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Differentiability of solutions with respect to the initial data in differential equations with state-dependent delays*

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Abstract

In this paper we consider a class of differential equations with state-dependent delays. We show differentiability of the solution with respect to the initial function and the initial time for each fixed time value assuming that the state-dependent time lag function is strictly monotone increasing.

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1 Introduction

In this paper we consider a class of state-dependent delay equations (SD-DDEs) of the form

$$\dot{x}(t) = f(t, x_t, x(t - \tau(t, x_t))), \qquad t \in [\sigma, T], \tag{1.1}$$

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where the associated initial condition is

$$x(t) = \varphi(t - \sigma), \qquad t \in [\sigma - r, \sigma]. \tag{1.2}$$

Here and throughout the manuscript $r > 0$ is a fixed constant and $x_t\colon [-r, 0] \to \mathbb{R}^n$, $x_t(\theta) := x(t + \theta)$ is the segment function. The particular form of the Eq. (1.1) assumes that the growth rate of the system depends on past values of the state, and one delay function is state-dependent, i.e., the delay function depends not only on time, but also on past values of the state. We suppose that this function is given explicitly, and it is denoted by τ in (1.1). It is easy to see that a function of the form $C \ni \psi \mapsto \psi(-\tau(t, \psi)) \in \mathbb{R}^n$ is not smooth, even if τ is a smooth function (see [9]). On the other hand, if τ in the above map is independent of ψ, then the map is linear, and hence it is smooth. The dependence of f on x_t represents all the other delayed terms in the equation. Later we will assume that f depends smoothly on its second argument, so this dependence will represent "non state-dependent" delayed terms in the equation. We note that, for simplicity, Eq. (1.1) contains only one state-dependent term, but all the results can be easily generalized to the case when in Eq. (1.1) there are several state-dependent delays. We refer the reader to [9] for a survey, which contains a brief summary of the general theory and also several applications of SD-DDEs.

In this paper we consider the initial time σ and the initial function φ as parameters in the initial value problem (IVP) (1.1)-(1.2), and we denote the corresponding solution by $x(t, \sigma, \varphi)$. The main goal of this paper is to discuss the differentiability of $x(t, \sigma, \varphi)$ with respect to (wrt) σ and φ. More precisely, we study the differentiability of the following four types of parameter maps: $\varphi \mapsto x(t, \sigma, \varphi)$, $\sigma \mapsto x(t, \sigma, \varphi)$, $\varphi \mapsto x_t(\cdot, \sigma, \varphi)$, and $\sigma \mapsto x_t(\cdot, \sigma, \varphi)$. In the first two cases differentiability is considered in a pointwise sense, and in the last two cases the differentiability of the solution segment functions is studied. Clearly, in the last two cases the differentiability depends on the selection of the state space, i.e., the space of the segment functions x_t. By differentiability we mean Fréchet-differentiability throughout this paper. We note that a natural application of differentiability of solutions wrt to parameters was given in [8], where a parameter estimation method was formulated and numerically tested which uses point evaluations of the derivatives of the solution wrt parameters.

Differentiability of solutions wrt parameters for SD-DDEs was studied in [2, 7, 10, 12, 13]. In all these papers the initial time σ was fixed to $\sigma = 0$, but other parameters in the formula of f and the delay function τ were also considered. Next we formulate these results for the IVP (1.1)-(1.2).

In [7] differentiability of the parameter maps $W^{1,\infty} \ni \varphi \mapsto x(t, 0, \varphi) \in \mathbb{R}^n$, $W^{1,\infty} \ni \varphi \mapsto x_t(\cdot, 0, \varphi) \in C$ and $W^{1,\infty} \ni \varphi \mapsto x_t(\cdot, 0, \varphi) \in W^{1,\infty}$ was shown. (For the definition of the spaces and norms see Section 2 below.) Here differentiability was obtained using strong norms, the C- and $W^{1,\infty}$-norms on the state space, but a strong assumption was needed to prove the result: it was assumed that $\varphi \in C^1$ and it satisfies the compatibility condition

$$\dot{\varphi}(0-) = f(0, \varphi, \varphi(-\tau(0, \varphi))). \tag{1.3}$$

Note that this condition yields that the corresponding solution is continuously differentiable wrt time on its whole domain. On the other hand, differentiability was shown only at a particular parameter value φ where the above compatibility condition is satisfied.

Walther in [12] and [13] obtained C^1-smoothness of the solution semiflow for large classes of SD-DDEs restricting the set of initial functions to those which satisfy the compatibility condition (1.3).

In [10] differentiability of the map $W^{1,\infty} \ni \varphi \mapsto x_t(\cdot, 0, \varphi) \in W^{1,p}$ $(1 \le p < \infty)$ was proved without the compatibility condition (1.3), but it was assumed that the time lag function $t \mapsto t-\tau(t, x_t)$ corresponding to a fixed solution x is strictly monotone increasing, more precisely,

$$\operatorname*{ess\,inf}_{0\le t\le\alpha} \frac{d}{dt}(t - \tau(t, x_t)) > 0, \tag{1.4}$$

when $\alpha > 0$ is such that the solution exists on $[-r, \alpha]$. It was shown that the differentiability holds in a small neighborhood of the fixed initial function where the monotonicity condition is also satisfied. On the other hand, the differentiability was obtained using only a weak norm, the $W^{1,p}$-norm $(1 \le p < \infty)$ on the state space.

Chen, Hu and Wu in [2] extended the above result to proving second-order differentiability of the parameter map using the monotonicity condition (1.4) of the state-dependent time lag function, the $W^{1,p}$-norm on the state space, and the $W^{2,p}$-norm on the space of initial functions. Note that τ was not given explicitly in [2], it was defined through a coupled differential equation, but it satisfied the monotonicity condition.

In this paper we combine the techniques of [7] and [10], and assuming the monotonicity condition (1.4) of the state-dependent time lag function we show (see Theorem 4.7 below) the continuous differentiability of the parameter maps

$$W^{1,\infty} \ni \varphi \mapsto x(t, \sigma, \varphi) \in \mathbb{R}^n \qquad \text{and} \qquad W^{1,\infty} \ni \varphi \mapsto x_t(\cdot, \sigma, \varphi) \in C$$

for a fixed t and σ. Note that here differentiability is proved in a pointwise sense and in the C-norm, respectively, like in [7], but without assuming the compatibility condition (1.3).

In Theorem 5.1 below we show that the parameter maps

$$\mathbb{R} \ni \sigma \mapsto x(t, \sigma, \varphi) \in \mathbb{R}^n \qquad \text{and} \qquad \mathbb{R} \ni \sigma \mapsto x_t(\cdot, \sigma, \varphi) \in C$$

are both differentiable at a fixed t, σ and φ, where a compatibility condition similar to (1.3) is satisfied. Assuming some additional conditions on f and τ, in Theorem 5.3 below we prove the differentiability of $\sigma \mapsto x(t,\sigma,\varphi)$ using the monotonicity assumption (1.4), but without the compatibility condition (1.3). Note that in this case a similar result does not hold for the map $\sigma \mapsto x_t(\cdot,\sigma,\varphi)$ using the C-norm, which is not surprising, since the map $\sigma \mapsto x(t,\sigma,\varphi)$ is differentiable at the point $t=\sigma$ if and only if a compatibility condition similar to (1.3) is satisfied (see Remark 5.4 below). We note that the derivative of the solution wrt the initial function and the initial time is a solution of an associated linear variational equation (see (4.27)-(4.28) and (5.9)-(5.11) below). We comment that differentiability wrt the initial time has not been studied before for SD-DDEs.

This paper is organized as follows. Section 2 introduces notations and some preliminary results, Section 3 discusses the well-posedness of the IVP (1.1)-(1.2), and Sections 4 and 5 study differentiability of the parameter map wrt the initial function and the initial time, respectively.

2 Notations and preliminaries

Throughout the manuscript $r>0$ is a fixed constant and $x_t\colon [-r,0]\to\mathbb{R}^n$, $x_t(\theta):=x(t+\theta)$ is the segment function. To avoid confusion with the notation of the segment function, sequences of functions are denoted using the upper index: x^k.

A fixed norm on $\mathbb{R}^n$ and its induced matrix norm on $\mathbb{R}^{n\times n}$ are both denoted by $|\cdot|$. C denotes the Banach space of continuous functions $\psi\colon [-r,0]\to\mathbb{R}^n$ equipped with the norm $|\psi|_C=\sup\{|\psi(s)|:\ s\in[-r,0]\}$. C^1 is the space of continuously differentiable functions $\psi\colon [-r,0]\to\mathbb{R}^n$ where the norm is defined by $|\psi|_{C^1}=\max\{|\psi|_C,|\dot\psi|_C\}$. L^∞ is the space of Lebesgue-measurable functions $\psi\colon [-r,0]\to\mathbb{R}^n$ which are essentially bounded. The norm on L^∞ is denoted by $|\cdot|_{L^\infty}$. $W^{1,p}$ denotes the Banach space of absolutely continuous functions $\psi\colon [-r,0]\to\mathbb{R}^n$ of finite norm defined by

$$|\psi|_{W^{1,p}}:=\left(\int_{-r}^0|\psi(s)|^p+|\dot\psi(s)|^p\,ds\right)^{1/p},\qquad 1\le p<\infty,$$

and for $p=\infty$

$$|\psi|_{W^{1,\infty}}:=\max\left\{|\psi|_C,|\dot\psi|_{L^\infty}\right\}.$$

We note that $W^{1,\infty}$ is equal to the space of Lipschitz-continuous functions from $[-r,0]$ to $\mathbb{R}^n$. If the domain or the range of the functions is different from $[-r,0]$ and $\mathbb{R}^n$, respectively, we will use a more detailed notation. E.g., $C(X,Y)$ denotes the space of continuous functions mapping from X to Y. Finally, $\mathcal{L}(X,Y)$ denotes the space of bounded linear operators from X to Y, where X and Y are normed linear spaces.

An open ball in the normed linear space X centered at a point $x \in X$ with radius δ is denoted by $\mathcal{B}_X(x;\,\delta) := \{y \in Y\colon\, |x-y| < \delta\}$. The corresponding closed ball is denoted by $\overline{\mathcal{B}}_X(x;\,\delta)$.

The partial derivatives of a function $g\colon\, X \times Y \to Z$ wrt the first and second variable will be denoted by $D_1 g$ and $D_2 g$, respectively. All derivatives in this paper are Fréchet-derivatives.

The following result is a simple consequence of Gronwall's lemma.

Lemma 2.1 *Suppose $a \geq 0$, $b\colon\, [\sigma,\alpha] \to [0,\infty)$ and $u\colon\, [\sigma-r,\alpha] \to \mathbb{R}^n$ are continuous functions such that $a \geq |u_\sigma|_C$, and*

$$|u(t)| \leq a + \int_\sigma^t b(s)|u_s|_C\,ds, \qquad t \in [\sigma,\alpha]. \tag{2.5}$$

Then

$$|u(t)| \leq |u_t|_C \leq a e^{\int_\sigma^\alpha b(s)\,ds}, \qquad t \in [\sigma,\alpha]. \tag{2.6}$$

Proof (2.5) yields

$$|u(t+\theta)| \leq a + \int_\sigma^{t+\theta} b(s)|u_s|_C\,ds \leq a + \int_\sigma^t b(s)|u_s|_C\,ds$$

for $t \in [\sigma,\alpha]$ and $\theta \in [-r,0]$ such that $t+\theta \geq \sigma$, and

$$|u(t+\theta)| \leq |u_\sigma|_C \leq a$$

for $t \in [\sigma,\alpha]$ and $\theta \in [-r,0]$ such that $t+\theta \leq \sigma$. Therefore (2.5) implies

$$|u_t|_C \leq a + \int_\sigma^t b(s)|u_s|_C\,ds, \qquad t \in [\sigma,\alpha],$$

and Gronwall's lemma yields (2.6). □

We recall the following result from [1], which was essential to prove differentiability wrt parameters in SD-DDEs in [2] and [10]. Note that the second part of the lemma was stated in [1] under the assumption $|u^k - u|_{W^{1,\infty}([\sigma,\alpha],\mathbb{R})} \to 0$ as $k \to \infty$, but this stronger assumption on the convergence is not needed in the proof. See also the proof of Lemma 4.26 in [6].

Lemma 2.2 ([1]) *Let $p \in [1,\infty)$, $g \in L^p([\sigma - r, \alpha], \mathbb{R}^n)$, $\varepsilon > 0$, and $u \in \mathcal{A}(\varepsilon)$, where*

$$\mathcal{A}(\varepsilon) := \{v \in W^{1,\infty}([\sigma,\alpha],[\sigma - r,\alpha]) \,:\, \dot{v}(s) \geq \varepsilon \text{ for a.e. } s \in [\sigma,\alpha]\}. \tag{2.7}$$

Then

$$\int_\sigma^\alpha |g(u(s))|^p\, ds \leq \frac{1}{\varepsilon}\int_{\sigma-r}^\alpha |g(s)|^p\, ds.$$

Moreover, if the sequence $u^k \in \mathcal{A}(\varepsilon)$ is such that $|u^k - u|_{C([\sigma,\alpha],\mathbb{R})} \to 0$ as $k \to \infty$, then

$$\lim_{k\to\infty}\int_\sigma^\alpha \Big|g(u^k(s)) - g(u(s))\Big|^p\, ds = 0. \tag{2.8}$$

3 Well-posedness

Consider the nonlinear SD-DDE

$$\dot{x}(t) = f(t, x_t, x(t - \tau(t, x_t))), \qquad t \in [\sigma, T], \tag{3.1}$$

and the corresponding initial condition

$$x(t) = \varphi(t - \sigma), \qquad t \in [\sigma - r, \sigma]. \tag{3.2}$$

Let $\Omega_1 \subset C$, $\Omega_2 \subset \mathbb{R}^n$ be open subsets of the respective spaces. $T > 0$ is finite or $T = \infty$, in which case $[0, T]$ denotes the interval $[0, \infty)$, and $\sigma \in [0, T)$.

Next we list our assumptions used later in our results.

(A1) (i) $f \,:\, \mathbb{R} \times C \times \mathbb{R}^n \supset [0, T] \times \Omega_1 \times \Omega_2 \to \mathbb{R}^n$ is continuous,

(ii) $f(t, \psi, u)$ is locally Lipschitz-continuous in ψ and u, i.e., for every finite $\alpha > 0$, $M_1 \subset \Omega_1$ and $M_2 \subset \Omega_2$, where M_1 and M_2 are compact subsets of C and $\mathbb{R}^n$, respectively, there exists a constant $L_1 = L_1(\alpha, M_1, M_2)$ such that

$$|f(t, \psi, u) - f(t, \bar{\psi}, \bar{u})| \leq L_1\Big(|\psi - \bar{\psi}|_C + |u - \bar{u}|\Big),$$

for $t \in [0, \alpha]$, $\psi, \bar{\psi} \in M_1$ and $u, \bar{u} \in M_2$,

(iii) $f : \mathbb{R} \times C \times \mathbb{R}^n \supset [0,T] \times \Omega_1 \times \Omega_2 \to \mathbb{R}^n$ is continuously differentiable wrt its second and third arguments,

(A2) (i) $\tau : \mathbb{R} \times C \supset [0,T] \times \Omega_1 \to [0,r]$ is continuous,

(ii) $\tau(t,\psi)$ is locally Lipschitz-continuous in ψ, i.e., for every $\alpha > 0$ and $M_1 \subset \Omega_1$ compact subset of C there exists a constant $L_2 = L_2(\alpha, M_1)$ such that

$$|\tau(t,\psi) - \tau(t,\bar{\psi})| \le L_2 |\psi - \bar{\psi}|_C$$

for $t \in [0,\alpha]$, $\psi, \bar{\psi} \in M_1$,

(iii) $\tau : \mathbb{R} \times C \supset [0,T] \times \Omega_1 \to \mathbb{R}$ is continuously differentiable wrt both arguments.

We introduce the set of admissible parameters

$$\Pi := \Big\{(\sigma,\varphi) \in [0,T) \times W^{1,\infty} \colon\ \varphi \in \Omega_1,\ \varphi(-\tau(\sigma,\varphi)) \in \Omega_2\Big\}.$$

The next theorem shows that every admissible parameter $(\hat{\sigma}, \hat{\varphi}) \in \Pi$ has a neighborhood P and there exists a constant $\alpha > \hat{\sigma}$ such that the IVP (3.1)-(3.2) has a unique solution on $[\sigma - r, \alpha]$ corresponding to all parameters $(\sigma, \varphi) \in P$. This solution will be denoted by $x(t,\sigma,\varphi)$, and its segment function at t is denoted by $x_t(\cdot,\sigma,\varphi)$.

