 *Assume (A0), (A1) (i)-(ii), (A2) (i)–(v), (H) and* $\gamma \in P_2 \cap \Gamma_2 \cap \mathcal{P}$. *Let* $F^0 := F$. *Then there exist* $K_{11} = K_{11}(\gamma) \geq 0$ *and a nonnegative sequence* $c_{3,k} = c_{3,k}(\gamma, h_k)$ *satisfying* $c_{3,k} \to 0$ *as* $k \to \infty$ *such that*

$$|F^k(s, z^h_s, h^\xi)| \leq K_{11}|h|_\Gamma, \qquad a.e.\ s \in [0,\alpha],\ h \in \Gamma,\ k \in \mathbb{N}_0 \tag{5.44}$$

$$|E^k(s, z^{k,h}_s, h^\xi) - E(s, z^h_s, h^\xi)| \leq c_{3,k}|h|_\Gamma, \qquad a.e.\ s \in [0,\alpha],\ k \in \mathbb{N}, \tag{5.45}$$

and, if in addition, (A2) (vi) holds, there exists a nonnegative sequence $c_{4,k} = c_{4,k}(\gamma, h_k)$ *satisfying* $c_{4,k} \to 0$ *as* $k \to \infty$ *such that*

$$\int_0^\alpha |F^k(s, z^h_s, h^\xi) - F(s, z^h_s, h^\xi)|\, ds \leq c_{4,k}|h|_{\Gamma_2}, \qquad k \in \mathbb{N},\ h \in \Gamma_2. \tag{5.46}$$

Proof Let L_2^* be defined by (4.20). A simple generalization of (5.10) yields

$$|A^k(s, z^{k,h}_s, h^\xi)| \leq K_6^*|h|_\Gamma, \qquad s \in [0,\alpha],\ h \in \Gamma,\ k \in \mathbb{N}_0. \tag{5.47}$$

with $K_6^* := L_2^*(N_1 + 1)$. The definition of F, (5.3) and (5.47) imply immediately (5.44) with $K_{11} := K_4K_6^* + 1$.

Relations (3.4), (3.5), (4.16), (4.17), (4.18), (4.34), (5.37), (5.43) and (5.47) yield for a.e. $s \in [0,\alpha]$

$$\begin{aligned} &|E^k(s, z^{k,h}_s, h^\xi) - E(s, z^h_s, h^\xi)| \\ &\quad\leq |\dot{x}^k(u^k(s)) - \dot{x}(u(s))||A^k(s, z^{k,h}_s, h^\xi)| \\ &\qquad + |\dot{x}(u(s))|\Big|A^k(s, z^{k,h}_s, h^\xi) - A(s, z^h_s, h^\xi)\Big| + |z^{k,h}(u^k(s)) - z^h(u^k(s))| \\ &\qquad + |z^h(u^k(s)) - z^h(u(s))| \\ &\quad\leq (L_0 + K_4K_0)|h_k|_\Gamma K_6^*|h|_\Gamma + Nc_{2,k}|h|_\Gamma + c_{1,k}N_5|h|_\Gamma + N_2|h|_\Gamma K_0|h_k|_\Gamma, \end{aligned}$$

which proves (5.45).

$$\begin{aligned} &|F^k(s, z^h_s, h^\xi) - F(s, z^h_s, h^\xi)| \\ &\quad\leq \Big(|\ddot{x}^k(u^k(s)) - \ddot{x}(u^k(s))| + |\ddot{x}(u^k(s)) - \ddot{x}(u(s))|\Big)|A^k(s, z^h_s, h^\xi)| \\ &\qquad + |\ddot{x}(u(s))|\Big|A^k(s, z^h_s, h^\xi) - A(s, z^h_s, h^\xi)\Big| + |\dot{z}^h(u^k(s)) - \dot{z}^h(u(s))|. \end{aligned}$$

Let $L_6^* = L_6(\alpha, M_1^*, M_2^*, M_3^*, M_4^*, \gamma, x)$ be the Lipschitz constant from (A2) (vi). Then for $t \in (0,\alpha]$ we have by (A2) (vi) that

$$\begin{aligned} |\ddot{x}^k(t) - \ddot{x}(t)| &= \Big|\frac{d}{dt}f(t, x^k_t, x^k(u^k(t)), \theta + h^\theta_k) - \frac{d}{dt}f(t, x_t, x(u(t)), \theta)\Big| \\ &\leq L_6^*(|x^k_t - x_t|_{W^{1,\infty}} + |h^\theta_k|_\Theta + |h^\xi_k|_\Xi) \\ &\leq L_6^*(L_0 + 1)|h_k|_\Gamma \\ &\leq L_6^*(L_0 + 1)|h_k|_{\Gamma_2}. \end{aligned}$$

For $t \in [-r, 0)$ and $h \in \Gamma_2$ we get

$$|\ddot{x}^k(t) - \ddot{x}(t)| = |\ddot{h}^\varphi_k(t)| \le |h_k|_{\Gamma_2}.$$

Using that $\ddot{x} \in L^\infty([-r, \alpha], \mathbb{R}^n)$, similarly to (4.33) we can argue that

$$\lim_{k\to\infty} \int_0^\alpha |\ddot{x}(u^k(s)) - \ddot{x}(u(s))|\, ds = 0.$$

Then the above relations, $|\ddot{x}(u(s))| \le K_4$ for a.e. $s \in [0, \alpha]$, (5.47), (5.18) and (5.36) yield (5.46). □