The well-posedness of several classes of SD-DDEs was studied in many papers (see, e.g., [4, 9, 10, 11]). The next result is an extension of a result from [7] to the case when the initial time σ is also considered as a parameter. The notations and estimates introduced in the next theorem will be essential in the following sections.

Theorem 3.1 *Assume (A1) (i), (ii), (A2) (i), (ii), and let $(\hat{\sigma}, \hat{\varphi}) \in \Pi$. Then there exist $\delta > 0$, $0 \le \sigma_0 \le \hat{\sigma}$, $\hat{\sigma} < \alpha \le T$ finite numbers such that $0 \le \sigma_0 < \hat{\sigma}$ if $\hat{\sigma} > 0$, and $\sigma_0 = 0$ if $\hat{\sigma} = 0$, and*

(i) for all $(\sigma,\varphi) \in P := [\sigma_0, \alpha) \times \mathcal{B}_{W^{1,\infty}}(\hat{\varphi}; \delta)$ the IVP (3.1)-(3.2) has a unique solution $x(t,\sigma,\varphi)$ on $[\sigma - r, \alpha]$;

(ii) there exist $M_1 \subset C$ and $M_2 \subset \mathbb{R}^n$ compact subsets of the respective spaces such that $x_t(\cdot,\sigma,\varphi) \in M_1$ and $x(t - \tau(t, x_t(\cdot,\sigma,\varphi)),\sigma,\varphi) \in M_2$ for $(\sigma,\varphi) \in P$ and $t \in [\sigma,\alpha]$; and

(iii) $x_t(\cdot,\sigma,\varphi) \in W^{1,\infty}$ *for* $(\sigma,\varphi) \in P$ *and* $t \in [\sigma,\alpha]$*, and there exist constants* $N = N(\sigma_0,\alpha,\delta,\hat\varphi)$ *and* $L = L(\sigma_0,\alpha,\delta,\hat\varphi)$ *such that*

$$|x_t(\cdot,\sigma,\varphi)|_{W^{1,\infty}} \le N, \qquad (\sigma,\varphi) \in P,\ t \in [\sigma,\alpha], \tag{3.3}$$

and

$$|x_t(\cdot,\sigma,\varphi) - x_t(\cdot,\bar\sigma,\bar\varphi)|_{W^{1,\infty}} \le L(|\sigma-\bar\sigma| + |\varphi-\bar\varphi|_{W^{1,\infty}}) \tag{3.4}$$

for $(\sigma,\varphi),(\bar\sigma,\bar\varphi) \in P$ *and* $t \in [\max\{\sigma,\bar\sigma\},\alpha]$.

Proof Let $(\hat\sigma,\hat\varphi) \in \Pi$, and let $\delta_1 > 0$ and $\delta_2 > 0$ be such that $\mathcal{B}_C(\hat\varphi;\,\delta_1) \subset \Omega_1$ and $\overline{\mathcal{B}}_{\mathbb{R}_n}(\hat\varphi(-\tau(\hat\sigma,\hat\varphi));\,\delta_2) \subset \Omega_2$. Let $\varepsilon_0 > 0$ be fixed. The definition of Π and the continuity of the maps $[0,T]\times\Omega_1 \to \mathbb{R}^n$, $(\sigma,\psi) \mapsto \psi(-\tau(\sigma,\psi))$ and $f\colon\ [0,T]\times\Omega_1\times\Omega_2 \to \mathbb{R}^n$ yield that there exist finite numbers σ_0, T_1 and $0 < \delta_3 < \delta_1$ such that $0 \le \sigma_0 \le \hat\sigma < T_1 \le T$, and $|\psi(-\tau(\sigma,\psi)) - \hat\varphi(-\tau(\hat\sigma,\hat\varphi))| < \delta_2$ and $|f(\sigma,\psi,\psi(-\tau(\sigma,\psi))) - f(\hat\sigma,\hat\varphi,\hat\varphi(-\tau(\hat\sigma,\hat\varphi)))| < \varepsilon_0$ for $\sigma \in [\sigma_0,T_1]$ and $\psi \in \mathcal{B}_C(\hat\varphi;\,\delta_3)$. Note that if $\hat\sigma > 0$, then σ_0 can be selected so that $0 \le \sigma_0 < \hat\sigma$.

For a fixed $\sigma \in [\sigma_0,T_1]$ and for a function $\varphi \in W^{1,\infty}$ we define the notation

$$\tilde\varphi(s) = \begin{cases} \varphi(s-\sigma), & s \in [\sigma-r,\sigma], \\ \varphi(0), & s > \sigma. \end{cases}$$

The new variable $y(t) = x(t) - \tilde\varphi(t)$ transforms Eq. (3.1) to

$$\dot y(t) = f\Big(t, y_t + \tilde\varphi_t, y(t-\tau(t,y_t+\tilde\varphi_t)) + \tilde\varphi(t-\tau(t,y_t+\tilde\varphi_t))\Big), \qquad t \ge \sigma.$$

We define the constants $K := |f(\hat\sigma,\hat\varphi,\hat\varphi(-\tau(\hat\sigma,\hat\varphi)))| + \varepsilon_0$, $\delta := \frac{\delta_3}{3}$, $\beta := \delta$, $\alpha := \sigma_0 + \min\left\{\frac{\beta}{K}, T_1-\sigma_0, \frac{\beta}{|\dot{\hat\varphi}|_{L^\infty}+\delta}\right\}$ and the set

$$E := \Big\{y \in C([\sigma-r,\alpha],\mathbb{R}^n)\colon\ y(s) = 0 \text{ for } s \in [\sigma-r,\sigma] \text{ and } |y(s)| \le \beta \text{ for } s \in [\sigma,\alpha]\Big\}.$$

Then for $y \in E$, $\varphi \in \mathcal{B}_{W^{1,\infty}}(\hat\varphi;\,\delta)$, $s \in [\sigma,\alpha]$ and $\theta \in [-r,0]$ we have

$$\begin{aligned} |y(s+\theta) + \tilde\varphi(s+\theta) - \hat\varphi(\theta)| &\le |y(s+\theta)| + |\tilde\varphi(s+\theta) - \varphi(\theta)| + |\varphi(\theta) - \hat\varphi(\theta)| \\ &< \beta + s|\dot\varphi|_{L^\infty} + \delta \\ &\le \beta + s(|\dot{\hat\varphi}|_{L^\infty} + |\dot\varphi - \dot{\hat\varphi}|_{L^\infty}) + \delta \\ &\le \delta_3, \end{aligned}$$

and hence $|y_s + \tilde{\varphi}_s - \hat{\varphi}|_C < \delta_3$. Consequently, $y_s + \tilde{\varphi}_s \in \mathcal{B}_C(\hat{\varphi};\, \delta_3) \subset \Omega_1$, and so

$$(y_s + \tilde{\varphi}_s)(-\tau(s, y_s + \tilde{\varphi}_s)) \in \mathcal{B}_{\mathbb{R}^n}(\hat{\varphi}(-\tau(\hat{\sigma}, \hat{\varphi}));\, \delta_2)$$

and

$$\Big| f\Big(t, y_t + \tilde{\varphi}_t, y(t - \tau(t, y_t + \tilde{\varphi}_t)) + \tilde{\varphi}(t - \tau(t, y_t + \tilde{\varphi}_t))\Big)\Big| \le K$$

for $y \in E$, $\varphi \in \mathcal{B}_{W^{1,\infty}}(\hat{\varphi};\, \delta)$ and $s \in [\sigma, \alpha]$.

Now, it is easy to show that the operator $T(\cdot, \sigma, \varphi)$ defined by

$$T(y, \sigma, \varphi)(t) := \begin{cases} 0, & t \in [\sigma - r, \sigma], \\ \int_\sigma^t f\Big(s, y_s + \tilde{\varphi}_s, y(s - \tau(s, y_s + \tilde{\varphi}_s)) + \tilde{\varphi}(s - \tau(s, y_s + \tilde{\varphi}_s))\Big) ds \end{cases}$$

for $t \in [\sigma, \alpha]$ maps the closed bounded convex set E into E. Then the Schauder fixed point theorem provides the existence of the fixed point y of $T(\cdot, \sigma, \varphi)$, i.e., a solution $x(t) = y(t) + \tilde{\varphi}(t)$ of the IVP (3.1)-(3.2) on $[\sigma - r, \alpha]$. This concludes the existence of the solution in part (i). The uniqueness of the solution corresponding to a fixed parameter $(\sigma, \varphi) \in P$ will follow from (3.4) with $\sigma = \bar{\sigma}$ and $\varphi = \bar{\varphi}$. So for the rest of the proof now $x(t, \sigma, \varphi)$ will denote *any* fixed solution of the IVP (3.1)-(3.2) corresponding to parameter $(\sigma, \varphi) \in P$.

To prove (ii) define the sets

$$M_1 := \Big\{\psi \in \overline{\mathcal{B}}_C(\hat{\varphi};\, \delta_3) \cap W^{1,\infty} \colon\, |\dot{\psi}|_{L^\infty} \le \max\{|\dot{\hat{\varphi}}|_{L^\infty} + \delta, K\}\Big\}$$

and $M_2 := \overline{\mathcal{B}}_{\mathbb{R}^n}(\hat{\varphi}(-\tau(\hat{\sigma}, \hat{\varphi}));\, \delta_2)$. Then $M_1 \subset \overline{\mathcal{B}}_C(\hat{\varphi};\, \delta_3) \subset \mathcal{B}_C(\hat{\varphi};\, \delta_1) \subset \Omega_1$ and the Arzelà-Ascoli Theorem yields that M_1 is compact in C. The proof of part (i) implies that M_1 and M_2 satisfy (ii).

To prove the first part of (iii), let $(\sigma, \varphi) \in P$. Then by part (ii) and by the definition of K we have

$$|\dot{x}(t, \sigma, \varphi)| = |f(t, x_t(\cdot, \sigma, \varphi), x(t - \tau(t, x_t(\cdot, \sigma, \varphi))), \sigma, \varphi))| \le K, \qquad t \in [\sigma, \alpha],$$

$|\varphi|_{W^{1,\infty}} \le |\hat{\varphi}|_{W^{1,\infty}} + |\varphi - \hat{\varphi}|_{W^{1,\infty}} \le |\hat{\varphi}|_{W^{1,\infty}} + \delta$, and

$$\begin{aligned} |x(t, \sigma, \varphi)| &\le |\varphi(0)| + \int_\sigma^t |f(s, x_s(\cdot, \sigma, \varphi), x(s - \tau(s, x_s(\cdot, \sigma, \varphi))), \sigma, \varphi))|\, ds \\ &\le |\varphi|_C + K\alpha \\ &\le |\hat{\varphi}|_{W^{1,\infty}} + \delta + K\alpha, \qquad t \in [\sigma, \alpha]. \end{aligned}$$

So $x_t(\cdot,\sigma,\varphi) \in W^{1,\infty}$ for all $(\sigma,\varphi) \in P$ and $t \in [\sigma,\alpha]$, and (3.3) holds with $N := \max\{K, |\hat\varphi|_{W^{1,\infty}} + \delta + K\alpha\}$.

Finally, to prove (3.4) let $(\sigma,\varphi),(\bar\sigma,\bar\varphi) \in P$, and for a shorter notation let $x(t) := x(t,\sigma,\varphi)$, $\bar x(t) := \bar x(t,\bar\sigma,\bar\varphi)$, $u(s) := s - \tau(s,x_s)$ and $\bar u(s) := s - \tau(s,\bar x_s)$. Then

$$x(t) = \varphi(0) + \int_\sigma^t f(s,x_s,x(u(s)))\,ds, \qquad t \in [\sigma,\alpha]$$

and

$$\bar x(t) = \bar\varphi(0) + \int_{\bar\sigma}^t f(s,\bar x_s,\bar x(\bar u(s)))\,ds, \qquad t \in [\bar\sigma,\alpha].$$

Suppose that $\bar\sigma \geq \sigma$. (The opposite case is identical.) Then

$$\begin{aligned} x(t) - \bar x(t) &= \varphi(0) - \bar\varphi(0) + \int_{\bar\sigma}^t \Big(f(s,x_s,x(u(s))) - f(s,\bar x_s,\bar x(\bar u(s)))\Big) ds \\ &\quad + \int_\sigma^{\bar\sigma} f(s,x_s,x(u(s)))\,ds, \qquad t \in [\bar\sigma,\alpha]. \end{aligned}$$

Hence, using part (ii), assumption (A1) (ii) and the definition of K, we get

$$|x(t)-\bar x(t)| \leq |\varphi(0)-\bar\varphi(0)| + \int_{\bar\sigma}^t L_1\Big(|x_s-\bar x_s|_C + |x(u(s)) - \bar x(\bar u(s))|\Big)ds + K|\sigma-\bar\sigma| \quad (3.5)$$

for $t \in [\bar\sigma,\alpha]$. The Mean Value Theorem, (3.3), part (ii) and assumption (A2) (ii) yield

$$\begin{aligned} |x(u(s)) - \bar x(\bar u(s))| &\leq |x(u(s)) - x(\bar u(s))| + |x(\bar u(s)) - \bar x(\bar u(s))| \\ &\leq N|u(s) - \bar u(s)| + |x(\bar u(s)) - \bar x(\bar u(s))| \\ &= N|\tau(s,x_s) - \tau(s,\bar x_s)| + |x(\bar u(s)) - \bar x(\bar u(s))| \\ &\leq (NL_2+1)|x_s - \bar x_s|_C, \qquad s \in [\bar\sigma,\alpha]. \end{aligned} \quad (3.6)$$

Introduce the constants

$$N_0 := NL_2 + 2, \qquad N_1 := e^{L_1 N_0 \alpha}. \quad (3.7)$$

Then combining (3.5) and (3.6) together with the definition of N_0 we obtain

$$|x(t) - \bar x(t)| \leq |\varphi - \bar\varphi|_{W^{1,\infty}} + K|\sigma - \bar\sigma| + \int_{\bar\sigma}^t L_1 N_0 |x_s - \bar x_s|_C\,ds, \qquad t \in [\bar\sigma,\alpha]. \quad (3.8)$$

Next we estimate $|x_{\bar\sigma} - \bar x_{\bar\sigma}|_C$. Let $\theta \in [-r, 0] \cap [\sigma - \bar\sigma, 0]$. Then

$$
\begin{aligned}
|x(\bar\sigma+\theta) - \bar x(\bar\sigma+\theta)| &= \left|\varphi(0) + \int_\sigma^{\bar\sigma+\theta} f(s, x_s, x(u(s)))\,ds - \bar\varphi(\theta)\right| \\
&\le |\varphi(0) - \bar\varphi(\theta)| + K(\bar\sigma+\theta-\sigma) \\
&\le |\varphi(0) - \bar\varphi(0)| + |\bar\varphi(0) - \bar\varphi(\theta)| + K|\sigma-\bar\sigma| \\
&\le |\varphi - \bar\varphi|_{W^{1,\infty}} + (|\bar\varphi|_{W^{1,\infty}} + K)|\sigma-\bar\sigma| \\
&\le |\varphi - \bar\varphi|_{W^{1,\infty}} + (N+K)|\sigma-\bar\sigma|.
\end{aligned}
$$

Now let $\theta \in [-r, 0] \cap (-\infty, \sigma - \bar\sigma]$. Then

$$
\begin{aligned}
|x(\bar\sigma+\theta) - \bar x(\bar\sigma+\theta)| &= |\varphi(\bar\sigma+\theta-\sigma) - \bar\varphi(\theta)| \\
&\le |\varphi(\bar\sigma+\theta-\sigma) - \varphi(\theta)| + |\varphi(\theta) - \bar\varphi(\theta)| \\
&\le |\varphi|_{W^{1,\infty}}|\sigma-\bar\sigma| + |\varphi-\bar\varphi|_C \\
&\le N|\sigma-\bar\sigma| + |\varphi-\bar\varphi|_{W^{1,\infty}}.
\end{aligned}
$$

Therefore $|x_{\bar\sigma} - \bar x_{\bar\sigma}| \le |\varphi - \bar\varphi|_{W^{1,\infty}} + (N+K)|\sigma-\bar\sigma|$, and so (3.8) implies

$$
|x(t) - \bar x(t)| \le |\varphi - \bar\varphi|_{W^{1,\infty}} + (N+K)|\sigma-\bar\sigma| + \int_{\bar\sigma}^t L_1 N_0 |x_s - \bar x_s|_C\,ds, \qquad t \in [\bar\sigma, \alpha].
$$

Employing Lemma 2.1 we get

$$
|x(t) - \bar x(t)| \le (|\varphi - \bar\varphi|_{W^{1,\infty}} + (N+K)|\sigma-\bar\sigma|)N_1, \qquad t \in [\bar\sigma, \alpha], \tag{3.9}
$$

where N_1 is defined by (3.7). For $t \in [\bar\sigma, \alpha]$ relations (3.6), (3.7) and (3.9) and assumption (A1) (ii) yield

$$
\begin{aligned}
|\dot x(t) - \dot{\bar x}(t)| &= \Big| f(t, x_t, x(u(t))) - f(t, \bar x_t, \bar x(\bar u(t)))\Big| \\
&\le L_1\Big(|x_t - \bar x_t|_C + |x(u(t)) - \bar x(\bar u(t))|\Big) \\
&\le L_1 N_0 |x_t - \bar x_t|_C \\
&\le L_1 N_0 \max\{1, N+K\} N_1 (|\sigma-\bar\sigma| + |\varphi-\bar\varphi|_{W^{1,\infty}}),
\end{aligned}
$$

hence (3.4) holds with $L = \max\{1, N+K\} N_1 \max\{1, L_1 N_0\}$. $\square$

Let $(\hat\sigma,\hat\varphi)\in\Pi$ be fixed, and let σ_0,α,δ and P be given by Theorem 3.1. We introduce the following set:

$$H := \{(t,\sigma,\varphi)\in\mathbb{R}^2\times W^{1,\infty}\colon\ (\sigma,\varphi)\in P,\ t\in[\sigma,\alpha]\}. \tag{3.10}$$

Theorem 3.1 has the following corollary.

Corollary 3.2 *Let $(\hat\sigma,\hat\varphi)\in\Pi$ be fixed, and let σ_0,α and δ be given by Theorem 3.1, and let H be defined by* (3.10). *Then there exists $L^*\ge 0$ such that*

$$|x_t(\cdot,\sigma,\varphi)-x_{\bar t}(\cdot,\bar\sigma,\bar\varphi)|_C \le L^*(|t-\bar t|+|\sigma-\bar\sigma|+|\varphi-\bar\varphi|_{W^{1,\infty}})$$

for $(t,\sigma,\varphi),(\bar t,\bar\sigma,\bar\varphi)\in H$.

Proof Let $(t,\sigma,\varphi),(\bar t,\bar\sigma,\bar\varphi)\in H$, and suppose $\bar\sigma\ge\sigma$. (The opposite case is similar.) Then $t,\bar t\in[\sigma,\alpha]$ and $\bar t\in[\max\{\bar\sigma,\sigma\},\alpha]$, therefore Theorem 3.1 yields

$$\begin{aligned}
&|x_t(\cdot,\sigma,\varphi)-x_{\bar t}(\cdot,\bar\sigma,\bar\varphi)|_C\\
&\quad\le\ |x_t(\cdot,\sigma,\varphi)-x_{\bar t}(\cdot,\sigma,\varphi)|_C+|x_{\bar t}(\cdot,\sigma,\varphi)-x_{\bar t}(\cdot,\bar\sigma,\bar\varphi)|_C\\
&\quad\le\ N|t-\bar t|+L(|\sigma-\bar\sigma|+|\varphi-\bar\varphi|_{W^{1,\infty}}).
\end{aligned}$$

Hence the statement follows with $L^*=\max\{N,L\}$. □

The following result is obvious.

Remark 3.3 *Suppose the conditions of Theorem 3.1 hold, P and α are defined by Theorem 3.1, and let*

$$\mathcal{P} := \Big\{(\sigma,\varphi)\in P\colon\ \varphi\in C^1,\quad \dot\varphi(0-)=f(\sigma,\varphi,\varphi(\sigma-\tau(\sigma,\varphi)))\Big\}. \tag{3.11}$$

Then for all parameter values $(\sigma,\varphi)\in\mathcal{P}$ the corresponding solution $x(t,\sigma,\varphi)$ is continuously differentiable wrt t for $t\in[\sigma-r,\alpha]$.

4 Differentiability wrt the initial function

In this section we study the differentiability of the solution $x(t,\sigma,\varphi)$ of the IVP (3.1)-(3.2) wrt φ. The differentiability of $x(t,\sigma,\varphi)$ wrt σ will be discussed in Section 5.

Throughout the rest of the manuscript we will use the following notations. The parameters $(\hat\sigma,\hat\varphi)\in\Pi$ are fixed, and the constants $\delta>0$, $0\le\sigma_0\le\hat\sigma$ and $\hat\sigma<\alpha\le T$ are defined by Theorem 3.1, and let $P:=[\sigma_0,\alpha)\times\mathcal{B}_{W^{1,\infty}}(\hat\varphi;\delta)$. The sets $M_1\subset C$ and $M_2\subset\mathbb{R}^n$ are defined by Theorem 3.1 (ii), $L_1=L_1(\alpha,M_1,M_2)$ and $L_2=L_2(\alpha,M_1)$ denote the corresponding Lipschitz constants from (A1) (ii) and (A2) (ii), respectively, and the constants $N=N(\alpha,\sigma_0,\delta)$ and $L=L(\alpha,\sigma_0,\delta)$ are defined by Theorem 3.1 (iii).

The proof of our differentiability results will be based on the following lemmas.

Lemma 4.1 *Let $x\in W^{1,\infty}([\sigma-r,\alpha],\mathbb{R}^n)$, and let $\omega_k\in(0,\infty)$ $(k\in\mathbb{N})$ be a sequence satisfying $\omega_k\to0$ as $k\to\infty$. Let $\varepsilon>0$, $\mathcal{A}(\varepsilon)$ be defined by (2.7), and $u,u^k\in\mathcal{A}(\varepsilon)$ be such that*

$$|u^k-u|_{C([\sigma,\alpha],\mathbb{R})}\le\omega_k,\qquad k\in\mathbb{N}. \tag{4.1}$$

Then

$$\lim_{k\to\infty}\frac{1}{\omega_k}\int_\sigma^\alpha|x(u^k(s))-x(u(s))-\dot x(u(s))(u^k(s)-u(s))|\,ds=0. \tag{4.2}$$

Proof Simple manipulations, (4.1) and Fubini's theorem yield

$$\begin{aligned}
&\int_\sigma^\alpha|x(u^k(s))-x(u(s))-\dot x(u(s))(u^k(s)-u(s))|\,ds\\
&\quad=\int_\sigma^\alpha\Big|\int_{u(s)}^{u^k(s)}\Big(\dot x(v)-\dot x(u(s))\Big)dv\Big|\,ds\\
&\quad=\int_\sigma^\alpha\Big|\int_0^1\Big[\dot x\Big(u(s)+\nu(u^k(s)-u(s))\Big)-\dot x(u(s))\Big](u^k(s)-u(s))\,d\nu\Big|\,ds\\
&\quad\le|u^k-u|_{C([\sigma,\alpha],\mathbb{R})}\int_\sigma^\alpha\int_0^1\Big|\dot x\Big(u(s)+\nu(u^k(s)-u(s))\Big)-\dot x(u(s))\Big|\,d\nu\,ds\\
&\quad\le\omega_k\int_0^1\int_\sigma^\alpha\Big|\dot x\Big(u(s)+\nu(u^k(s)-u(s))\Big)-\dot x(u(s))\Big|\,ds\,d\nu.
\end{aligned}$$

It follows from Lemma 2.2 that for every $\nu\in[0,1]$

$$\lim_{k\to\infty}\int_\sigma^\alpha\Big|\dot x\Big(u(s)+\nu(u^k(s)-u(s))\Big)-\dot x(u(s))\Big|\,ds=0,$$

hence we conclude (4.2) by using the Lebesgue's Dominated Convergence Theorem.

□

We introduce the notations

$$\omega_f(t,\bar\psi,\bar u,\psi,u) := f(t,\psi,u) - f(t,\bar\psi,\bar u) - D_2 f(t,\bar\psi,\bar u)(\psi-\bar\psi) - D_3 f(t,\bar\psi,\bar u)(u-\bar u) \quad (4.3)$$

and

$$\omega_\tau(\bar t,\bar\psi,t,\psi) := \tau(t,\psi) - \tau(\bar t,\bar\psi) - D_1\tau(\bar t,\bar\psi)(t-\bar t) - D_2\tau(\bar t,\bar\psi)(\psi-\bar\psi) \quad (4.4)$$

for $\bar t, t \in [0,T]$, $\bar\psi,\psi \in \Omega_1$ and $\bar u, u \in \Omega_2$.

Lemma 4.2 *Suppose (A1) (i)–(iii), (A2) (i)–(iii). Let P and $\alpha > 0$ be defined by Theorem 3.1, let $(\sigma,\varphi) \in P$ be fixed, and $(\sigma_k,\varphi^k) \in P$ ($k \in \mathbb{N}$) be a sequence satisfying $|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}} \to 0$ as $k \to \infty$. Let $x(t) := x(t,\sigma,\varphi)$ and $x^k(t) := x(t,\sigma_k,\varphi^k)$. Then*

$$\lim_{k\to\infty} \frac{1}{|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}} \int_{\max\{\sigma,\sigma_k\}}^{\alpha} |\omega_f(s, x_s, x(s-\tau(s,x_s)), x^k_s, x^k(s-\tau(s,x^k_s)))|\, ds = 0 \quad (4.5)$$

and

$$\lim_{k\to\infty} \frac{1}{|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}} \int_{\max\{\sigma,\sigma_k\}}^{\alpha} |\omega_\tau(s, x_s, s, x^k_s)|\, ds = 0. \quad (4.6)$$

Proof It follows from the definition of ω_f that

$$\begin{aligned}\omega_f(t,\bar\psi,\bar u,\psi,u) &= \int_0^1 \Big[\Big(D_2 f(t,\bar\psi+\nu(\psi-\bar\psi),\bar u+\nu(u-\bar u)) - D_2 f(t,\bar\psi,\bar u)\Big)(\psi-\bar\psi) \\ &\quad + \Big(D_3 f(t,\bar\psi+\nu(\psi-\bar\psi),\bar u+\nu(u-\bar u)) - D_3 f(t,\bar\psi,\bar u)\Big)(u-\bar u)\Big]\, d\nu,\end{aligned}$$

therefore

$$\begin{aligned}&|\omega_f(t,\bar\psi,\bar u,\psi,u)| \\ &\qquad \le \sup_{0<\nu<1}\Big(\Big|D_2 f(t,\bar\psi+\nu(\psi-\bar\psi),\bar u+\nu(u-\bar u)) - D_2 f(t,\bar\psi,\bar u)\Big|_{\mathcal{L}(C,\mathbb{R}^n)}|\psi-\bar\psi|_C \\ &\qquad\quad + \Big|D_3 f(t,\bar\psi+\nu(\psi-\bar\psi),\bar u+\nu(u-\bar u)) - D_3 f(t,\bar\psi,\bar u)\Big||u-\bar u|\Big) \quad (4.7)\end{aligned}$$

for $t \in [0,T]$, $\bar\psi,\psi \in \Omega_1$ and $\bar u, u \in \Omega_2$. Define

$$\begin{aligned}\Omega_f(\varepsilon) &:= \sup\Big\{\max\Big(|D_2 f(t,\psi,u) - D_2 f(t,\tilde\psi,\tilde u)|_{\mathcal{L}(C,\mathbb{R}^n)}, |D_3 f(t,\psi,u) - D_3 f(t,\tilde\psi,\tilde u)|\Big) : \\ &\qquad |\psi-\tilde\psi|_C + |u-\tilde u| \le \varepsilon,\ \ t \in [\sigma_0,\alpha],\ \psi,\tilde\psi \in M_1,\ u,\tilde u \in M_2\Big\}.\end{aligned}$$

Note that Ω_f is well-defined and $\Omega_f(\varepsilon) \to 0$ as $\varepsilon \to 0$, since M_1 and M_2 are compact, and $D_2 f$ and $D_3 f$ are continuous and, therefore, uniformly continuous on $[\sigma_0, \alpha] \times M_1 \times M_2$.

By our assumptions and Theorem 3.1 we have

$$x_t, x_t^k \in M_1, \quad x(t - \tau(t, x_t)), x^k(t - \tau(t, x_t^k)) \in M_2,$$

and

$$|x_t - x_t^k|_{W^{1,\infty}} \le L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}})$$

for $t \in [\max\{\sigma, \sigma_k\}, \alpha]$, $k \in \mathbb{N}$. Then the definitions of L_2, L and N and the Mean Value Theorem yield

$$\begin{aligned}
&|x(t - \tau(t, x_t)) - x^k(t - \tau(t, x_t^k))| \\
&\qquad \le \ |x(t - \tau(t, x_t)) - x(t - \tau(t, x_t^k))| + |x(t - \tau(t, x_t^k)) - x^k(t - \tau(t, x_t^k))| \\
&\qquad \le \ N|\tau(t, x_t) - \tau(t, x_t^k)| + L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}) \\
&\qquad \le \ N L_2 |x_t - x_t^k|_C + L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}) \\
&\qquad \le \ (N L_2 + 1) L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}) \qquad (4.8)
\end{aligned}$$

for $t \in [\sigma, \alpha]$ and $k \in \mathbb{N}$. Then (3.7), (4.7), (4.8) and the definition of Ω_f imply

$$\begin{aligned}
&\int_{\max\{\sigma,\sigma_k\}}^{\alpha} |\omega_f(s, x_s, x(s - \tau(s, x_s)), x_s^k, x^k(s - \tau(s, x_s^k)))|\, ds \\
&\qquad \le \ \alpha \Omega_f\Big(N_0 L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}})\Big) N_0 L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}),
\end{aligned}$$

which proves (4.5), since $\Omega_f\Big(N_0 L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}})\Big) \to 0$ as $k \to \infty$.

Define

$$\begin{aligned}
\Omega_\tau(\varepsilon) \ := \ \sup\Big\{ &\max\Big(|D_1\tau(t, \psi) - D_1\tau(\bar{t}, \bar{\psi})|, |D_2\tau(t, \psi) - D_2\tau(\bar{t}, \bar{\psi})|_{\mathcal{L}(C,\mathbb{R})}\Big) : \\
&|t - \bar{t}| + |\psi - \bar{\psi}|_C \le \varepsilon, \ t, \bar{t} \in [\sigma_0, \alpha], \ \psi, \bar{\psi} \in M_1 \Big\}. \qquad (4.9)
\end{aligned}$$

Similarly to (4.7) we can obtain

$$\begin{aligned}
|\omega_\tau(\bar{t}, \bar{\psi}, t, \psi)| \ \le \sup_{0<\nu<1} \Big(&\Big| D_1\tau(\bar{t} + \nu(t - \bar{t}), \bar{\psi} + \nu(\psi - \bar{\psi})) - D_1\tau(\bar{t}, \bar{\psi}) \Big| |t - \bar{t}| \\
&+ \Big| D_2\tau(\bar{t} + \nu(t - \bar{t}), \bar{\psi} + \nu(\psi - \bar{\psi})) - D_2\tau(\bar{t}, \bar{\psi}) \Big|_{\mathcal{L}(C,\mathbb{R})} |\psi - \bar{\psi}|_C \Big), \quad (4.10)
\end{aligned}$$

for $\bar{t}, t \in [0, T]$ and $\bar{\psi}, \psi \in \Omega_1$. Then it is easy to see that

$$\int_{\max\{\sigma,\sigma_k\}}^{\alpha} |\omega_\tau(s, x_s, s, x_s^k)|\, ds \le \alpha \Omega_\tau\Big(L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}})\Big) L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}),$$

which, together with the assumed continuity of $D_1\tau$ and $D_2\tau$, implies (4.6). □

Let $\alpha_\sigma^* := \min\{\sigma + r, \alpha\}$. We introduce a certain class of functions

$$X(\sigma,\alpha) \;:=\; \Big\{x \in W^{1,\infty}([\sigma - r, \alpha], \mathbb{R}^n)\colon\ x_t \in \Omega_1,\ x(t - \tau(t, x_t)) \in \Omega_2 \text{ for } t \in [\sigma, \alpha], \\ \text{and } \operatorname{ess\,inf}\Big\{\frac{d}{dt}(t - \tau(t, x_t))\colon\ \text{a.e. } t \in [\sigma, \alpha_\sigma^*]\Big\} > 0\Big\}. \tag{4.11}$$

The next lemma shows that if a solution $\hat{x}(t) := x(t, \hat{\sigma}, \hat{\varphi})$ of the IVP (3.1)-(3.2) belongs to $X(\sigma, \alpha)$, then there exists a small neighborhood P_1 of the parameters $(\hat{\sigma}, \hat{\varphi})$ so that all solutions which correspond to parameters from P_1 will belong to $X(\sigma, \alpha)$, as well. The next result is a generalization of Lemma 5.2 of [10] for the case when σ varies, and here assuming less smoothness of τ wrt its second argument.