For a.e. $s \in [0, \alpha]$, $h, y \in \Gamma$ and $k \in \mathbb{N}$ we introduce the bilinear operators by

$$\begin{aligned}
G^k(s)\langle (h^\varphi, h^\xi), (y^\varphi, y^\xi)\rangle \;&:=\; D_{22}\tau(s, x^k_s, \xi + h^\xi_k)\langle h^\varphi, y^\varphi\rangle + D_{23}\tau(s, x^k_s, \xi + h^\xi_k)\langle h^\varphi, y^\xi\rangle \\
&\quad + D_{32}\tau(s, x^k_s, \xi + h^\xi_k)\langle h^\xi, y^\varphi\rangle + D_{33}\tau(s, x^k_s, \xi + h^\xi_k)\langle h^\xi, y^\xi\rangle, \\
H^k(s)\langle (h^\varphi, h^\xi), (y^\varphi, y^\xi)\rangle \;&:=\; -A^k(s, h^\varphi, h^\xi) F^k(s, y^\varphi, y^\xi) \\
&\quad - \dot{x}^k(u^k(s)) G^k(s)\langle (h^\varphi, h^\xi), (y^\varphi, y^\xi)\rangle \\
&\quad - \dot{h}^\varphi(-\tau(s, x^k_s, \xi + h^\xi_k)) A^k(s, y^\varphi, y^\xi),
\end{aligned}$$

and

$$\begin{aligned}
B^k(s)\langle h, y\rangle \;:=\;& D_{22}f(\mathbf{v}^k(s))\langle h^\varphi, y^\varphi\rangle + D_{23}f(\mathbf{v}^k(s))\langle h^\varphi, E^k(s, y^\varphi, y^\xi)\rangle \\
&+ D_{24}f(\mathbf{v}^k(s))\langle h^\varphi, y^\theta\rangle + D_{32}f(\mathbf{v}^k(s))\langle E^k(s, h^\varphi, h^\xi), y^\varphi\rangle \\
&+ D_{33}f(\mathbf{v}^k(s))\langle E^k(s, h^\varphi, h^\xi), E^k(s, y^\varphi, y^\xi)\rangle \\
&+ D_{34}f(\mathbf{v}^k(s))\langle E^k(s, h^\varphi, h^\xi), y^\theta\rangle + D_{42}f(\mathbf{v}^k(s))\langle h^\theta, y^\varphi\rangle \\
&+ D_{43}f(\mathbf{v}^k(s))\langle h^\theta, E^k(s, y^\varphi, y^\xi)\rangle + D_{44}f(\mathbf{v}^k(s))\langle h^\theta, y^\theta\rangle \\
&+ D_3 f(\mathbf{v}^k(s)) H^k(s)\langle (h^\varphi, h^\xi), (y^\varphi, y^\xi)\rangle.
\end{aligned}$$

The next result shows the boundedness of B^k and the convergence of B^k to B.

Lemma 5.12 *Assume (A0), (A1) (i)–(iv), (A2) (i)–(v), (H) and $\gamma \in P_2 \cap \Gamma_2 \cap \mathcal{P}$. Let $B^0 := B$. Then there exists $K_{12} = K_{12}(\gamma, h_k) \ge 0$ such that*

$$|B^k(s)\langle (h^\varphi, h^\xi), (y^\varphi, y^\xi)\rangle| \le K_{12}|h|_\Gamma |y|_\Gamma, \quad \text{a.e. } s \in [0, \alpha],\; h, y \in \Gamma,\; \gamma \in P_2,\; k \in \mathbb{N}_0. \tag{5.48}$$

If in addition (A2) (vi) and $\gamma \in P_2 \cap \Gamma_2 \cap \mathcal{P}$ hold, then there exists a nonnegative sequence $c_{5,k} = c_{5,k}(\gamma)$ such that $c_{5,k} \to 0$ as $k \to \infty$, and

$$\int_0^\alpha \Big| B^k(s)\langle (z^h_s, h^\theta, h^\xi), (z^y_s, y^\theta, y^\xi)\rangle - B(s)\langle (z^h_s, h^\theta, h^\xi), (z^y_s, y^\theta, y^\xi)\rangle \Big|\, ds \le c_{5,k}|h|_{\Gamma_2}|y|_{\Gamma_2}, \tag{5.49}$$

for $h, y \in \Gamma_2$.

Proof Let L_4^* be defined by

$$L_4^* := \max_{i,j=2,3,4} \max\{|D_{ij}f(t,\psi,u,\theta)|_{\mathcal{L}^2(Y_i\times Y_j,\mathbb{R}^n)}\colon\ t\in[0,\alpha],\ \psi\in M_1^*,\ u\in M_2^*,\ \theta\in M_3^*\}, \quad (5.50)$$

where $Y_2 = C$, $Y_3 = \mathbb{R}^n$, $Y_4 = \Theta$, and L_5^* be defined by (5.38). Introduce the notations $G^0 := G$ and $H^0 := H$. Then the definitions of G^k and L_5^* yield

$$\begin{aligned} |G^k(s)\langle(h^\varphi,h^\xi),(y^\varphi,y^\xi)\rangle| &\le L_5^*|h^\varphi|_C|y^\varphi|_C + L_5^*|h^\varphi|_C|y^\xi|_\Xi + L_5^*|h^\xi|_\Xi|y^\varphi|_C + L_5^*|h^\xi|_\Xi|y^\xi|_\Xi \\ &\le L_5^*(|h^\varphi|_C + |h^\xi|_\Xi)(|y^\varphi|_C + |y^\xi|_\Xi) \\ &\le L_5^*|h|_\Gamma|y|_\Gamma, \qquad h,y\in\Gamma,\ s\in[0,\alpha],\ k\in\mathbb{N}_0. \end{aligned} \quad (5.51)$$

Then definition of H^k, (3.4), (4.16), (5.2), (5.44), (5.51) and (5.47) imply

$$|H^k(s)\langle(z_s^h,h^\xi),(z_s^y,y^\xi)\rangle| \le K_{13}|h|_\Gamma|y|_\Gamma, \quad \text{a.e. } s\in[0,\alpha],\ h,y\in\Gamma,\ k\in\mathbb{N}_0 \quad (5.52)$$

with $K_{13} = K_{13}(\gamma) := K_6^*K_{11} + NL_5^* + K_6^*$.

Let $E^0 := E$. An easy generalization of (5.16) yields

$$|E^k(s,z_s^h,h^\xi)| \le K_8^*|h|_\Gamma, \qquad k\in\mathbb{N}_0$$

with $K_8^* := NK_6^* + N_1$. Then we have by the definitions of B^k and L_4^*, (5.16) and (5.52)

$$|B^k(s)\langle h,y\rangle| \le L_4^*(4 + 4K_8^* + (K_8^*)^2 + K_{13})|h|_\Gamma|y|_\Gamma, \qquad \text{a.e. } s\in[0,\alpha],\ h,y\in\Gamma,\ k\in\mathbb{N}_0,$$

which yields (5.48).

Define

$$\begin{aligned} \Omega_{2,\tau}(\varepsilon) \ :=\ \max_{i,j=2,3}\sup\Big\{&|D_{ij}\tau(s,\tilde\psi,\tilde\eta) - D_{ij}\tau(s,\bar\psi,\bar\eta)|_{\mathcal{L}^2(Z_i\times Z_j,\mathbb{R})}\colon \\ & s\in[0,\alpha],\ \tilde\psi,\bar\psi\in M_1^*,\tilde\eta,\bar\eta\in M_4^*,\ |\tilde\psi-\bar\psi|_C + |\tilde\eta-\bar\eta|_\Xi \le \varepsilon\Big\}, \end{aligned}$$

where $Z_2 := C$ and $Z_3 := \Xi$. Assumption (A2) (iv) and the compactness of $[0,\alpha]\times M_1^*\times M_4^*$ yield that $\Omega_{2,\tau}(\varepsilon)\to 0$ as $\varepsilon\to 0+$. Then (4.16) and (4.25) give

$$\begin{aligned} |[G^k(s) - G(s)]\langle(z_s^h,h^\xi),(z_s^y,y^\xi)\rangle| &\le |[D_{22}\tau(s,x_s^k,\xi+h_k^\xi) - D_{22}\tau(s,x_s,\xi)]\langle z_s^h,z_s^y\rangle| \\ &\quad + |[D_{23}\tau(s,x_s^k,\xi+h_k^\xi) - D_{23}\tau(s,x_s^k,\xi+h_k^\xi)]\langle z_s^h,y^\xi\rangle| \\ &\quad + |[D_{32}\tau(s,x_s^k,\xi+h_k^\xi) - D_{32}\tau(s,x_s^k,\xi+h_k^\xi)]\langle h^\xi,z_s^y\rangle| \\ &\quad + |[D_{33}\tau(s,x_s^k,\xi+h_k^\xi) - D_{33}\tau(s,x_s^k,\xi+h_k^\xi)]\langle h^\xi,y^\xi\rangle| \\ &\le \Omega_{2,\tau}\Big((L_0+1)|h_k|_\Gamma\Big)(N_1+1)^2|h|_\Gamma|y|_\Gamma, \qquad s\in[0,\alpha]. \end{aligned} \quad (5.53)$$

Relations (3.4), (3.5), (4.18), (4.16), (4.17), (5.36), (5.43), (5.44), (5.46), (5.51), (5.53) and

(5.47) imply

$$
\begin{aligned}
&\int_0^\alpha |[H^k(s)-H(s)]\langle(z^h_s,h^\xi),(z^y_s,y^\xi)\rangle|\,ds \\
&\quad\le \int_0^\alpha \Big(|[A^k(s,z^h_s,h^\xi)-A(s,z^h_s,h^\xi)]F(s,z^y_s,y^\xi)| \\
&\qquad +|A^k(s,z^h_s,h^\xi)[F^k(s,z^y_s,y^\xi)-F(s,z^y_s,y^\xi)]| \\
&\qquad +|[\dot{x}^k(u^k(s))-\dot{x}(u(s))]G^k(s)\langle(z^h_s,h^\xi),(z^y_s,y^\xi)\rangle| \\
&\qquad +|\dot{x}(u(s))[G^k(s)-G(s)]\langle(z^h_s,h^\xi),(z^y_s,y^\xi)\rangle| \\
&\qquad +|[\dot{z}^h(u^k(s))-\dot{z}^h(u(s))]A^k(s,z^y_s,y^\xi)| \\
&\qquad +|\dot{z}^h(u(s))[A^k(s,z^y_s,y^\xi)-A(s,z^y_s,y^\xi)]|\Big)\,ds \\
&\quad\le \alpha K_{10}|h|_\Gamma|h_k|_\Gamma K_{11}|y|_\Gamma+K_6^*|h|_\Gamma c_{4,k}|y|_{\Gamma_2}+(L_0+K_4K_0)|h_k|_\Gamma L_5^*N_2^2|h|_\Gamma|y|_\Gamma \\
&\qquad +N\Omega_{2,\tau}\Big((L_0+1)|h_k|_\Gamma\Big)(N_1+1)^2|h|_\Gamma|y|_\Gamma \\
&\qquad +\int_0^\alpha |\dot{z}^h(u^k(s))-\dot{z}^h(u(s))|\,ds\,K_6^*|y|_\Gamma+\alpha N_2|h|_\Gamma K_{10}|h_k|_\Gamma|y|_\Gamma \\
&\quad\le c_{6,k}|h|_{\Gamma_2}|y|_{\Gamma_2} \qquad (5.54)
\end{aligned}
$$

with some appropriate sequence $c_{6,k}=c_{6,k}(\gamma)$ satisfying $c_{6,k}\to 0$ as $k\to\infty$, where in the last estimate we used (5.18).

Simple manipulations give

$$
\begin{aligned}
&|[B^k(s)-B(s)]\langle(z^h_s,h^\theta,h^\xi),(z^y_s,y^\theta,y^\xi)\rangle| \\
&\quad\le |[D_{22}f(\mathbf{v}^k(s))-D_{22}f(\mathbf{v}(s))]\langle z^h_s,z^y_s\rangle| \\
&\qquad +|[D_{23}f(\mathbf{v}^k(s))-D_{23}f(\mathbf{v}(s))]\langle z^h_s,E^k(s,z^y_s,y^\xi)\rangle| \\
&\qquad +|D_{23}f(\mathbf{v}(s))\langle z^h_s,E^k(s,z^y_s,y^\xi)-E(s,z^y_s,y^\xi)\rangle| \\
&\qquad +|[D_{24}f(\mathbf{v}^k(s))-D_{24}f(\mathbf{v}(s))]\langle z^h_s,y^\theta\rangle| \\
&\qquad +|[D_{32}f(\mathbf{v}^k(s))-D_{32}f(\mathbf{v}(s))]\langle E^k(s,z^h_s,h^\xi),z^y_s\rangle| \\
&\qquad +|D_{32}f(\mathbf{v}(s))\langle E^k(s,z^h_s,h^\xi)-E(s,z^h_s,h^\xi),z^y_s\rangle| \\
&\qquad +|[D_{33}f(\mathbf{v}^k(s))-D_{33}f(\mathbf{v}(s))]\langle E^k(s,z^h_s,h^\xi),E^k(s,z^y_s,y^\xi)\rangle| \\
&\qquad +|D_{33}f(\mathbf{v}(s))\langle E^k(s,z^h_s,h^\xi)-E(s,z^h_s,h^\xi),E^k(s,z^y_s,y^\xi)\rangle| \\
&\qquad +|D_{33}f(\mathbf{v}(s))\langle E(s,z^h_s,h^\xi),E^k(s,z^y_s,y^\xi)-E(s,z^y_s,y^\xi)\rangle| \\
&\qquad +|[D_{34}f(\mathbf{v}^k(s))-D_{34}f(\mathbf{v}(s))]\langle E^k(s,z^h_s,h^\xi),y^\theta\rangle| \\
&\qquad +|D_{34}f(\mathbf{v}(s))\langle E^k(s,z^h_s,h^\xi)-E(s,z^h_s,h^\xi),y^\theta\rangle| \\
&\qquad +|[D_{42}f(\mathbf{v}^k(s))-D_{42}f(\mathbf{v}(s))]\langle h^\theta,z^y_s\rangle| \\
&\qquad +|[D_{43}f(\mathbf{v}^k(s))-D_{43}f(\mathbf{v}(s))]\langle h^\theta,E^k(s,z^y_s,y^\xi)\rangle \\
&\qquad +|D_{43}f(\mathbf{v}(s))\langle h^\theta,E^k(s,z^y_s,y^\xi)-E(s,z^y_s,y^\xi)\rangle \\
&\qquad +|[D_{44}f(\mathbf{v}^k(s))-D_{44}f(\mathbf{v}(s))]\langle h^\theta,y^\theta\rangle| \\
&\qquad +|[D_3f(\mathbf{v}^k(s))-D_3f(\mathbf{v}(s))]H^k(s)\langle(z^h_s,h^\xi),(z^y_s,y^\xi)\rangle| \\
&\qquad +|D_3f(\mathbf{v}(s))[H^k(s)-H(s)]\langle(z^h_s,h^\xi),(z^y_s,y^\xi)\rangle|. \qquad (5.55)
\end{aligned}
$$