Lemma 4.3 *Suppose (A2) (i)–(iii), $(\hat{\sigma}, \hat{\varphi}) \in \Pi$, and let the constants $\delta > 0$, $0 \le \sigma_0 \le \hat{\sigma}$ and $\hat{\sigma} < \alpha \le T$ and the set P be defined by Theorem 3.1, and let $\hat{x}(t) := x(t, \hat{\sigma}, \hat{\varphi})$ for $t \in [\hat{\sigma} - r, \alpha]$. Suppose $\hat{x} \in X(\hat{\sigma}, \alpha)$. Then there exist $\gamma^* > 0$ and $0 < \delta^* \le \delta$ such that for*

$$P_1 := \Big([\sigma_0, \alpha) \cap (\hat{\sigma} - \gamma^*, \hat{\sigma} + \gamma^*)\Big) \times \mathcal{B}_{W^{1,\infty}}(\hat{\varphi};\, \delta^*)$$

we have

$$x(\cdot, \sigma, \varphi) \in X(\sigma, \alpha), \qquad (\sigma, \varphi) \in P_1.$$

Proof Let $x(t) := x(t, \sigma, \varphi)$ for $(\sigma, \varphi) \in P$ and $t \in [\sigma, \alpha]$. We have $\frac{d}{dt}(t - \tau(t, x_t)) = 1 - \frac{d}{dt}\tau(t, x_t)$, therefore $x \in X(\sigma, \alpha)$ is equivalent to that there exists $\varepsilon > 0$ such that

$$\frac{d}{dt}\tau(t, x_t) \le 1 - \varepsilon, \qquad \text{a.e. } t \in [\sigma, \alpha_\sigma^*]. \tag{4.12}$$

Let $L_3 := \max\{|D_1\tau(s, \psi)|\colon\ s \in [\sigma_0, \alpha],\ \psi \in M_1\}$. Then the definitions of L_2, L_3 and N yield

$$|\tau(t, x_t) - \tau(\bar{t}, x_{\bar{t}})| \le |\tau(t, x_t) - \tau(\bar{t}, x_t)| + |\tau(\bar{t}, x_t) - \tau(\bar{t}, x_{\bar{t}})| \le (L_3 + L_2 N)|t - \bar{t}|,$$

for $t, \bar{t} \in [\sigma, \alpha]$, hence $t \mapsto \tau(t, x_t)$ is Lipschitz-continuous, and so it is almost everywhere differentiable on $[\sigma, \alpha]$. To prove (4.12), it is enough to argue that there exist $\gamma^* > 0$, $0 < \delta^* \le \delta$ and $\varepsilon > 0$ such that for every $\sigma \in [\sigma_0, \alpha) \cap (\hat{\sigma} - \gamma^*, \hat{\sigma} + \gamma^*)$, $\varphi \in \mathcal{B}_{W^{1,\infty}}(\hat{\varphi};\, \delta^*)$ and a.e. $t \in (\sigma, \alpha_\sigma^*)$ there exists $\eta = \eta(\sigma, \varphi, t) > 0$ such that

$$\frac{\tau(t + h, x_{t+h}) - \tau(t, x_t)}{h} < 1 - \varepsilon, \qquad 0 < |h| < \eta. \tag{4.13}$$

Since $\hat{x} \in X(\hat{\sigma}, \alpha)$, there exists $\varepsilon_1 > 0$ such that $\frac{d}{dt}\tau(t, \hat{x}_t) < 1 - \varepsilon_1$ for a.e. $t \in [\hat{\sigma}, \alpha^*_\sigma]$, so for a.e. $t \in (\hat{\sigma}, \alpha^*_\sigma)$ there exists $\eta_1 = \eta_1(t) > 0$ such that

$$\frac{\tau(t+h, \hat{x}_{t+h}) - \tau(t, \hat{x}_t)}{h} < 1 - \varepsilon_1, \qquad 0 < |h| < \eta_1. \tag{4.14}$$

Using the definition of ω_τ introduced in (4.4) we get for $t \in (\hat{\sigma}, \alpha^*_\sigma)$

$$\frac{\tau(t+h, \hat{x}_{t+h}) - \tau(t, \hat{x}_t)}{h} = \frac{1}{h}\Big(D_1\tau(t, \hat{x}_t)h + D_2\tau(t, \hat{x}_t)(\hat{x}_{t+h} - \hat{x}_t)\Big) + \frac{1}{h}\omega_\tau(t, \hat{x}_t, t+h, \hat{x}_{t+h}),$$

and $\frac{1}{h}\omega_\tau(t, \hat{x}_t, t+h, \hat{x}_{t+h}) \to 0$ as $h \to 0$, since $|\hat{x}_{t+h} - \hat{x}_t|_C \to 0$ as $h \to 0$, and τ is Fréchet-differentiable at $(t, \hat{x}_t)$ by (A2) (iii). Therefore for every $0 < \varepsilon_2 < \varepsilon_1$ and a.e. $t \in (\hat{\sigma}, \alpha^*_\sigma)$ there exists $\eta_2 = \eta_2(\varepsilon_2, t)$ such that $0 < \eta_2 < \eta_1$ and

$$\frac{1}{h}\Big(D_1\tau(t, \hat{x}_t)h + D_2\tau(t, \hat{x}_t)(\hat{x}_{t+h} - \hat{x}_t)\Big) < 1 - \varepsilon_2, \qquad 0 < |h| < \eta_2. \tag{4.15}$$

We will distinguish three cases.

Case (i): Suppose $\sigma \geq \hat{\sigma}$. Then for $t \in (\sigma, \alpha^*_\sigma)$ there exists $0 < \eta_3 < \eta_2$ such that $t + h \in (\sigma, \alpha^*_\sigma)$, and so $x_t, x_{t+h}, \hat{x}_t, \hat{x}_{t+h}$ are all defined, for $|h| < \eta_3$. For $t \in (\sigma, \alpha^*_\sigma)$ and $0 < |h| < \eta_3$ consider

$$\begin{aligned}
&\frac{\tau(t+h, x_{t+h}) - \tau(t, x_t)}{h} \\
&\quad= \frac{1}{h}\Big(D_1\tau(t, x_t)h + D_2\tau(t, x_t)(x_{t+h} - x_t) + \omega_\tau(t, x_t, t+h, x_{t+h})\Big) \\
&\quad= \frac{1}{h}\Big(D_1\tau(t, \hat{x}_t)h + D_2\tau(t, \hat{x}_t)(\hat{x}_{t+h} - \hat{x}_t)\Big) \\
&\qquad+\Big(D_1\tau(t, x_t) - D_1\tau(t, \hat{x}_t)\Big) + \frac{1}{h}\Big(D_2\tau(t, x_t) - D_2\tau(t, \hat{x}_t)\Big)(x_{t+h} - x_t) \\
&\qquad+\frac{1}{h}D_2\tau(t, \hat{x}_t)\Big(x_{t+h} - \hat{x}_{t+h} - x_t + \hat{x}_t\Big) + \frac{1}{h}\omega_\tau(t, x_t, t+h, x_{t+h}).
\end{aligned} \tag{4.16}$$

We have $|x_t - \hat{x}_t|_{W^{1,\infty}} \leq L(|\sigma - \hat{\sigma}| + |\varphi - \hat{\varphi}|_{W^{1,\infty}})$ from (3.4). Using the definition of Ω_τ (see (4.9)), the Mean Value Theorem, (3.3) and (4.10), we obtain

$$\begin{aligned}
\left|\frac{1}{h}\Big(D_2\tau(t, x_t) - D_2\tau(t, \hat{x}_t)\Big)(x_{t+h} - x_t)\right| &\leq \Omega_\tau\Big(L(|\sigma - \hat{\sigma}| + |\varphi - \hat{\varphi}|_{W^{1,\infty}})\Big)|\dot{x}|_{L^\infty([\sigma-r,\alpha],\mathbb{R}^n)} \\
&\leq \Omega_\tau\Big(L(|\sigma - \hat{\sigma}| + |\varphi - \hat{\varphi}|_{W^{1,\infty}})\Big)N \\
&\to 0, \qquad \text{as } \sigma \to \hat{\sigma}+ \text{ and } |\varphi - \hat{\varphi}|_{W^{1,\infty}} \to 0.
\end{aligned} \tag{4.17}$$

Similarly, for $t \in (\sigma, \alpha^*_\sigma)$

$$\Big| D_1\tau(t, x_t) - D_1\tau(t, \hat{x}_t) \Big| \le \Omega_\tau\Big(L(|\sigma - \hat{\sigma}| + |\varphi - \hat{\varphi}|_{W^{1,\infty}}) \Big) \to 0, \tag{4.18}$$

as $\sigma \to \hat{\sigma}+$ and $|\varphi - \hat{\varphi}|_{W^{1,\infty}} \to 0$. By employing the Mean Value Theorem and (3.4) we find for $t \in (\sigma, \alpha^*_\sigma)$ and $|h| < \eta_3$ that

$$\begin{aligned} \Big| x_{t+h} - \hat{x}_{t+h} - x_t + \hat{x}_t \Big|_C &\le \max_{\sigma \le t \le \alpha} |\dot{x}_t - \dot{\hat{x}}_t|_{L^\infty} |h| \\ &\le \max_{\sigma \le t \le \alpha} |x_t - \hat{x}_t|_{W^{1,\infty}} |h| \\ &\le L(|\sigma - \hat{\sigma}| + |\varphi - \hat{\varphi}|_{W^{1,\infty}})|h|. \end{aligned} \tag{4.19}$$

Therefore for $t \in (\sigma, \alpha^*_\sigma)$ and $0 < |h| < \eta_3$

$$\frac{1}{h} D_2\tau(t, \hat{x}_t)\Big(x_{t+h} - \hat{x}_{t+h} - x_t + \hat{x}_t \Big) \to 0, \qquad \text{as } \sigma \to \hat{\sigma}+ \text{ and } |\varphi - \hat{\varphi}|_{W^{1,\infty}} \to 0, \tag{4.20}$$

since $|D_2\tau(t, \hat{x}_t)|_{\mathcal{L}(C,\mathbb{R})} \le L_2$ for $t \in [\hat{\sigma}, \alpha^*_\sigma]$.

Finally, with the help of (4.10) and (3.3), we get for $t \in (\sigma, \alpha^*_\sigma)$ and $|h| < \eta_3$

$$\begin{aligned} \frac{1}{|h|} |\omega_\tau(t, x_t, t+h, x_{t+h})| &\le \frac{1}{|h|} \Omega_\tau(|h| + |x_{t+h} - x_t|_C)(|h| + |x_{t+h} - x_t|_C) \\ &\le \Omega_\tau((1+N)|h|)(1+N) \\ &\to 0, \qquad \text{as } |h| \to 0. \end{aligned} \tag{4.21}$$

Therefore, combining relations (4.17)–(4.21) with (4.15) and (4.16) yields that for every $0 < \varepsilon_3 < \varepsilon_2$ there exists $0 < \delta_1 < \delta$ and $0 < \gamma_1 < \alpha - \hat{\sigma}$ such that for every $\sigma \in [\hat{\sigma}, \hat{\sigma} + \gamma_1)$ and $\varphi \in \mathcal{B}_{W^{1,\infty}}(\hat{\varphi}; \delta_1)$, and for a.e. $t \in (\sigma, \alpha^*_\sigma)$ there exist $0 < \eta_4 < \eta_3$ such that (4.13) holds with $\varepsilon = \varepsilon_3$ and $\eta = \eta_4$.

Case (ii): Suppose $\sigma < \hat{\sigma}$ and $t \in (\hat{\sigma}, \alpha^*_{\hat{\sigma}})$. Then again $x_t, x_{t+h}, \hat{x}_t$ and $\hat{x}_{t+h}$ are all defined for small h, therefore the argument of Case (i) can be repeated, and we get that for every $0 < \varepsilon_4 < \varepsilon_3$, there exist $0 < \delta_3 < \delta_2$ and $0 < \gamma_2 < \gamma_1$ such that for $\sigma \in [\sigma_0, \alpha) \cap (\hat{\sigma} - \gamma_2, \hat{\sigma} + \gamma_2)$ and a.e. $t \in (\hat{\sigma}, \alpha^*_{\hat{\sigma}})$ there exists $0 < \eta_5 < \eta_4$ such that (4.13) holds with $\varepsilon^* = \varepsilon_4$ and $\eta = \eta_5$.

Case (iii): Suppose $\sigma < \hat{\sigma}$ and $t \in (\sigma, \hat{\sigma})$, and let σ_1 be such that $\sigma_1 \in (\hat{\sigma}, \alpha^*_{\hat{\sigma}})$. Then let $0 < \eta_6 < \eta_5$ be such that $t + h \in (\sigma, \hat{\sigma})$ and $\sigma_1 + h \in (\hat{\sigma}, \alpha^*_{\hat{\sigma}})$ for $|h| < \eta_6$. Then for $0 < |h| < \eta_6$

$$\begin{aligned} \frac{\tau(t+h, x_{t+h}) - \tau(t, x_t)}{h} &= \frac{\tau(\sigma_1 + h, \hat{x}_{\sigma_1+h}) - \tau(\sigma_1, \hat{x}_{\sigma_1})}{h} \\ &\quad + \frac{\tau(t+h, x_{t+h}) - \tau(\sigma_1 + h, \hat{x}_{\sigma_1+h}) - \tau(t, x_t) + \tau(\sigma_1, \hat{x}_{\sigma_1})}{h}. \end{aligned}$$

Since $\sigma_1 \in (\hat{\sigma}, \alpha)$, by (4.14) we know that σ_1 can be selected so that σ_1 is arbitrary close to $\hat{\sigma}$, and

$$\frac{\tau(\sigma_1 + h, \hat{x}_{\sigma_1+h}) - \tau(\sigma_1, \hat{x}_{\sigma_1})}{h} < 1 - \varepsilon_4, \qquad |h| < \eta_6. \tag{4.22}$$

Using the definition of ω_τ we get

$$\begin{aligned}
&\frac{\tau(t+h, x_{t+h}) - \tau(t, x_t) - \tau(\sigma_1 + h, \hat{x}_{\sigma_1+h}) + \tau(\sigma_1, \hat{x}_{\sigma_1})}{h} \\
&= \frac{1}{h}\Big(D_1\tau(t, x_t)h + D_2\tau(t, x_t)(x_{t+h} - x_t)\Big) \\
&\quad -\frac{1}{h}\Big(D_1\tau(\sigma_1, \hat{x}_{\sigma_1})h + D_2\tau(\sigma_1, \hat{x}_{\sigma_1})(\hat{x}_{\sigma_1+h} - \hat{x}_{\sigma_1})\Big) \\
&\quad +\frac{1}{h}\omega_\tau(t, x_t, t+h, x_{t+h}) - \frac{1}{h}\omega_\tau(\sigma_1, \hat{x}_{\sigma_1}, \sigma_1 + h, \hat{x}_{\sigma_1+h}) \\
&= \frac{1}{h}\Big(D_1\tau(t, x_t) - D_1\tau(\sigma_1, \hat{x}_{\sigma_1})\Big)h \\
&\quad +\frac{1}{h}\Big(D_2\tau(t, x_t) - D_2\tau(\sigma_1, \hat{x}_{\sigma_1})\Big)(\hat{x}_{\sigma_1+h} - \hat{x}_{\sigma_1}) \\
&\quad +\frac{1}{h}D_2\tau(t, x_t)\Big(x_{t+h} - x_t - (x_{\sigma_1+h} - x_{\sigma_1})\Big) \\
&\quad +\frac{1}{h}D_2\tau(t, x_t)\Big(x_{\sigma_1+h} - x_{\sigma_1} - (\hat{x}_{\sigma_1+h} - \hat{x}_{\sigma_1})\Big) \\
&\quad +\frac{1}{h}\omega_\tau(t, x_t, t+h, x_{t+h}) - \frac{1}{h}\omega_\tau(\sigma_1, \hat{x}_{\sigma_1}, \sigma_1 + h, \hat{x}_{\sigma_1+h}).
\end{aligned}$$