Define

$$\Omega_{2,f}(\varepsilon) := \max_{i,j=2,3,4} \sup\Big\{ |D_{ij}f(s,\tilde{\psi},\tilde{v},\tilde{\eta}) - D_{ij}f(s,\bar{\psi},\bar{v},\bar{\eta})|_{\mathcal{L}^2(Y_i\times Y_j,\mathbb{R})} : s\in[0,\alpha],\ \tilde{\psi},\bar{\psi}\in M_1^*, \tilde{v},\bar{v}\in M_2^*,\ \tilde{\eta},\bar{\eta}\in M_3^*, |\tilde{\psi}-\bar{\psi}|_C + |\tilde{v}-\bar{v}| + |\tilde{\eta}-\bar{\eta}|_\Theta \le \varepsilon \Big\},$$

where $Y_2 := C$, $Y_3 := \mathbb{R}^n$ and $Y_4 := \Theta$. Assumption (A1) (iii) and the compactness of $[0,\alpha]\times M_1^*\times M_2^*\times M_3^*$ yields that $\Omega_{2,f}(\varepsilon)\to 0$ as $\varepsilon\to 0+$. Let L_1^* be defined by (4.28). Then combining (5.55) with (4.24), $|D_{ij}f(\mathbf{v}^k(s)) - D_{ij}f(\mathbf{v}(s))|_{\mathcal{L}^2(Y_i\times Y_j,\mathbb{R}^n)} \le \Omega_{2,f}\Big(K_3|h_k|_\Gamma\Big)$ for $i,j=2,3,4$, $|D_if(\mathbf{v}^k(s))|_{\mathcal{L}(Y_i,\mathbb{R}^n)} \le L_1^*$ for $i=2,3,4$, $s\in[0,\alpha]$ and $k\in\mathbb{N}_0$, (4.16), (5.16), (5.45), (5.52), (5.54) and (5.55) yields (5.49). □

Now we prove the boundedness of the bilinear map $\Gamma^2\ni(h,y)\mapsto w_t^{h,y}\in W^{1,\infty}$ for any fixed t.

Lemma 5.13 *Assume (A0), (A1) (i)–(iv), (A2) (i)–(v) and $\gamma\in P_2\cap\Gamma_2$. Then there exists $N_8=N_8(\gamma)\ge 0$ such that the solution of the IVP* (5.34)-(5.35) *satisfies*

$$|w_t^{h,y}|_{W^{1,\infty}} \le N_8|h|_\Gamma|y|_\Gamma, \qquad t\in[0,\alpha], \quad h,y\in\Gamma. \tag{5.56}$$

Proof Let L_1^* and L_2^* be defined by (4.28) and (4.20), respectively. It follows from (5.34) and (5.35) that

$$w^{h,y}(t) = \int_0^t B(s)\langle(z_s^h,h^\theta,h^\xi),(z_s^y,y^\theta,y^\xi)\rangle\,ds + \int_0^t L(s,x)(w_s^{h,y},0,0)\,ds, \qquad t\in[0,\alpha].$$

Therefore (4.31) and (5.48) yield

$$|w^{h,y}(t)| \le \alpha K_{12}|h|_\Gamma|y|_\Gamma + N_4\int_0^t |w_s^{h,y}|_C\,ds, \qquad t\in[0,\alpha].$$

Since $w^{h,y}(t)=0$ for $t\in[-r,0]$, Lemma 2.1 gives

$$|w^{h,y}(t)| \le \alpha K_{12}e^{N_4\alpha}|h|_\Gamma|y|_\Gamma, \qquad t\in[0,\alpha], \quad h,y\in\Gamma. \tag{5.57}$$

Then (5.34) implies

$$|\dot{w}^{h,y}(t)| \le |L(t,x)(w_t^{h,y},0,0)| + |B(t)\langle(z_t^h,h^\theta,h^\xi),(z_t^y,y^\theta,y^\xi\rangle|,$$

hence (5.56) holds with $N_8 := \max\{\alpha K_{12}e^{N_4\alpha}, N_4\alpha K_{12}e^{N_4\alpha}+K_{12}\}$. □

Next we prove the continuity of the bilinear map $\Gamma_2^2\ni(h,y)\mapsto w_t^{h,y}\in C$ wrt γ.