Therefore using $|t - \sigma_1| \le |\sigma - \sigma_1|$, $|\hat{x}_{\sigma_1+h} - \hat{x}_{\sigma_1}|_C \le N|h|$, (4.19), and the estimates

$$|x_t - \hat{x}_{\sigma_1}|_C \le |x_t - x_{\sigma_1}|_C + |x_{\sigma_1} - \hat{x}_{\sigma_1}|_C \le N|\sigma - \sigma_1| + L(|\sigma - \hat{\sigma}| + |\varphi - \hat{\varphi}|_{W^{1,\infty}})$$

and

$$|x_{\sigma_1+h} - \hat{x}_{\sigma_1+h} - (x_{\sigma_1} - \hat{x}_{\sigma_1})|_C \le \max_{\hat{\sigma} \le s \le \alpha} |x_s - \hat{x}_s|_{W^{1,\infty}}|h| \le L(|\sigma - \hat{\sigma}| + |\varphi - \hat{\varphi}|_{W^{1,\infty}})|h|,$$

and the definitions of L_2, Ω_τ, we obtain

$$\begin{aligned}
&\Big|\frac{\tau(t+h, x_{t+h}) - \tau(t, x_t) - \tau(\sigma_1 + h, \hat{x}_{\sigma_1+h}) + \tau(\sigma_1, \hat{x}_{\sigma_1})}{h}\Big| \\
&\quad \le \Omega_\tau\Big(\max\{N, L\}(2|\sigma - \sigma_1| + |\varphi - \hat{\varphi}|_{W^{1,\infty}})\Big)(1+N) \\
&\qquad + \frac{1}{|h|}\Big|D_2\tau(t, x_t)\Big(x_{t+h} - x_t - (x_{\sigma_1+h} - x_{\sigma_1}))\Big)\Big| + L_2L(|\sigma - \hat{\sigma}| + |\varphi - \hat{\varphi}|_{W^{1,\infty}}) \\
&\qquad 2\Omega_\tau((1+N)h)(1+N).
\end{aligned} \tag{4.23}$$

We show that the second term on the right-hand-side of (4.23) also goes to 0 as $h \to 0$, $\sigma \to \hat\sigma$ and $\sigma_1 \to \hat\sigma+$. Since $D_2\tau(t, x_t) \in \mathcal{L}(C, \mathbb{R})$, it can be represented by a Riemann-Stieltjes integral, i.e., for each $t \in [\sigma, \alpha]$ there exists a function $\mu(t, \cdot)\colon [-r, 0] \to \mathbb{R}^{1\times n}$ of bounded variation such that

$$D_2\tau(t, x_t)\psi = \int_{-r}^{0} d_\theta\mu(t, \theta)\psi(\theta), \qquad \psi \in C, \quad t \in [\sigma, \alpha].$$

Then

$$\begin{aligned}
&\frac{1}{|h|}\Big|D_2\tau(t, x_t)\Big(x_{t+h} - x_t - (x_{\sigma_1+h} - x_{\sigma_1})\Big)\Big| \\
&\quad= \frac{1}{|h|}\Big|\int_{-r}^{0} d_\theta\mu(t, \theta)\Big(x(t+h+\theta) - x(t+\theta) - (x(\sigma_1+h+\theta) - x(\sigma_1+\theta))\Big)\Big| \\
&\quad= \frac{1}{|h|}\Big|\int_{-r}^{0} d_\theta\mu(t, \theta)\Big(\int_{t+\theta}^{t+\theta+h} \dot x(u)\, du - \int_{\sigma_1+\theta}^{\sigma_1+\theta+h} \dot x(u)\, du\Big)\Big| \\
&\quad= \Big|\int_{-r}^{0} d_\theta\mu(t, \theta)\int_0^1 (\dot x(t+\theta+\nu h) - \dot x(\sigma_1+\theta+\nu h))\, d\nu\Big| \\
&\quad\to 0, \qquad \text{as } \sigma \to \hat\sigma- \quad \text{and } \sigma_1 \to \hat\sigma+. \qquad (4.24)
\end{aligned}$$

The last relation holds, since the function

$$g_{t,\sigma_1,h}(\theta) := \int_0^1 (\dot x(t+\theta+\nu h) - \dot x(\sigma_1+\theta+\nu h))\, d\nu$$

is continuous on $[-r, 0]$, and for every fixed $\theta \in [-r, 0]$ Lemma 2.2 implies $|g_{t,\sigma_1,h}(\theta)| \le \int_0^1 |\dot x(t+\theta+\nu h) - \dot x(\sigma_1+\theta+\nu h)|\, d\nu \to 0$ as $\sigma \to \hat\sigma-$ (and hence $t \to \hat\sigma-$) and $\sigma_1 \to \hat\sigma+$, therefore the Dominated Convergence Theorem for Lebesgue-Stieltjes integrals yields (4.24).

Therefore, combining (4.22), (4.23) and the above relation, we get that for every $0 < \varepsilon < \varepsilon_4$ there exist $0 < \delta^* < \delta_3$ and $0 < \gamma^* < \gamma_2$ such that for $\sigma \in [\sigma_0, \alpha) \cap (\hat\sigma - \gamma^*, \hat\sigma + \gamma^*)$ and $\varphi \in \mathcal{B}_{W^{1,\infty}}(\hat\varphi; \delta^*)$ (4.13) holds, and therefore the lemma is established. $\square$

If $x(t) = x(t, \sigma, \varphi)$ is a solution of the IVP (3.1)-(3.2) for $(\sigma, \varphi) \in P$, then x is, in general, only a $W^{1,\infty}$-function on the interval $[\sigma - r, \sigma]$, but it is continuously differentiable for $t \ge \sigma$. Therefore Lemma 2.2 yields that if $x \in X(\sigma, \alpha)$, then the composite function $\dot x(t - \tau(t, x_t))$ is defined for a.e. $t \in [\sigma, \alpha^*_\sigma]$, it is integrable on $[\sigma, \alpha^*_\sigma]$, and it is always well-defined and continuous for $t \in (\alpha^*_\sigma, \alpha]$, where $\alpha^*_\sigma := \min\{\sigma + r, \alpha\}$.

For a.e. $t \in [\sigma, \alpha]$ and for any $x \in X(\sigma, \alpha)$ we introduce the linear operator $L(t, x) \colon C \to \mathbb{R}^n$ by

$$L(t,x)\psi := D_2 f(t, x_t, x(t-\tau(t,x_t)))\psi \\ +D_3 f(t, x_t, x(t-\tau(t,x_t)))\Big(-\dot{x}(t-\tau(t,x_t))D_2\tau(t,x_t)\psi + \psi(-\tau(t,x_t))\Big) \quad (4.25)$$

for $\psi \in C$. We have

$$|L(t,x)\psi| \le m(t)|\psi|_C, \qquad \text{a.e. } t \in [\sigma, \alpha],$$

where

$$\begin{aligned} m(t) \;:=\; & |D_2(t, x_t, x(t-\tau(t,x_t)))|_{\mathcal{L}(C,\mathbb{R}^n)} \\ & +|D_3 f(t, x_t, x(t-\tau(t,x_t)))|\Big(|\dot{x}(t-\tau(t,x_t))||D_2\tau(t,x_t)|_{\mathcal{L}(C,\mathbb{R})} + 1\Big) \end{aligned}$$

for a.e. $t \in [\sigma, \alpha]$. Note that Lemma 2.2 implies $m \in L^1([\sigma, \alpha], \mathbb{R})$. Hence $L(t, x)$ is a bounded linear operator for all t for which $\dot{x}(t - \tau(t, x_t))$ exists. Moreover, if for some $(\sigma, \varphi) \in P$ the function $x(t) = x(t, \sigma, \varphi)$ is the solution of the IVP (3.1)-(3.2), then

$$m(t) \le L_1 N_0, \qquad \text{a.e. } t \in [\sigma, \alpha], \tag{4.26}$$

where N_0 is defined by (3.7).

Let P_1 be defined by Lemma 4.3. Then for $(\sigma, \varphi) \in P_1$ we define the variational equation associated to $x = x(\cdot, \sigma, \varphi)$ as

$$\dot{z}(t) = L(t,x)z_t, \qquad \text{a.e. } t \in [\sigma, \alpha], \tag{4.27}$$

$$z(t) = h(t-\sigma), \qquad t \in [\sigma - r, \sigma], \tag{4.28}$$

where the initial function is $h \in C$. The IVP (4.27)-(4.28) is a Carathéodory type linear delay equation. By its solution we mean a continuous function $z\colon [\sigma - r, \alpha]$ which is absolutely continuous on $[\sigma, \alpha]$, and it satisfies (4.27) for a.e. $t \in [\sigma, \alpha]$ and (4.28) for all $t \in [\sigma - r, \sigma]$. A standard argument ([3], [5]) shows that the IVP (4.27)-(4.28) has a unique solution $z(t) = z(t, \sigma, \varphi, h)$ for $(\sigma, \varphi) \in P_1$ and $t \in [\sigma - r, \alpha]$.

Lemma 4.4 *Assume (A1) (i)–(iii), (A2) (i)–(iii). Let P_1 be defined by Lemma 4.3, $(\sigma, \varphi) \in P_1$, and $x(t) = x(t, \sigma, \varphi)$ for $t \in [\sigma - r, \alpha]$. Let $h \in C$ and let $z(t, \sigma, \varphi, h)$ be the corresponding solution of the IVP* (4.27)-(4.28) *on $[\sigma - r, \alpha]$. Then* *(i)* $z(t,\sigma,\varphi,\cdot)\in\mathcal{L}(C,\mathbb{R}^n)$, *the map* $C\to C$, $h\mapsto z_t(\cdot,\sigma,\varphi,h)$ *is in* $\mathcal{L}(C,C)$, *and*

$$|z(t,\sigma,\varphi,\cdot)|_{\mathcal{L}(C,\mathbb{R}^n)}\le |z_t(\cdot,\sigma,\varphi,\cdot)|_{\mathcal{L}(C,C)}\le N_1,\qquad (\sigma,\varphi)\in P_1,\ t\in[\sigma,\alpha],\quad (4.29)$$

where N_1 *is defined by* (3.7);

(ii) there exists $N_2\ge 0$ *such that*

$$|z_t(\cdot,\sigma,\varphi,\cdot)|_{\mathcal{L}(W^{1,\infty},W^{1,\infty})}\le N_2,\qquad (\sigma,\varphi)\in P_1,\ t\in[\sigma,\alpha].\qquad (4.30)$$

Proof (i) The linearity of $z(t,\sigma,\varphi,\cdot)$ is obvious. To show the boundedness fix $h\in C$, and for simplicity let $z(t):=z(t,\sigma,\varphi,h)$. Then integrating (4.27) we get

$$z(t)=h(0)+\int_\sigma^t L(s,x)z_s\,ds,\qquad t\in[\sigma,\alpha].$$

Hence (4.26) yields

$$|z(t)|\le |h|_C+\int_\sigma^t L_1N_0|z_s|_C\,ds,\qquad t\in[\sigma,\alpha],$$

but then (4.29) follows from Lemma 2.1 with N_1 defined by (3.7).

To prove (ii) fix $h\in W^{1,\infty}$, and note that for $z(t)=z(t,\sigma,\varphi,h)$ we have $\dot z(t)=L(t,x)z_t$ for a.e. $t\in[\sigma,\alpha]$, therefore (4.26) and part (i) give $|\dot z(t)|\le L_1N_0N_1|h|_C$ for a.e. $t\in[\sigma,\alpha]$. On the other hand, $|\dot z(t)|\le |h|_{W^{1,\infty}}$ for a.e. $t\in[\sigma-r,\sigma]$, and since $N_1\ge 1$, $N_2:=\max\{N_1,L_1N_0N_1\}$ satisfies (4.30). $\square$

The following estimate will be used in the proof of the next lemma.

Lemma 4.5 *Assume (A1) (i)–(iii), (A2) (i)–(iii). Let* P_1 *be defined by Lemma 4.3,* (σ_k,φ^k), $(\sigma,\varphi)\in P_1$, $x^k(s):=x(s,\sigma_k,\varphi^k)$, $x(s):=x(s,\sigma,\varphi)$, $u^k(s):=s-\tau(s,x^k_s)$ *and* $u(s):=s-\tau(s,x_s)$. *Then*

$$\begin{aligned}|L(s,x^k)\psi-L(s,x)\psi| \;\le\;& N_0\Omega_f\Big(N_0L(|\sigma_k-\sigma|+|\varphi^k-\varphi|_{W^{1,\infty}})\Big)|\psi|_C\\ &+L_1N\Omega_\tau\Big(L(|\sigma_k-\sigma|+|\varphi^k-\varphi|_{W^{1,\infty}})\Big)|\psi|_C\\ &+L_1L_2L(|\sigma_k-\sigma|+|\varphi^k-\varphi|_{W^{1,\infty}})|\psi|_C\\ &+L_1L_2\Big|\dot x(u^k(s))-\dot x(u(s))\Big||\psi|_C\\ &+L_1\Big|\psi(-\tau(s,x^k_s))-\psi(-\tau(s,x_s))\Big|,\qquad s\in[\nu_k,\alpha],\quad (4.31)\end{aligned}$$

where $\nu_k:=\max\{\sigma,\sigma_k\}$ *and* N_0 *is defined by* (3.7).

Proof Let $\psi \in C$ be fixed. Then (4.25) and standard manipulations imply

$$\begin{aligned}
&L(s, x^k)\psi - L(s,x)\psi \\
&\quad = D_2 f(s, x_s^k, x^k(u^k(s)))\psi - D_2 f(s, x_s, x(u(s)))\psi \\
&\quad\quad + \Big(D_3 f(s, x_s^k, x^k(u^k(s))) - D_3 f(s, x_s, x(u(s))) \Big) \Big(-\dot{x}^k(u^k(s)) D_2\tau(s, x_s^k)\psi \Big) \\
&\quad\quad + D_3 f(s, x_s, x(u(s))) \Big(-\dot{x}^k(u^k(s)) + \dot{x}(u^k(s)) \Big) D_2\tau(s, x_s^k)\psi \\
&\quad\quad + D_3 f(s, x_s, x(u(s))) \Big(-\dot{x}(u^k(s)) + \dot{x}(u(s)) \Big) D_2\tau(s, x_s^k)\psi \\
&\quad\quad + D_3 f(s, x_s, x(u(s))) \Big(-\dot{x}(u(s)) \Big) \Big(D_2\tau(s, x_s^k) - D_2\tau(s, x_s) \Big)\psi \\
&\quad\quad + \Big(D_3 f(s, x_s^k, x^k(u^k(s))) - D_3 f(s, x_s, x(u(s))) \Big) \psi(-\tau(s, x_s^k)) \\
&\quad\quad + D_3 f(s, x_s, x(u(s))) \Big(\psi(-\tau(s, x_s^k)) - \psi(-\tau(s, x_s)) \Big), \qquad s \in [\nu_k, \alpha]. \quad (4.32)
\end{aligned}$$

We have by (3.4), (3.6) and (3.7) for $s \in [\nu_k, \alpha]$

$$|x_s^k - x_s|_C + |x^k(u^k(s)) - x(u(s))| \le N_0 |x_s^k - x_s|_C \le N_0 L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}).$$

Therefore, using (3.3), (4.29) and the definitions of L_1, L_2, Ω_f and Ω_τ, we get from (4.32)

$$\begin{aligned}
|L(s, x^k)\psi - L(s,x)\psi| \;=\; & \Omega_f\Big(N_0 L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}) \Big) |\psi|_C \\
& + \Omega_f\Big(N_0 L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}) \Big) N L_2 |\psi|_C \\
& + L_1 L_2 L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}) |\psi|_C \\
& + L_1 L_2 \Big| \dot{x}(u^k(s)) - \dot{x}(u(s)) \Big| |\psi|_C \\
& + L_1 N \Omega_\tau\Big(L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}) \Big) |\psi|_C \\
& + \Omega_f\Big(N_0 L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}) \Big) |\psi|_C \\
& + L_1 \Big| \psi(-\tau(s, x_s^k)) - \psi(-\tau(s, x_s)) \Big| \qquad s \in [\nu_k, \alpha],
\end{aligned}$$

which implies (4.31). □

Next we show that the linear operators $z(t, \sigma, \varphi, \cdot)$ and $z_t(\cdot, \sigma, \varphi, \cdot)$ are continuous in t, σ and φ, assuming that (σ, φ) belongs to P_1.

Lemma 4.6 *Assume (A1) (i)–(iii), (A2) (i)–(iii). Let P_1 be defined by Lemma 4.3, and*

$$H_1 := \{(t,\sigma,\varphi)\colon\ (\sigma,\varphi) \in P_1,\ t \in [\sigma,\alpha]\}, \tag{4.33}$$

let $(\sigma,\varphi) \in P_1$, and $x(t) := x(t,\sigma,\varphi)$ for $t \in [\sigma - r, \alpha]$. Let $h \in W^{1,\infty}$ and let $z(t,\sigma,\varphi,h)$ be the corresponding solution of the IVP (4.27)-(4.28) *on $[\sigma - r, \alpha]$. Then the maps*

$$\mathbb{R}^2 \times W^{1,\infty} \supset H_1 \to \mathcal{L}(W^{1,\infty}, \mathbb{R}^n), \quad (t,\sigma,\varphi) \mapsto z(t,\sigma,\varphi,\cdot)$$

and

$$\mathbb{R}^2 \times W^{1,\infty} \supset H_1 \to \mathcal{L}(W^{1,\infty}, C), \quad (t,\sigma,\varphi) \mapsto z_t(\cdot,\sigma,\varphi,\cdot)$$

are continuous.