Lemma 5.14 *Assume (A0), (A1) (i)–(iv), (A2) (i)–(vi), (H) and $\gamma\in P_2\cap\Gamma_2\cap\mathcal{P}$. For $h,y\in\Gamma_2$ and $k\in\mathbb{N}$ let $w^{h,y}(t):=w(t,\gamma,h,y)$ and $w^{k,h,y}(t):=w(t,\gamma+h_k,h,y)$ be the solutions of the IVP* (5.34)-(5.35)*. Then there exists a nonnegative sequence $c_{7,k}=c_{7,k}(\gamma)$ such that $c_{7,k}\to 0$ as $k\to\infty$ and*

$$|w_t^{k,h,y}-w_t^{h,y}|_C \le c_{7,k}|h|_{\Gamma_2}|y|_{\Gamma_2}, \qquad t\in[0,\alpha], \quad h,y\in\Gamma_2. \tag{5.58}$$

Proof It follows from (5.22) using (4.18) and (5.3) that

$$|\dot z^{k,h}(t)-\dot z^h(t)|\le\Big(N_4c_{1,k}N_5+c_{0,k}(N_2+1)+N_3K_4K_0(N_2+1)|h_k|_\Gamma\Big)|h|_\Gamma,\quad t\in[0,\alpha],$$

therefore (4.34) gives

$$|z^{k,h}_t-z^h_t|_{W^{1,\infty}}\le c_{8,k}|h|_\Gamma,\qquad t\in[0,\alpha],\ h\in\Gamma,\ k\in\mathbb{N}. \tag{5.59}$$

with $c_{8,k}=\max\{c_{1,k}N_5,N_4c_{1,k}N_5+c_{0,k}(N_2+1)+N_3K_4K_0|h_k|_\Gamma\}$.

Let L_1^* and L_2^* be defined by (4.28) and (4.20), respectively. It follows from (4.22), (4.31), (4.35), (5.34), (5.48), (5.43), (5.56) and (5.59)

$$\begin{aligned}
&|w^{k,h,y}(t)-w^{h,y}(t)|\\
&\le \int_0^t\Big(|[L(s,x^k)-L(s,x)](w^{k,h,y}_s,0,0)|+|L(s,x)(w^{k,h,y}_s-w^{h,y}_s,0,0)|\Big)ds\\
&\quad+\int_0^t\Big(|B^k(s)\langle(z^{k,h}_s,h^\theta,h^\xi),(z^{k,y}_s-z^y_s,0,0)\rangle|+|B^k(s)\langle(z^{k,h}_s-z^h_s,0,0),(z^y_s,y^\theta,y^\xi)\rangle|\\
&\qquad+|B^k(s)\langle(z^h_s,h^\theta,h^\xi),(z^y_s,y^\theta,y^\xi)\rangle-B(s)\langle(z^h_s,h^\theta,h^\xi),(z^y_s,y^\theta,y^\xi)\rangle|\Big)ds\\
&\le \alpha c_{0,k}N_8|h|_\Gamma|y|_\Gamma+N_3\int_0^\alpha|\dot x(u^k(s))-\dot x(u(s))|\,dsN_8|h|_\Gamma|y|_\Gamma\\
&\quad+N_4\int_0^t|w^{k,h,y}_s-w^{h,y}_s|_C\,ds+2\alpha K_{12}(N_2+1)c_{8,k}|h|_\Gamma|y|_\Gamma+\alpha c_{5,k}|h|_{\Gamma_2}|y|_{\Gamma_2}\\
&\le c_{9,k}|h|_{\Gamma_2}|y|_{\Gamma_2}+N_4\int_0^t|w^{k,h,y}_s-w^{h,y}_s|_C\,ds,
\end{aligned}$$

where $c_{9,k}=c_{9,k}(\gamma):=\alpha c_{0,k}N_8+\alpha N_3K_4K_0N_8|h_k|_\Gamma+2\alpha K_{12}(N_2+1)c_{8,k}+\alpha c_{5,k}$. Then Lemma 2.1 is applicable, since $|w^{k,h,y}_0-w^{h,y}_0|_C=0$, and it yields (5.58) with $c_{7,k}:=c_{9,k}e^{N_4\alpha}$. □

We define

$$\begin{aligned}
\omega_{D_2f}(\mathbf{v}(s),\mathbf{v}^k(s),\psi)&:=D_2f(\mathbf{v}^k(s))\psi-D_2f(\mathbf{v}(s))\psi-D_{22}f(\mathbf{v}(s))\langle\psi,x^k_s-x_s\rangle\\
&\quad-D_{23}f(\mathbf{v}(s))\langle\psi,x^k(u^k(s))-x(u(s))\rangle-D_{24}f(\mathbf{v}(s))\langle\psi,h^\theta_k\rangle,\\
\omega_{D_3f}(\mathbf{v}(s),\mathbf{v}^k(s),v)&:=D_3f(\mathbf{v}^k(s))v-D_3f(\mathbf{v}(s))v-D_{32}f(\mathbf{v}(s))\langle v,x^k_s-x_s\rangle\\
&\quad-D_{33}f(\mathbf{v}(t))\langle v,x^k(u^k(s))-x(u(s))\rangle-D_{34}f(\mathbf{v}(s))\langle v,h^\theta_k\rangle,\\
\omega_{D_4f}(\mathbf{v}(s),\mathbf{v}^k(s),\eta)&:=D_4f(\mathbf{v}^k(s))\eta-D_4f(\mathbf{v}(s))\eta-D_{42}f(\mathbf{v}(s))\langle\eta,x^k_s-x_s\rangle\\
&\quad-D_{43}f(\mathbf{v}(s))\langle\eta,x^k(u^k(s))-x(u(s))\rangle-D_{44}f(\mathbf{v}(s))\langle\eta,h^\theta_k\rangle
\end{aligned}$$

for $s\in[0,\alpha]$, $\psi\in C$, $v\in\mathbb{R}^n$ and $\eta\in\Theta$.

The proof of the following lemma is similar to that of Lemma 5.8.

Lemma 5.15 *Assume (A0), (A1) (i)–(vi), (A2) (i) and (H). Then*

$$\lim_{k\to\infty}\sup_{\substack{h\neq0\\h\in\Gamma}}\frac{1}{|h|_\Gamma|h_k|_\Gamma}\int_0^\alpha|\omega_{D_2f}(s,x_s,x(u(s)),\theta,x^k_s,x^k(u^k(s)),\theta+h^\theta_k,z^{k,h}_s)|\,ds=0, \tag{5.60}$$

$$\lim_{k\to\infty}\sup_{\substack{h\neq 0\\ h\in\Gamma}}\frac{1}{|h|_\Gamma|h_k|_\Gamma}\int_0^\alpha|\omega_{D_3f}(s,x_s,x(u(s)),\theta,x_s^k,x^k(u^k(s)),\theta+h_k^\theta,E^k(s,z_s^{k,h},h^\xi))|\,ds=0, \tag{5.61}$$

and

$$\lim_{k\to\infty}\sup_{\substack{h\neq 0\\ h\in\Gamma}}\frac{1}{|h|_\Gamma|h_k|_\Gamma}\int_0^\alpha|\omega_{D_4f}(s,x_s,x(u(s)),\theta,x_s^k,x^k(u^k(s)),\theta+h_k^\theta,h_k^\theta)|\,ds=0. \tag{5.62}$$

The proof of the main Theorem 5.17 will be based on the following relation.

Lemma 5.16 *Assume (A0), (A1) (i)–(iv), (A2) (i)–(v), (H),* $\gamma\in P_2\cap\Gamma_2\cap\mathcal{P}$ *and* $h_k\in\Gamma_2$ *for* $k\in\mathbb{N}$. *Then*

$$\begin{aligned}
&L(s,x^k)(z_s^{k,h},h^\theta,h^\xi)-L(s,x)(z_s^h+w_s^{h,h_k},h^\theta,h^\xi)-B(s)\Big\langle(z_s^h,h^\theta,h^\xi),(z_s^{h_k},h_k^\theta,h_k^\xi)\Big\rangle\\
&\quad= L(s,x)(q_s^{k,h},0,0)+g_5^{k,h}(s),\qquad a.e.\ s\in[0,\alpha],
\end{aligned} \tag{5.63}$$

where

$$\begin{aligned}
g_5^{k,h}(s) := {} & D_{22}f(\mathbf{v}(s))\langle z_s^{k,h}-z_s^h,x_s^k-x_s\rangle+D_{22}f(\mathbf{v}(s))\langle z_s^h,p_s^k\rangle\\
&+D_{23}f(\mathbf{v}(s))\langle z_s^{k,h}-z_s^h,x^k(u^k(s))-x(u(s))\rangle+D_{23}f(\mathbf{v}(s))\langle z_s^h,g_1^k(s)\rangle\\
&+D_{24}f(\mathbf{v}(s))\langle z_s^{k,h}-z_s^h,h_k^\theta\rangle+D_{32}f(\mathbf{v}(s))\langle E(s,z_s^h,h^\xi),p_s^k\rangle\\
&+D_{32}f(\mathbf{v}(s))\langle E^k(s,z_s^{k,h},h^\xi)-E(s,z_s^h,h^\xi),x_s^k-x_s\rangle\\
&+D_{33}f(\mathbf{v}(s))\langle E^k(s,z_s^{k,h},h^\xi)-E(s,z_s^h,h^\xi),x^k(u^k(s))-x(u(s))\rangle\\
&+D_{33}f(\mathbf{v}(s))\langle E(s,z_s^h,h^\xi),g_1^k(s)\rangle+D_3f(\mathbf{v}(s))g_4^{k,h}(s)\\
&+D_{34}f(\mathbf{v}(s))\langle E^k(s,z_s^{k,h},h^\xi)-E(s,z_s^h,h^\xi),h_k^\theta\rangle\\
&+D_{42}f(\mathbf{v}(s))\langle h^\theta,p_s^k\rangle+D_{43}f(\mathbf{v}(s))\langle h^\theta,g_1^k(s)\rangle+\omega_{D_2f}(\mathbf{v}(s),\mathbf{v}^k(s),z_s^{k,h})\\
&+\omega_{D_3f}(\mathbf{v}(s),\mathbf{v}^k(s),E^k(s,z_s^{k,h},h^\xi))+\omega_{D_4f}(\mathbf{v}(s),\mathbf{v}^k(s),h^\theta)
\end{aligned}$$

satisfies

$$\lim_{k\to\infty}\sup_{\substack{h\neq 0\\ h\in\Gamma_2}}\frac{1}{|h|_{\Gamma_2}|h_k|_{\Gamma_2}}\int_0^\alpha|g_5^{k,h}(s)|\,ds=0. \tag{5.64}$$

Proof Straightforward manipulations yield for a.e. $s\in[0,\alpha]$

$$\begin{aligned}
&L(s,x^k)(z_s^{k,h},h^\theta,h^\xi)-L(s,x)(z_s^h+w_s^{h,h_k},h^\theta,h^\xi)-B(s)\Big\langle(z_s^h,h^\theta,h^\xi),(z_s^{h_k},h_k^\theta,h_k^\xi)\Big\rangle\\
&= D_2f(\mathbf{v}^k(s))z_s^{k,h}-D_2f(\mathbf{v}(s))z_s^{k,h}+D_2f(\mathbf{v}(s))(z_s^{k,h}-z_s^h-w_s^{h,h_k})\\
&\quad+D_3f(\mathbf{v}^k(s))E^k(s,z_s^{k,h},h^\xi)-D_3f(\mathbf{v}(s))E^k(s,z_s^{k,h},h^\xi)\\
&\quad+D_3f(\mathbf{v}(s))\Big(E^k(s,z_s^{k,h},h^\xi)-E(s,z_s^h,h^\xi)\Big)+D_4f(\mathbf{v}^k(s))h^\theta-D_4f(\mathbf{v}(s))h^\theta\\
&\quad-D_3f(\mathbf{v}(s))E(s,w_s^{h,h_k},0)-B(s)\Big\langle(z_s^h,h^\theta,h^\xi),(z_s^{h_k},h_k^\theta,h_k^\xi)\Big\rangle
\end{aligned}$$