Proof Let $(\sigma,\varphi) \in P_1$ be fixed, and let $(\sigma_k,\varphi^k) \in P_1$ ($k \in \mathbb{N}$) be a sequence such that $|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}} \to 0$ as $k \to \infty$. We introduce the short notations $x^k(t) := x(t,\sigma_k,\varphi^k)$, $x(t) := x(t,\sigma,\varphi)$, $u^k(t) := t - \tau(t, x^k_t)$, $u(t) := t - \tau(t,x_t)$, $z^k(t) := z(t,\sigma_k,\varphi^k,h)$ and $z(t) := z(t,\sigma,\varphi,h)$ for a fixed $h \in W^{1,\infty}$. The functions z and z^k satisfy

$$\begin{aligned} z^k(t) &= h(0) + \int_{\sigma_k}^t L(s,x^k)z^k_s\,ds, \qquad t \in [\sigma_k,\alpha], \\ z(t) &= h(0) + \int_{\sigma}^t L(s,x)z_s\,ds, \qquad t \in [\sigma,\alpha]. \end{aligned}$$

Suppose first $\sigma \le \sigma_k$. Then

$$z^k(t) - z(t) = -\int_\sigma^{\sigma_k} L(s,x)z_s\,ds + \int_{\sigma_k}^t \Big(L(s,x^k)z^k_s - L(s,x)z_s\Big)ds, \qquad t \in [\sigma_k,\alpha].$$

On the other hand, if $\sigma \ge \sigma_k$, we get

$$z^k(t) - z(t) = \int_{\sigma_k}^{\sigma} L(s,x^k)z^k_s\,ds + \int_{\sigma}^t \Big(L(s,x^k)z^k_s - L(s,x)z_s\Big)ds, \qquad t \in [\sigma,\alpha].$$

Therefore in both cases (4.26) and (4.29) yield for $t \in [\max\{\sigma,\sigma_k\},\alpha]$

$$|z^k(t) - z(t)| \le |\sigma_k - \sigma| L_1 N_0 N_1 |h|_C + \int_{\nu_k}^t \Big| L(s,x^k)(z^k_s - z_s) + (L(s,x^k) - L(s,x))z_s \Big|\,ds, \tag{4.34}$$

where $\nu_k := \max\{\sigma,\sigma_k\}$. Then Lemma 4.5, (4.29) and

$$|z(u^k(s)) - z(u(s))| \le N_2|u^k(s) - u(s)| \le N_2 L_2 |x^k_s - x_s|_C \le N_2 L_2 L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}})$$

imply

$$|z^k(t) - z(t)| \leq A_k |h|_{W^{1,\infty}} + \int_{\nu_k}^t L_1 N_0 |z_s^k - z_s|_C \, ds, \qquad t \in [\nu_k, \alpha], \tag{4.35}$$

where A_k is defined by

$$\begin{aligned} A_k \;:=\; & (1 + L_1 N_0 N_1 + \alpha L_1 L_2 L N_1 + \alpha L_1 N_2 L_2 L)(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}) \\ & + \alpha N_0 N_1 \Omega_f \Big(N_0 L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}) \Big) \\ & + \alpha L_1 N N_1 \Omega_\tau \Big(L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}) \Big) \\ & + L_1 L_2 N_1 \int_{\nu_k}^{\alpha} |\dot{x}(u^k(s)) - \dot{x}(u(s))| \, ds. \end{aligned}$$

Next we estimate $|z^k_{\nu_k} - z_{\nu_k}|_C$. Let $\theta \in [-r, 0] \cap [\min\{\sigma, \sigma_k\} - \nu_k, 0]$. Suppose first that $\sigma_k \leq \sigma$. Then (4.26) and (4.29) imply

$$\begin{aligned} |z^k(\nu_k + \theta) - z(\nu_k + \theta)| &= \Big| h(0) + \int_{\sigma_k}^{\nu_k + \theta} L(s, x^k) z_s^k \, ds - h(\nu_k + \theta - \sigma) \Big| \\ &\leq |h(0) - h(\nu_k + \theta - \sigma)| + L_1 N_0 N_1 |h|_C (\nu_k + \theta - \sigma_k) \\ &\leq |h|_{W^{1,\infty}} (1 + L_1 N_0 N_1) |\sigma_k - \sigma|. \end{aligned}$$

In the opposite case when $\sigma_k > \sigma$ we get for $\theta \in [-r, 0] \cap [\min\{\sigma, \sigma_k\} - \nu_k, 0]$

$$\begin{aligned} |z^k(\nu_k + \theta) - z(\nu_k + \theta)| &= \Big| h(\nu_k + \theta - \sigma_k) - h(0) - \int_{\sigma}^{\nu_k + \theta} L(s, x) z_s \, ds \Big| \\ &\leq |h|_{W^{1,\infty}} (1 + L_1 N_0 N_1) |\sigma_k - \sigma|. \end{aligned}$$

For $\theta \in [-r, 0] \cap (-\infty, \min\{\sigma, \sigma_k\} - \nu_k]$ we have

$$|z^k(\nu_k + \theta) - z(\nu_k + \theta)| = |h(\nu_k + \theta - \sigma_k) - h(\nu_k + \theta - \sigma)| \leq |h|_{W^{1,\infty}} |\sigma_k - \sigma|.$$

Combining the above three cases we get

$$|z^k_{\nu_k} - z_{\nu_k}|_C \leq |h|_{W^{1,\infty}} (1 + L_1 N_0 N_1) |\sigma_k - \sigma|,$$

and therefore the definition of A_k implies $|z^k_{\nu_k} - z_{\nu_k}|_C \leq A_k$. Hence Lemma 2.1 is applicable for (4.35), and it gives

$$|z^k(t) - z(t)| \leq |z_t^k - z_t|_C \leq A_k N_1 |h|_{W^{1,\infty}}, \qquad t \in [\nu_k, \alpha],$$

where N_1 is defined by (3.7). Therefore we get for $t \in [\nu_k, \alpha]$

$$|z(t,\sigma_k,\varphi^k,\cdot) - z(t,\sigma,\varphi,\cdot)|_{\mathcal{L}(W^{1,\infty},\mathbb{R}^n)} \le |z_t(\cdot,\sigma_k,\varphi^k,\cdot) - z_t(\cdot,\sigma,\varphi,\cdot)|_{\mathcal{L}(W^{1,\infty},C)} \le A_k N_1 \tag{4.36}$$

for all $k \in \mathbb{N}$. If $\sigma_k \le \sigma$, then (4.36) holds for all $t \in [\sigma,\alpha]$. If $\sigma < \sigma_k$ and $t \in (\sigma,\alpha]$, then (4.36) holds for such t for large enough k, since for large enough k the convergence of σ_k to σ yields $\sigma_k < t$ for large enough k. Finally, for $\sigma < \sigma_k$ and $t = \sigma$ (4.36) holds for all k, since $z^k_\sigma - z_\sigma = h - h = 0$. Consequently, (4.36) holds for all $t \in [\sigma,\alpha]$ and for large enough k.

To show that $A_k \to 0$ as $k \to \infty$ it is enough to argue that the last term in the definition of A_k tends to 0 as $k \to \infty$. If $\sigma_k \le \sigma$, then $\nu_k = \sigma$, and so

$$\int_{\nu_k}^{\alpha} |\dot{x}(u^k(s)) - \dot{x}(u(s))|\, ds = \int_{\sigma}^{\alpha} |\dot{x}(u^k(s)) - \dot{x}(u(s))|\, ds \to 0, \qquad k \to \infty$$

by Lemma 2.2. If $\sigma_k > \sigma$, then $\nu_k = \sigma_k$, and for every $\varepsilon > 0$ there exists k_0 such that $|\sigma_k - \sigma| \le \varepsilon$ for all $k \ge k_0$. But then for $k \ge k_0$ (3.3) yields

$$\begin{aligned}
&\int_{\nu_k}^{\alpha} |\dot{x}(u^k(s)) - \dot{x}(u(s))|\, ds \\
&\qquad \le \int_{\sigma_k}^{\sigma+\varepsilon} |\dot{x}(u^k(s)) - \dot{x}(u(s))|\, ds + \int_{\sigma+\varepsilon}^{\alpha} |\dot{x}(u^k(s)) - \dot{x}(u(s))|\, ds \\
&\qquad \le 2\varepsilon N + \int_{\sigma+\varepsilon}^{\alpha} |\dot{x}(u^k(s)) - \dot{x}(u(s))|\, ds.
\end{aligned}$$

Lemma 2.1 yields $\int_{\sigma+\varepsilon}^{\alpha} |\dot{x}(u^k(s)) - \dot{x}(u(s))|\, ds \to 0$ as $k \to \infty$, therefore

$$\limsup_{k\to\infty} \int_{\nu_k}^{\alpha} |\dot{x}(u^k(s)) - \dot{x}(u(s))|\, ds \le 2\varepsilon N.$$

Since $\varepsilon > 0$ can be arbitrary small, we get $A_k \to 0$ as $k \to \infty$.

Let $t \in (\sigma,\alpha]$ be fixed, and let $t_k \in [\sigma_k,\alpha]$ be a sequence such that $t_k \to t$ as $k \to \infty$. Since $\sigma_k \to \sigma$, there exists $k_0 \in \mathbb{N}$ such that $\sigma_k < t$ for $k \ge k_0$. Then (4.30) and the Mean Value Theorem yield

$$|z_{t_k}(\cdot,\sigma_k,\varphi^k,\cdot) - z_t(\cdot,\sigma_k,\varphi^k,\cdot)|_{\mathcal{L}(W^{1,\infty},C)} \le N_2|t_k - t|, \qquad k \ge k_0.$$

Combining this relation together with (4.36) and $A_k \to 0$ we get

$$\begin{aligned}
&|z(t_k, \sigma_k, \varphi^k, \cdot) - z(t, \sigma, \varphi, \cdot)|_{\mathcal{L}(W^{1,\infty}, \mathbb{R}^n)} \\
&\quad \leq \; |z_{t_k}(\cdot, \sigma_k, \varphi^k, \cdot) - z_t(\cdot, \sigma, \varphi, \cdot)|_{\mathcal{L}(W^{1,\infty}, C)} \\
&\quad \leq \; |z_{t_k}(\cdot, \sigma_k, \varphi^k, \cdot) - z_t(\cdot, \sigma_k, \varphi^k, \cdot)|_{\mathcal{L}(W^{1,\infty}, C)} + |z_t(\cdot, \sigma_k, \varphi^k, \cdot) - z_t(\cdot, \sigma, \varphi, \cdot)|_{\mathcal{L}(W^{1,\infty}, C)} \\
&\quad \leq \; N_2 |t_k - t| + A_k N_1 \\
&\quad \to \; 0, \qquad \text{as } k \to \infty.
\end{aligned}$$

For the case $t = \sigma$ and $\sigma_k \leq \sigma$, the segment function $z_\sigma(\cdot, \sigma_k, \varphi^k, h)$ is defined, so the previous argument works in this case, as well. For the case $t = \sigma$ and $\sigma_k > \sigma$ let $\varepsilon > 0$ be fixed, and let $k_0 \in \mathbb{N}$ be such that $\sigma_k < \sigma + \varepsilon$ for $k \geq k_0$. Let $t_k \in [\sigma_k, \alpha]$ be such that $t_k \to \sigma$ as $k \to \infty$. Then

$$\begin{aligned}
|z(t_k, \sigma_k, \varphi^k, \cdot) - z(\sigma, \sigma, \varphi, \cdot)|_{\mathcal{L}(W^{1,\infty}, \mathbb{R}^n)} &\leq |z_{t_k}(\cdot, \sigma_k, \varphi^k, \cdot) - z_\sigma(\cdot, \sigma, \varphi, \cdot)|_{\mathcal{L}(W^{1,\infty}, C)} \\
&\leq |z_{t_k}(\cdot, \sigma_k, \varphi^k, \cdot) - z_{\sigma+\varepsilon}(\cdot, \sigma_k, \varphi^k, \cdot)|_{\mathcal{L}(W^{1,\infty}, C)} \\
&\quad + |z_{\sigma+\varepsilon}(\cdot, \sigma_k, \varphi^k, \cdot) - z_{\sigma+\varepsilon}(\cdot, \sigma, \varphi, \cdot)|_{\mathcal{L}(W^{1,\infty}, C)} \\
&\quad + |z_{\sigma+\varepsilon}(\cdot, \sigma, \varphi, \cdot) - z_\sigma(\cdot, \sigma, \varphi, \cdot)|_{\mathcal{L}(W^{1,\infty}, C)} \\
&\leq N_2 |t_k - \sigma - \varepsilon| + A_k N_1 + N_2 \varepsilon.
\end{aligned}$$

In this case we also get that $|z(t_k, \sigma_k, \varphi^k, \cdot) - z(\sigma, \sigma, \varphi, \cdot)|_{\mathcal{L}(W^{1,\infty}, \mathbb{R}^n)} \to 0$ as $k \to \infty$, since ε is arbitrarily close to 0. This concludes the proof. □

Now we are ready to prove the Fréchet-differentiability of the function $x(t, \sigma, \varphi)$ wrt φ. We will denote this derivative by $D_3 x(t, \sigma, \varphi)$. The next theorem shows that not only the map $W^{1,\infty} \ni \varphi \mapsto x(t, \sigma, \varphi) \in \mathbb{R}^n$, but also the map $W^{1,\infty} \ni \varphi \mapsto x_t(\cdot, \sigma, \varphi) \in C$ is differentiable. We denote the derivative of this latter map by $D_3 x_t(\cdot, \sigma, \varphi)$, as well.

Theorem 4.7 *Assume (A1) (i)–(iii), (A2) (i)–(iii). Suppose $(\hat{\sigma}, \hat{\varphi}) \in \Pi$ is such that $x(\cdot, \hat{\sigma}, \hat{\varphi}) \in X(\hat{\sigma}, \alpha)$. Let P_1 and H_1 be the sets defined by Lemma 4.3 and* (4.33)*, respectively. Then the functions*

$$\mathbb{R}^2 \times W^{1,\infty} \supset H_1 \to \mathbb{R}^n, \qquad (t, \sigma, \varphi) \mapsto x(t, \sigma, \varphi)$$

and

$$\mathbb{R}^2 \times W^{1,\infty} \supset H_1 \to C, \qquad (t, \sigma, \varphi) \mapsto x_t(\cdot, \sigma, \varphi),$$

are both continuously differentiable wrt φ, and

$$D_3x(t,\sigma,\varphi)h = z(t,\sigma,\varphi,h), \qquad h \in W^{1,\infty},\ (t,\sigma,\varphi) \in H_1, \tag{4.37}$$

and

$$D_3x_t(\cdot,\sigma,\varphi)h = z_t(\cdot,\sigma,\varphi,h), \qquad h \in W^{1,\infty},\ (t,\sigma,\varphi) \in H_1, \tag{4.38}$$

where $z(t,\sigma,\varphi,h)$ is the solution of the IVP (4.27)-(4.28) *for $(t,\sigma,\varphi) \in H_1$ and $h \in W^{1,\infty}$.*

Proof Let $(\sigma,\varphi) \in P_1$ be fixed, and let $h^k \in W^{1,\infty}$ ($k \in \mathbb{N}$) be a sequence with $|h^k|_{W^{1,\infty}} \to 0$ as $k \to \infty$, and to simplify notation, let $x^k(t) := x(t,\sigma,\varphi + h^k)$, $x(t) := x(t,\sigma,\varphi)$, $u(s) := s - \tau(s,x_s)$, $u^k(s) := s - \tau(s,x^k_s)$ and $z(t) := z(t,\sigma,\varphi,h^k)$. Then

$$\begin{aligned} x^k(t) &= x^k(\sigma) + \int_\sigma^t f(s,x^k_s,x^k(u^k(s)))\,ds, \qquad t \in [\sigma,\alpha], \\ x(t) &= x(\sigma) + \int_\sigma^t f(s,x_s,x(u(s)))\,ds, \qquad t \in [\sigma,\alpha], \end{aligned}$$

and

$$z(t) = z(\sigma) + \int_\sigma^t L(s,x)z_s\,ds, \qquad t \in [\sigma,\alpha].$$