$$
\begin{aligned}
= \;& D_2f(\mathbf{v}^k(s))z_s^{k,h} - D_2f(\mathbf{v}(s))z_s^{k,h} - D_{22}f(\mathbf{v}(s))\langle z_s^{k,h}, x_s^k - x_s\rangle \\
& -D_{23}f(\mathbf{v}(s))\langle z_s^{k,h}, x^k(u^k(s)) - x(u(s))\rangle - D_{24}f(\mathbf{v}(s))\langle z_s^{k,h}, h_k^\theta\rangle \\
& +D_2f(\mathbf{v}(s))q_s^{k,h} + D_{22}f(\mathbf{v}(s))\langle z_s^{k,h} - z_s^h, x_s^k - x_s\rangle + D_{22}f(\mathbf{v}(s))\langle z_s^h, p_s^k\rangle \\
& +D_{23}f(\mathbf{v}(s))\langle z_s^{k,h} - z_s^h, x^k(u^k(s)) - x(u(s))\rangle \\
& +D_{23}f(\mathbf{v}(s))\langle z_s^h, x^k(u^k(s)) - x(u(s)) - E(s, z_s^{h_k}, h_k^\xi)\rangle + D_{24}f(\mathbf{v}(s))\langle z_s^{k,h} - z_s^h, h_k^\theta\rangle \\
& +D_3f(\mathbf{v}^k(s))E^k(s, z_s^{k,h}, h^\xi) - D_3f(\mathbf{v}(s))E^k(s, z_s^{k,h}, h^\xi) \\
& -D_{32}f(\mathbf{v}(s))\langle E^k(s, z_s^{k,h}, h^\xi), x_s^k - x_s\rangle \\
& -D_{33}f(\mathbf{v}(s))\langle E^k(s, z_s^{k,h}, h^\xi), x^k(u^k(s)) - x(u(s))\rangle - D_{34}f(\mathbf{v}(s))\langle E^k(s, z_s^{k,h}, h^\xi), h_k^\theta\rangle \\
& +D_{32}f(\mathbf{v}(s))\langle E^k(s, z_s^{k,h}, h^\xi) - E(s, z_s^h, h^\xi), x_s^k - x_s\rangle + D_{32}f(\mathbf{v}(s))\langle E(s, z_s^h, h^\xi), p_s^k\rangle \\
& +D_{33}f(\mathbf{v}(s))\langle E^k(s, z_s^{k,h}, h^\xi) - E(s, z_s^h, h^\xi), x^k(u^k(s)) - x(u(s))\rangle \\
& +D_{33}f(\mathbf{v}(s))\langle E(s, z_s^h, h^\xi), x^k(u^k(s)) - x(u(s)) - E(s, z_s^{h_k}, h_k^\xi)\rangle \\
& +D_{34}f(\mathbf{v}(s))\langle E^k(s, z_s^{k,h}, h^\xi) - E(s, z_s^h, h^\xi), h_k^\theta\rangle \\
& +D_3f(\mathbf{v}(s))[E^k(s, z_s^{k,h}, h^\xi) - E(s, z_s^h, h^\xi) - H(s)\langle (z_s^h, h^\xi), (z_s^{k,h}, h_k^\xi)\rangle - E(s, w_s^{h,h_k}, 0)] \\
& +D_4(\mathbf{v}^k(s))h^\theta - D_4(\mathbf{v}(s))h^\theta - D_{42}f(\mathbf{v}(s))\langle h^\theta, x_s^k - x_s\rangle \\
& -D_{43}f(\mathbf{v}(s))\langle h^\theta, x^k(u^k(s)) - x(u(s))\rangle - D_{44}f(\mathbf{v}(s))\langle h^\theta, h_k^\theta\rangle \\
& +D_{42}f(\mathbf{v}(s))\langle h^\theta, p_s^k\rangle + D_{43}f(\mathbf{v}(s))\langle h^\theta, x^k(u^k(s)) - x(u(s)) - E(s, z_s^{h_k}, h_k^\xi)\rangle,
\end{aligned}
$$

which implies (5.63), using (5.27) and (5.41). Let L_1^* and L_4^* be defined by (4.28) and (5.50), respectively. Then (3.5), (4.17), (4.23), (4.34), (5.16), (5.45) and (5.50) yield

$$
\begin{aligned}
& \int_0^\alpha |g_5^{k,h}(s)|\, ds \\
\le \;& \alpha L_4^* c_{1,k} N_5 |h|_\Gamma L_0 |h_k|_\Gamma + \alpha L_4^* N_1 |h|_\Gamma \max_{s\in[0,\alpha]} |p_s^k|_C + \alpha L_4^* c_{1,k} N_5 |h|_\Gamma K_2 |h_k|_\Gamma \\
& + L_4^* N_1 |h|_\Gamma \int_0^\alpha |g_1^k(s)|\, ds + \alpha L_4^* c_{1,k} N_5 |h|_\Gamma |h_k|_\Gamma + \alpha L_4^* K_8 |h|_\Gamma \max_{s\in[0,\alpha]} |p_s^k|_C \\
& + \alpha L_4^* c_{3,k} |h|_\Gamma L_0 |h_k|_\Gamma + \alpha L_4^* c_{3,k} |h|_\Gamma K_2 |h_k|_\Gamma \\
& + L_4^* K_8 |h|_\Gamma \int_0^\alpha |g_1^k(s)|\, ds + L_1^* \int_0^\alpha |g_4^{k,h}(s)|\, ds + \alpha L_4^* c_{3,k} |h|_\Gamma |h_k|_\Gamma \\
& + \alpha L_4^* |h|_\Gamma \max_{s\in[0,\alpha]} |p_s^k|_C + L_4^* |h|_\Gamma \int_0^\alpha |g_1^k(s)|\, ds + \int_0^\alpha |\omega_{D_2f}(\mathbf{v}(s), \mathbf{v}^k(s), z_s^{k,h})|\, ds \\
& + \int_0^\alpha |\omega_{D_3f}(\mathbf{v}(s), \mathbf{v}^k(s), E^k(s, z_s^{k,h}, h^\xi))|\, ds + \int_0^\alpha |\omega_{D_4f}(\mathbf{v}(s), \mathbf{v}^k(s), h^\theta)|\, ds.
\end{aligned}
$$

Hence $c_{1,k} \to 0$, $c_{3,k} \to 0$ as $k \to \infty$, (5.24), (5.28), (5.40), (5.60), (5.61) and (5.62) imply (5.64). □

Now we are ready to prove the main result of this section.

Theorem 5.17 *Assume (A0), (A1) (i)–(iv), (A2) (i)–(v). Then for $t \in [0, \alpha]$ the maps*

$$
\Gamma_2 \supset (P_2 \cap \Gamma_2) \to \mathbb{R}^n, \quad \gamma \mapsto x(t, \gamma)
$$

and

$$\Gamma_2 \supset (P_2 \cap \Gamma_2) \to C, \quad \gamma \mapsto x_t(\cdot, \gamma)$$

are twice differentiable wrt γ *for every* $\gamma \in P_2 \cap \Gamma_2 \cap \mathcal{P}$*, and*

$$D_{22}x(t, \gamma)\langle h, y\rangle = w^{h,y}(t), \qquad h, y \in \Gamma_2,$$

and

$$D_{22}x_t(\cdot, \gamma)\langle h, y\rangle = w_t^{h,y}, \qquad h, y \in \Gamma_2,$$

where $w^{h,y}$ *is the solution of the IVP* (5.34)-(5.35). *Moreover, if in addition, (A2) (vi) holds, then the maps*

$$\mathbb{R} \times \Gamma_2 \supset \Big([0, \alpha] \times (P_2 \cap \Gamma_2 \cap \mathcal{P})\Big) \to \mathcal{L}^2(\Gamma_2 \times \Gamma_2, \mathbb{R}^n), \quad (t, \gamma) \mapsto D_{22}x(t, \gamma)$$

and

$$\mathbb{R} \times \Gamma_2 \supset \Big([0, \alpha] \times (P_2 \cap \Gamma_2 \cap \mathcal{P})\Big) \to \mathcal{L}^2(\Gamma_2 \times \Gamma_2, C), \quad (t, \gamma) \mapsto D_{22}x_t(\cdot, \gamma)$$

are continuous.

Proof It follows from Theorem 4.9 that $D_2x(t, \gamma) \in \mathcal{L}(\Gamma, \mathbb{R}^n)$ exists for all $\gamma \in P_2$ and $t \in [0, \alpha]$. Since $|h|_\Gamma \le |h|_{\Gamma_2}$ for all $h \in \Gamma_2$, it follows that $D_2x(t, \gamma)\Big|_{\Gamma_2} \in \mathcal{L}(\Gamma_2, \mathbb{R}^n)$, and $D_2x(t, \gamma)\Big|_{\Gamma_2}$ is the derivtive of the map $\Gamma_2 \supset (P_2 \cap \Gamma_2) \to \mathbb{R}^n$, $\gamma \to x(t, \gamma)$. For simplicity, the restiction of $D_2x(t, \gamma)$ to Γ_2 will be denoted by $D_2x(t, \gamma)$, as well. Theorem 4.9 yields that $D_2x(t, \gamma)h = z(t, \gamma, h)$, where $z(t, \gamma, h)$ is the solution of the IVP (4.14)-(4.15) for $h \in \Gamma_2$.

Let $\gamma \in P_2 \cap \Gamma_2 \cap \mathcal{P}$ be fixed, $h_k = (h_k^\varphi, h_k^\theta, h_k^\xi) \in \Gamma_2$ ($k \in \mathbb{N}$) be a sequence such that $h_k \ne 0$ and $\gamma + h_k \in P_2$ for $k \in \mathbb{N}$, $0 \ne h = (h^\varphi, h^\theta, h^\xi) \in \Gamma_2$. Let $x(t) := x(t, \gamma)$ and $x^k(t) := x(t, \gamma + h_k)$ be the solutions of the IVP (1.1)-(1.2), $z^h(t) := D_2x(t, \gamma)h$ and $z^{k,h}(t) := D_2x(t, \gamma + h_k)h$ be the solution of the IVP (4.14)-(4.15), and $w^{h,h_k}(t)$ be the solution of the IVP (5.34)-(5.35) corresponding to parameters h and h_k. Then we have for $t \in [0, \alpha]$

$$\begin{aligned}
z^{k,h}(t) &= h^\varphi(0) + \int_0^t L(s, x^k)(z_s^{k,h}, h^\theta, h^\xi)\, ds, \\
z^h(t) &= h^\varphi(0) + \int_0^t L(s, x)(z_s^h, h^\theta, h^\xi)\, ds, \\
w^{h,h_k}(t) &= \int_0^t \Big(L(s, x)(w_s^{h,h_k}, 0, 0) + B(s)\Big\langle (z_s^h, h^\theta, h^\xi), (z_s^{h_k}, h_k^\theta, h_k^\xi)\Big\rangle\Big) ds.
\end{aligned}$$

Hence Lemma 5.16 and the definition of $q^{k,h}$ give

$$\begin{aligned}
q^{k,h}(t) &= \int_0^t \Big(L(s, x^k)(z_s^{k,h}, h^\theta, h^\xi) - L(s, x)(z_s^h + w_s^{h,h_k}, h^\theta, h^\xi) \\
&\qquad - B(s)\Big\langle (z_s^h, h^\theta, h^\xi), (z_s^{h_k}, h_k^\theta, h_k^\xi)\Big\rangle\Big)\, ds \\
&= \int_0^t g_5^{k,h}(s)\, ds + \int_0^t L(s, x)(q_s^{k,h}, 0, 0)\, ds, \qquad t \in [0, \alpha],
\end{aligned}$$

so (4.31) yields

$$|q^{k,h}(t)| \le \int_0^t |g_5^{k,h}(s)|\, ds + \int_0^t |L(s, x)(q_s^{k,h}, 0, 0)|\, ds \le \int_0^\alpha |g_5^{k,h}(s)|\, ds + N_4 \int_0^t |q_s^{k,h}|_C\, ds,$$

for $t \in [0, \alpha]$. Using that $q^{k,h}(t) = 0$ for $t \in [-r, 0]$, Lemma 2.1 implies

$$|q^{k,h}(t)| \leq |q_t^{k,h}|_C \leq N_5 \int_0^\alpha |g_5^{k,h}(s)|\, ds, \qquad t \in [0, \alpha],$$

where $N_5 := e^{N_4 \alpha}$. Therefore (5.64) yields for $t \in [0, \alpha]$

$$\lim_{k\to\infty} \sup_{\substack{h\neq 0 \\ h\in\Gamma_2}} \frac{|q^{k,h}(t)|}{|h|_{\Gamma_2}|h_k|_{\Gamma_2}} \leq \lim_{k\to\infty} \sup_{\substack{h\neq 0 \\ h\in\Gamma_2}} \frac{|q_t^{k,h}|_C}{|h|_{\Gamma_2}|h_k|_{\Gamma_2}} \leq \lim_{k\to\infty} \sup_{\substack{h\neq 0 \\ h\in\Gamma_2}} \frac{N_5}{|h|_{\Gamma_2}|h_k|_{\Gamma_2}} \int_0^\alpha |g_5^{k,h}(s)|\, ds = 0,$$

which completes the proof of the second-order differentiability wrt parameters. The continuity of $D_{22}x(t, \gamma)$ follows from Lemma 5.14. □

We note that the method used in this section to prove the existence of the second order derivative $D_{22}x(t, \gamma)$ relied on the assumption that the parameter γ satisfies the compatibility condition $\gamma \in \mathcal{P}$.

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