We have $x^k(\sigma) = \varphi(0) + h^k(0)$, $x(\sigma) = \varphi(0)$ and $z(\sigma) = h^k(0)$, therefore

$$x^k(t) - x(t) - z(t) = \int_\sigma^t \Big(f(s,x^k_s,x^k(u^k(s))) - f(s,x_s,x(u(s))) - L(s,x)z_s \Big) ds. \tag{4.39}$$

The definitions of ω_f and $L(s,x)$ defined in (4.3) and (4.25), respectively, and simple manipulations yield for $s \in [\sigma, \alpha]$

$$
\begin{aligned}
& f(s, x_s^k, x^k(u^k(s))) - f(s, x_s, x(u(s))) - L(s,x)z_s \\
&\quad = \; D_2 f(s, x_s, x(u(s)))(x_s^k - x_s) + D_3 f(s, x_s, x(u(s)))(x^k(u^k(s)) - x(u(s))) \\
&\qquad\quad + \omega_f(s, x_s, x(u(s)), x_s^k, x^k(u^k(s))) - L(s,x)z_s \\
&\quad = \; D_2 f(s, x_s, x(u(s)))(x_s^k - x_s - z_s) \\
&\qquad\quad + D_3 f(s, x_s, x(u(s)))\Big(x^k(u^k(s)) - x(u(s)) + \dot{x}(u(s)) D_2\tau(s, x_s) z_s - z(u(s))\Big) \\
&\qquad\quad + \omega_f(s, x_s, x(u(s)), x_s^k, x^k(u^k(s))) \\
&\quad = \; D_2 f(s, x_s, x(u(s)))(x_s^k - x_s - z_s) \\
&\qquad\quad + D_3 f(s, x_s, x(u(s)))\Big(x^k(u^k(s)) - x(u^k(s)) - z(u^k(s))\Big) \\
&\qquad\quad + D_3 f(s, x_s, x(u(s)))\Big(x(u^k(s)) - x(u(s)) - \dot{x}(u(s))(u^k(s) - u(s))\Big) \\
&\qquad\quad + D_3 f(s, x_s, x(u(s)))\dot{x}(u(s))\Big(u^k(s) - u(s) + D_2\tau(s, x_s)(x_s^k - x_s)\Big) \\
&\qquad\quad - D_3 f(s, x_s, x(u(s)))\dot{x}(u(s)) D_2\tau(s, x_s)(x_s^k - x_s - z_s) \\
&\qquad\quad + D_3 f(s, x_s, x(u(s)))(z(u^k(s)) - z(u(s))) \\
&\qquad\quad + \omega_f(s, x_s, x(u(s)), x_s^k, x^k(u^k(s))).
\end{aligned} \tag{4.40}
$$

Using (3.3) and the definitions of L_1 and L_2, we get from (4.39) and (4.40)

$$
\begin{aligned}
|x^k(t) - x(t) - z(t)| \;\le\; & \int_\sigma^t \Big[L_1 \Big(|x_s^k - x_s - z_s|_C + |x^k(u^k(s)) - x(u^k(s)) - z(u^k(s))| \\
& \quad + |x(u^k(s)) - x(u(s)) - \dot{x}(u(s))(u^k(s) - u(s))| \\
& \quad + N|u^k(s) - u(s) + D_2\tau(s, x_s)(x_s^k - x_s)| \\
& \quad + N L_2 |x_s^k - x_s - z_s|_C + |z(u^k(s)) - z(u(s))| \Big) \\
& \quad + |\omega_f(s, x_s, x(u(s)), x_s^k, x^k(u^k(s)))| \Big] ds, \qquad t \in [\sigma, \alpha].
\end{aligned}
$$

Let N_0 and ω_τ be defined by (3.7) and (4.4), respectively, then

$$
|x^k(t) - x(t) - z(t)| \le a_k + b_k + c_k + d_k + \int_\sigma^t L_1 N_0 |x_s^k - x_s - z_s|_C \, ds, \qquad t \in [\sigma, \alpha], \tag{4.41}
$$

where

$$a_k := \int_\sigma^\alpha |\omega_f(s, x_s, x(u(s)), x_s^k, x^k(u^k(s)))|\, ds, \tag{4.42}$$

$$b_k := L_1 N \int_\sigma^\alpha |\omega_\tau(s, x_s, s, x_s^k)|\, ds, \tag{4.43}$$

$$c_k := L_1 \int_\sigma^\alpha |x(u^k(s)) - x(u(s)) - \dot{x}(u(s))(u^k(s) - u(s))|\, ds, \tag{4.44}$$

and

$$d_k := L_1 \int_\sigma^\alpha |z(u^k(s)) - z(u(s))|\, ds. \tag{4.45}$$

Since $|x_\sigma^k - x_\sigma - z_\sigma|_C = 0$, Lemma 2.1 is applicable for (4.41), and it yields

$$|x^k(t) - x(t) - z(t)| \le |x_t^k - x_t - z_t|_C \le (a_k + b_k + c_k + d_k)N_1, \qquad t \in [\sigma, \alpha],$$

where N_1 is defined in (3.7). But then

$$\frac{|x^k(t) - x(t) - z(t)|}{|h^k|_{W^{1,\infty}}} \le \frac{|x_t^k - x_t - z_t|_C}{|h^k|_{W^{1,\infty}}} \le \frac{a_k + b_k + c_k + d_k}{|h^k|_{W^{1,\infty}}} N_1, \qquad t \in [\sigma, \alpha],$$

which proves both (4.37) and (4.38), since Lemmas 4.1 and 4.2 show that $\frac{a_k+b_k+c_k}{|h^k|_{W^{1,\infty}}} \to 0$ as $k \to \infty$, and (A2) (ii), (3.4) and (4.30) yield

$$\frac{d_k}{|h^k|_{W^{1,\infty}}} \le L_1 N_2 \int_\sigma^\alpha |u^k(s) - u(s)|\, ds \le \alpha L_1 N_2 L_2 L |h^k|_{W^{1,\infty}} \to 0, \quad \text{as } k \to \infty. \tag{4.46}$$

The continuity of $D_3 x(t, \sigma, \varphi)$ and $D_3 x_t(\cdot, \sigma, \varphi)$ follows from Lemma 4.6. $\Box$

5 Differentiability wrt the initial time

The next theorem shows the Fréchet-differentiability of $x(t, \sigma, \varphi)$ and $x_t(\cdot, \sigma, \varphi)$ with respect to σ at those parameter values which belong to the set $\mathcal{P}$ defined by (3.11). We will denote these derivatives by $D_2 x(t, \sigma, \varphi)$ and $D_2 x_t(\cdot, \sigma, \varphi)$, respectively. We recall that if $(\sigma, \varphi) \in \mathcal{P}$, then $x(\cdot, \sigma, \varphi)$ is continuously differentiable on $[\sigma - r, \alpha]$.

Theorem 5.1 *Assume (A1) (i)–(iii), (A2) (i)–(iii). Suppose* $(\hat{\sigma}, \hat{\varphi}) \in \Pi$ *is such that* $x(\cdot, \hat{\sigma}, \hat{\varphi}) \in X(\hat{\sigma}, \alpha)$*. Let* P_1 *and* H_1 *be the sets defined by Lemma 4.3 and* (4.33)*, respectively, let* $\mathcal{P}$ *be defined by* (3.11)*, and let*

$$\mathcal{H}_1 := \{(t, \sigma, \varphi) \colon\ (\sigma, \varphi) \in \mathcal{P} \cap P_1,\ t \in [\sigma, \alpha]\}.$$

Then the functions

$$\mathbb{R}^2 \times W^{1,\infty} \supset H_1 \to \mathbb{R}^n, \qquad (t,\sigma,\varphi) \mapsto x(t,\sigma,\varphi)$$

and

$$\mathbb{R}^2 \times W^{1,\infty} \supset H_1 \to C, \qquad (t,\sigma,\varphi) \mapsto x_t(\cdot,\sigma,\varphi)$$

are both differentiable wrt σ at every point $(t,\sigma,\varphi) \in \mathcal{H}_1$. Moreover,

$$D_2 x(t,\sigma,\varphi) = z(t,\sigma,\varphi,-\dot{\varphi}), \qquad (t,\sigma,\varphi) \in \mathcal{H}_1,$$

and

$$D_2 x_t(\cdot,\sigma,\varphi) = z_t(\cdot,\sigma,\varphi,-\dot{\varphi}), \qquad (t,\sigma,\varphi) \in \mathcal{H}_1,$$

where $z(t,\sigma,\varphi,-\dot{\varphi})$ is the solution of the IVP (4.27)-(4.28).

Proof Let $(\sigma,\varphi) \in P_1 \cap \mathcal{P}$, and let $h_k \in \mathbb{R}$ $(k \in \mathbb{N})$ be a sequence satisfying $h_k \to 0$ as $k \to \infty$ and $(\sigma + h_k, \varphi) \in P_1$ for $k \in \mathbb{N}$. To simplify notation, let $x^k(t) := x(t,\sigma+h_k,\varphi)$, $x(t) := x(t,\sigma,\varphi)$, $u(s) := s - \tau(s,x_s)$, $u^k(s) := s - \tau(s,x^k_s)$, and $z(t) := z(t,\sigma,\varphi,-\dot{\varphi})$. Then

$$\begin{aligned}
x^k(t) &= x^k(\sigma+h_k) + \int_{\sigma+h_k}^t f(s,x^k_s,x^k(u^k(s)))\,ds, \qquad t \in [\sigma+h_k,\alpha],\\
x(t) &= x(\sigma) + \int_\sigma^t f(s,x_s,x(u(s)))\,ds, \qquad t \in [\sigma,\alpha],\\
z(t) &= z(\sigma) + \int_\sigma^t L(s,x)z_s\,ds, \qquad t \in [\sigma,\alpha].
\end{aligned}$$

We distinguish two cases.

(i) We assume first that $h_k < 0$ for all $k \in \mathbb{N}$. Then

$$x^k(t) = x^k(\sigma) + \int_\sigma^t f(s,x^k_s,x^k(u^k(s)))\,ds, \qquad t \in [\sigma,\alpha],$$

and hence for $t \in [\sigma,\alpha]$

$$\begin{aligned}
x^k(t) - x(t) - z(t)h_k &= x^k(\sigma) - x(\sigma) - z(\sigma)h_k\\
&\quad + \int_\sigma^t \Big(f(s,x^k_s,x^k(u^k(s))) - f(s,x_s,x(u(s))) - L(s,x)z_s h_k \Big) ds.
\end{aligned}$$

Then, using (4.40) with z_s is replaced by $z_s h_k$, similarly to (4.41) we get for $t \in [\sigma,\alpha]$

$$|x^k(t) - x(t) - z(t)h_k| \le a_k + b_k + c_k + d_k|h_k| + e_k + \int_\sigma^t L_1 N_0 |x^k_s - x_s - z_s h_k|_C\,ds. \quad (5.1)$$

Here a_k, b_k and c_k, d_k are defined by formulas (4.42)–(4.45), respectively, and

$$e_k := |x^k_\sigma - x_\sigma - z_\sigma h_k|_C.$$

By the definition of e_k, Lemma 2.1 is applicable, so it implies

$$|x^k(t)-x(t)-z(t)h_k| \le |x^k_t - x_t - z_t h_k|_C \le (a_k+b_k+c_k+d_k|h_k|+e_k)N_1, \quad t \in [\sigma, \alpha], \tag{5.2}$$

where N_1 is defined by (3.7). Lemmas 4.1 and 4.2 yield that

$$\lim_{k\to\infty} \frac{a_k + b_k + c_k}{|h_k|} = 0. \tag{5.3}$$

Note that now (4.30) cannot be used to estimate d_k, since z restricted to $[\sigma - r, \sigma]$ is not in $W^{1,\infty}$, but Lemma 2.2 yields immediately that $d_k \to 0$ as $k \to \infty$.

Finally, we estimate e_k. Suppose first that $\theta \in [-r, h_k]$. Then

$$|x^k(\sigma+\theta) - x(\sigma+\theta) - z(\sigma+\theta)h_k| = |\varphi(\theta - h_k) - \varphi(\theta) - \dot\varphi(\theta)(-h_k)| \le \Omega_\varphi(|h_k|)|h_k|, \tag{5.4}$$

where

$$\Omega_\varphi(\varepsilon) := \max\{|\dot\varphi(\theta) - \dot\varphi(\bar\theta)| \colon\ \theta, \bar\theta \in [-r, 0],\ |\theta - \bar\theta| \le \varepsilon\}.$$

Since $\varphi \in C^1$, $\Omega_\varphi(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now suppose $\theta \in [h_k, 0]$. Then using that $(\sigma, \varphi) \in \mathcal{P}$ we get

$$\begin{aligned}
&|x^k(\sigma+\theta) - x(\sigma+\theta) - z(\sigma+\theta)h_k| \\
&\quad = \left|\varphi(0) + \int_{\sigma+h_k}^{\sigma+\theta} f(s, x^k_s, x^k(u^k(s)))\,ds - \varphi(\theta) + \dot\varphi(\theta)h_k\right| \\
&\quad = \Big|\varphi(0) - \varphi(\theta) + \dot\varphi(0-)\theta + (\dot\varphi(\theta) - \dot\varphi(0-))h_k \\
&\qquad\quad + \int_{\sigma+h_k}^{\sigma+\theta} f(s, x^k_s, x^k(u^k(s)))\,ds - \dot\varphi(0-)(\theta - h_k)\Big| \\
&\quad \le |\varphi(0) - \varphi(\theta) + \dot\varphi(0-)\theta| + |\dot\varphi(\theta) - \dot\varphi(0-)||h_k| \\
&\qquad\quad + \left|\int_{\sigma+h_k}^{\sigma+\theta} \Big(f(s, x^k_s, x^k(u^k(s))) - f(s, \varphi, \varphi(-\tau(\sigma,\varphi)))\Big)ds\right| \\
&\qquad\quad + \left|\int_{\sigma+h_k}^{\sigma+\theta} \Big(f(s, \varphi, \varphi(-\tau(\sigma,\varphi))) - f(\sigma, \varphi, \varphi(-\tau(\sigma,\varphi)))\Big)ds\right| \\
&\quad \le \Omega_\varphi(|\theta|)|\theta| + \Omega_\varphi(|\theta|)|h_k| \\
&\qquad\quad + \int_{\sigma+h_k}^{\sigma+\theta} L_1\Big(|x_s - x_\sigma|_C + |x^k(u^k(s)) - x(u(\sigma))|\Big)ds \\
&\qquad\quad + |h_k| \max_{\sigma+h_k \le s \le \sigma} \Big|f(s, \varphi, \varphi(-\tau(\sigma,\varphi))) - f(\sigma, \varphi, \varphi(-\tau(\sigma,\varphi)))\Big|.
\end{aligned} \tag{5.5}$$

Applying (3.3), (3.4) and (A2) (ii) we obtain

$$\begin{aligned}
&|x^k(u^k(s)) - x(u(\sigma))| \\
&\quad\le |x^k(u^k(s)) - x^k(u(\sigma))| + |x^k(u(\sigma)) - x(u(\sigma))| \\
&\quad\le N|s - \tau(s, x_s^k) - (\sigma - \tau(\sigma, x_\sigma))| + L|h_k| \\
&\quad\le N\Big(|s-\sigma| + |\tau(s, x_s^k) - \tau(s, x_s)| + |\tau(s, x_s) - \tau(s, x_\sigma)| + |\tau(s, x_\sigma) - \tau(\sigma, x_\sigma)|\Big) \\
&\qquad\quad + L|h_k| \\
&\quad\le N\Big(|h_k| + L_2|x_s^k - x_s|_C + L_2|x_s - x_\sigma|_C + |\tau(s,\varphi) - \tau(\sigma,\varphi)|\Big) + L|h_k| \\
&\quad\le N\Big(|h_k| + L_2L|h_k| + L_2N|h_k| + \max_{\sigma+h_k\le s\le\sigma}|\tau(s,\varphi) - \tau(\sigma,\varphi)|\Big) + L|h_k|,
\end{aligned}$$

for $s \in [\sigma + h_k, \sigma + \theta]$. Therefore (5.5) yields

$$\begin{aligned}
&|x^k(\sigma+\theta) - x(\sigma+\theta) - z(\sigma+\theta)h_k| \\
&\quad\le |h_k|\Big[2\Omega_\varphi(|h_k|) + |h_k|(2L_1N + L_1NL_2L + L_1N^2L_2 + L_1L) \\
&\qquad + L_1N \max_{\sigma+h_k\le s\le\sigma}|\tau(s,\varphi) - \tau(\sigma,\varphi)| \\
&\qquad + L_1 \max_{\sigma+h_k\le s\le\sigma}\Big|f(s,\varphi,\varphi(-\tau(\sigma,\varphi))) - f(\sigma,\varphi,\varphi(-\tau(\sigma,\varphi)))\Big|\Big],
\end{aligned}$$

which, together with (5.4), yields

$$\lim_{k\to\infty}\frac{e_k}{|h_k|} = 0.$$

Therefore (5.2) and (5.3) imply that $x(t,\sigma,\varphi)$ and $x_t(\cdot,\sigma,\varphi)$ are differentiable wrt σ from the left at (t,σ,φ) for all $t \in [\sigma,\alpha]$.

(ii) Now we assume that $h_k > 0$ for all $k \in \mathbb{N}$. Then for $t \in [\sigma + h_k, \alpha]$ we have

$$x(t) = x(\sigma + h_k) + \int_{\sigma+h_k}^t f(s, x_s, x(u(s)))\,ds$$

and

$$z(t) = z(\sigma + h_k) + \int_{\sigma+h_k}^t L(s,x)z_s\,ds,$$

therefore

$$\begin{aligned}
x^k(t) - x(t) - z(t)h_k &= x^k(\sigma+h_k) - x(\sigma+h_k) - z(\sigma+h_k)h_k \\
&\quad \int_{\sigma+h_k}^t \Big(f(s, x_s^k, x^k(u^k(s))) - f(s, x_s, x(u(s))) - L(s,x)z_sh_k\Big)ds.
\end{aligned} \tag{5.6}$$

Then similarly to (5.2) for $t \in [\sigma + h_k, \alpha]$ we get

$$|x^k(t) - x(t) - z(t)h_k| \le |x^k_t - x_t - z_t h_k|_C \le (\bar a_k + \bar b_k + \bar c_k + \bar d_k h_k + \bar e_k) N_1, \quad t \in [\sigma + h_k, \alpha], \tag{5.7}$$

where $\bar a_k, \bar b_k$ and $\bar c_k, \bar d_k$ are defined by formulas analogous to (4.42)–(4.45), respectively, but where σ in the lower limit of the integrals is replaced by $\sigma + h_k$, and

$$\bar e_k := |x^k_{\sigma+h_k} - x_{\sigma+h_k} - z_{\sigma+h_k} h_k|_C.$$

It follows from Lemmas 4.1 and 4.2 and $|u^k(s) - u(s)| \le L_2 L h_k$ for $s \in [\sigma + h_k, \alpha]$ that

$$\lim_{k\to\infty} \frac{\bar a_k + \bar b_k + \bar c_k}{h_k} = 0. \tag{5.8}$$

Let $\varepsilon > 0$ be fixed, and let $k_0 \in \mathbb{N}$ be such that $0 < h_k < \varepsilon$ for $k \ge k_0$. The definition of $\bar d_k$ and (4.29) yield

$$\begin{aligned} \bar d_k &= L_1 \int_{\sigma+h_k}^{\sigma+\varepsilon} |z(u^k(s)) - z(u(s))|\, ds + L_1 \int_{\sigma+\varepsilon}^{\alpha} |z(u^k(s)) - z(u(s))|\, ds \\ &\le L_1 \varepsilon 2 N_1 |\dot\varphi|_C + L_1 \int_{\sigma+\varepsilon}^{\alpha} |z(u^k(s)) - z(u(s))|\, ds. \end{aligned}$$

Therefore Lemma 2.2 gives $\bar d_k \to 0$ as $k \to \infty$, since ε can be arbitrary close to 0.

Now, we estimate $\bar e_k$. For $\theta \in [-r, -h_k]$ we have

$$\begin{aligned} &|x^k(\sigma + h_k + \theta) - x(\sigma + h_k + \theta) - z(\sigma + h_k + \theta)h_k| \\ &\quad = |\varphi(\theta) - \varphi(h_k + \theta) + \dot\varphi(h_k + \theta)h_k| \\ &\quad \le \Omega_\varphi(h_k)h_k. \end{aligned}$$

Let $\theta \in [-h_k, 0]$. Then

$$\begin{aligned} &|x^k(\sigma + h_k + \theta) - x(\sigma + h_k + \theta) - z(\sigma + h_k + \theta)h_k| \\ &= \Big|\varphi(\theta) - \varphi(0) - \int_\sigma^{\sigma+h_k+\theta} f(s, x_s, x(u(s)))\, ds + \dot\varphi(0-)h_k - \int_\sigma^{\sigma+h_k+\theta} L(s,x) z_s h_k\, ds\Big| \\ &\le |\varphi(\theta) - \varphi(0) - \dot\varphi(0-)\theta| + \Big| - \int_\sigma^{\sigma+h_k+\theta} f(s, x_s, x(u(s)))\, ds + \dot\varphi(0-)(h_k + \theta)\Big| \\ &\qquad + \Big|\int_\sigma^{\sigma+h_k+\theta} L(s,x) z_s h_k\, ds\Big| \\ &\le \Omega_\varphi(|\theta|)|\theta| + \Big|\int_\sigma^{\sigma+h_k+\theta} \Big(f(s, x_s, x(u(s))) - f(\sigma, x_\sigma, x(u(\sigma)))\Big)\, ds\Big| + (h_k)^2 L_1 N_0 N_1 |\dot\varphi|_C \\ &\le \Omega_\varphi(h_k)h_k + h_k \max_{\sigma \le s \le \sigma + h_k} \Big|f(s, x_s, x(u(s))) - f(\sigma, x_\sigma, x(u(\sigma)))\Big| + |h_k|^2 L_1 N_0 N_1 |\dot\varphi|_C. \end{aligned}$$

Combining the above estimates we get

$$\lim_{k\to\infty} \frac{\bar{e}_k}{h_k} = 0.$$

This relation, (5.7) and (5.8) yield the statement of the theorem. $\square$

In general, the partial derivatives $D_2 f$ and $D_2\tau$ can be represented by Riemann-Stieltjes integrals. Now we consider a special case of (3.1) where we assume a specific form of $D_2 f(s, x_s, x(s-\tau(s, x_s)))$ and $D_2\tau(s, x_s)$. Let $\alpha^*_\sigma := \min\{\sigma + r, \alpha\}$. We suppose

(A1) (iv) for every $(\sigma, \varphi) \in P$ there exist continuous functions $A^1, \dots, A^m : [\sigma, \alpha] \to \mathbb{R}^{n\times n}$ and $A\colon [\sigma, \alpha] \times [-r, 0] \to \mathbb{R}^{n\times n}$, and $\lambda^1, \dots, \lambda^m \in W^{1,\infty}([\sigma, \alpha], [0, r])$ such that

(a) for $x = x(\cdot, \sigma, \varphi)$ and for all $\psi \in C$, $s \in [\sigma, \alpha]$

$$D_2 f(s, x_s, x(s-\tau(s, x_s)))\psi = \sum_{i=1}^{m} A^i(s)\psi(-\lambda^i(s)) + \int_{-r}^{0} A(s, \theta)\psi(\theta)\, d\theta,$$

(b) $\operatorname{ess\,inf}\{\frac{d}{dt}(t - \lambda^i(t))\colon t \in [\sigma, \alpha^*_\sigma]\} > 0$ for $i = 1, \dots, m$, and

(c) $\sum_{i=1}^{m} |A^i(s)| + \int_{-r}^{0} |A(s, \theta)|\, d\theta \le L_1$ for $s \in [\sigma, \alpha]$;

(A2) (iv) for every $(\sigma, \varphi) \in P$ there exist continuous functions $b^1, \dots, b^\ell : [\sigma, \alpha] \to \mathbb{R}^{1\times n}$, $b\colon [\sigma, \alpha] \times [-r, 0] \to \mathbb{R}^{1\times n}$, and $\xi^1, \dots, \xi^\ell \in W^{1,\infty}([\sigma, \alpha], [0, r])$ such that

(a) for $x = x(\cdot, \sigma, \varphi)$ and for all $\psi \in C$, $s \in [\sigma, \alpha]$

$$D_2\tau(s, x_s)\psi = \sum_{j=1}^{\ell} b^j(s)\psi(-\xi^j(s)) + \int_{-r}^{0} b(s, \theta)\psi(s)\, ds,$$

(b) $\operatorname{ess\,inf}\{\frac{d}{dt}(t - \xi^j(t))\colon t \in [\sigma, \alpha^*_\sigma]\} > 0$ for $j = 1, \dots, \ell$, and

(c) $\sum_{j=1}^{\ell} |b^j(s)| + \int_{-r}^{0} |b(s, \theta)|\, d\theta \le L_2$ for $s \in [\sigma, \alpha]$.

Our additional assumptions can be naturally satisfied, e.g., for equations of the form

$$\begin{aligned} \dot{x}(t) &= \bar{f}\Big(t, x(t-\lambda^1(t)), \dots, x(t-\lambda^m(t)), \int_{-r}^{0} A(t, \theta)x(s+\theta)\, ds, \\ &\qquad x\Big(t - \bar{\tau}\Big[t, x(t-\xi^1(t)), \dots, x(t-\xi^\ell(t)), \int_{-r}^{0} b(t, \theta)x(s+\theta)\, ds\Big]\Big)\Big). \end{aligned}$$

Assuming (A1) (iv) and (A2) (iv), the operator $L(t,x)$ defined by (4.25) has the form

$$\begin{aligned} L(t,x)\psi &= \sum_{i=1}^{m} A^i(t)\psi(-\lambda^i(t)) + \int_{-r}^{0} A(t,\theta)\psi(\theta)\,d\theta + D_3 f(t,x_t,x(t-\tau(t,x_t))) \\ &\quad \times\Big(-\dot{x}(t-\tau(t,x_t))\Big(\sum_{j=1}^{\ell} b^j(t)\psi(-\xi^j(t)) + \int_{-r}^{0} b(t,\theta)\psi(\theta)\,d\theta\Big) + \psi(-\tau(t,x_t))\Big). \end{aligned}$$

Our assumptions and Lemma 2.2 yield that $L(t,x)z_t$ is well-defined for a.e. $t \in [\sigma,\alpha]$ for a function $z\colon [\sigma-r,\alpha] \to \mathbb{R}^n$, where z restricted to $[\sigma-r,\sigma]$ is in $L^\infty([\sigma-r,\sigma],\mathbb{R}^n)$, and z is continuous on $[\sigma,\alpha]$. We also have in this case that $|L(t,x)z_s| \le L_1 N_0 |z_s|_{L^\infty}$ for a.e. $t \in [\sigma,\alpha]$. We can extend the IVP (4.27)-(4.28) to this case by considering

$$\dot{z}(t) = L(t,x)z_s, \qquad \text{a.e. } t \in [\sigma,\alpha] \tag{5.9}$$

$$z(\sigma) = v, \tag{5.10}$$

$$z(t) = h(t-\sigma), \qquad \text{a.e. } t \in [\sigma-r,\sigma), \tag{5.11}$$

where $v \in \mathbb{R}^n$ and $h \in L^\infty$. By a solution of (5.9)-(5.11) we mean a function $z\colon [\sigma-r,\alpha] \to \mathbb{R}^n$, which is absolutely continuous on $[\sigma,\alpha]$, and satisfies (5.9)-(5.11). It is easy to show that (5.9)-(5.11), or equivalently, the integral equation

$$\begin{aligned} z(t) &= v + \int_{\sigma}^{t} L(s,x)z_s\,ds, \qquad t \in [\sigma,\alpha], \\ z(t) &= h(t-\sigma), \qquad \text{a.e. } t \in [\sigma-r,\sigma]. \end{aligned}$$

has a unique solution $z(t) = z(t,\sigma,\varphi,v,h)$ on $[\sigma-r,\alpha]$ for all $(v,h) \in \mathbb{R}^n \times L^\infty$ and $(\sigma,\varphi) \in P_1$. On $\mathbb{R}^n \times L^\infty$ we use the norm $|(v,h)|_{\mathbb{R}^n\times L^\infty} := |v| + |h|_{L^\infty}$.

Lemma 5.2 *Assume (A1) (i)–(iv), (A2) (i)–(iv). Let P_1 and H_1 be the sets defined by Lemma 4.3 and* (4.33)*, respectively. Then there exists $N_3 \ge 1$ such that for all $(\sigma,\varphi) \in P_1$ and $(v,h) \in \mathbb{R}^n \times L^\infty$ the corresponding solution $z(t,\sigma,\varphi,v,h)$ of the IVP* (5.9)-(5.11) *satisfies*

$$|z(t,\sigma,\varphi,v,h)| \le N_3(|v| + |h|_{L^\infty}), \quad t \in [\sigma-r,\alpha]. \tag{5.12}$$

Moreover, the function

$$\mathbb{R}^2 \times W^{1,\infty} \supset H_1 \to \mathbb{R}^n, \qquad (t,\sigma,\varphi) \mapsto z(t,\sigma,\varphi,v,h)$$

is continuous for all fixed $(v,h) \in \mathbb{R}^n \times L^\infty$.

Proof Let $z(t) := z(t,\sigma,\varphi,v,h)$ and $\alpha_\sigma^* := \min\{\sigma+r,\alpha\}$. We introduce the functions

$$\bar{h}(s) := \begin{cases} h(s-\sigma), & s\in[\sigma-r,\sigma), \\ 0, & s\in[\sigma,\alpha] \end{cases}$$

and $y(t) := z(t)-\bar{h}(t)$ for $t\in[-r,\alpha]$. Then $y(t)=0$ for a.e. $t\in[\sigma-r,\sigma)$, $y(t)=z(t)$ for $t\in[\sigma,\alpha]$, and it satisfies

$$y(t) = v + \int_\sigma^t L(s,x)\bar{h}_s\,ds + \int_\sigma^t L(s,x)y_s\,ds, \quad t\in[\sigma,\alpha]. \tag{5.13}$$

Define $w(t) := \max\{|y(s)|\colon \sigma\le s\le t\}$. We have that $\bar{h}_s=0$ for $s\ge\sigma+r$, and hence

$$\int_\sigma^t |L(s,x)\bar{h}_s|\,ds \le \int_\sigma^{\alpha_\sigma^*} |L(s,x)\bar{h}_s|\,ds \le rL_1N_0|h|_{L^\infty}, \qquad t\in[\sigma,\alpha],$$

and assumption (A1) (iv) and (A2) (iv) imply

$$\begin{aligned} |L(s,x)y_s| &\le \Big[\sum_{i=1}^m |A^i(s)| + \int_{-r}^0 |A(s,\theta)|ds + |D_3f(s,x_s,x(s-\tau(s,x_s)))| \\ &\quad \times\Big(|\dot{x}(s-\tau(s,x_s))|\Big\{\sum_{j=1}^\ell |b^j(s)| + \int_{-r}^0 |b(s,\theta)|\,d\theta\Big\}+1\Big)\Big]w(s) \\ &\le (L_1+L_1(NL_2+1))w(s) \\ &= L_1N_0w(s), \qquad s\in[\sigma,\alpha], \end{aligned}$$

hence (5.13) yields

$$w(t) \le |v| + rL_1N_0|h|_{L^\infty} + \int_\sigma^t L_1N_0w(s)\,ds, \qquad t\in[\sigma,\alpha].$$

Therefore, Gronwall's inequality gives

$$|z(t)| = |y(t)| \le w(t) \le (|v|+rL_1N_0|h|_{L^\infty})N_1, \qquad t\in[\sigma,\alpha],$$

where N_1 is defined by (3.7). Therefore, $N_3 := \max\{1, rL_1N_0\}N_1$ satisfies (5.12), since $N_1\ge 1$.

To show the continuity, let $(\sigma_k,\varphi^k)\in P_1$ be a sequence such that $|\sigma_k-\sigma|+|\varphi^k-\varphi|_{W^{1,\infty}}\to 0$ as $k\to\infty$, and let $x^k(t) := x(t,\sigma_k,\varphi^k)$, $x(t) := x(t,\sigma,\varphi)$, $u^k(s) := s-\tau(s,x_s^k)$, $u(s) := s-\tau(s,x_s)$, and for a fixed $(v,h)\in\mathbb{R}^n\times L^\infty$, let $z^k(t) := z(t,\sigma_k,\varphi^k,v,h)$ and $z(t) := z(t,\sigma,\varphi,v,h)$, and $y(t) := z(t)-\bar{h}(t)$. Then

$$z^k(t) = v + \int_{\sigma_k}^t L(s,x^k)z_s^k\,ds, \qquad t\in[\sigma_k,\alpha],$$

and

$$z(t) = v + \int_\sigma^t L(s,x)z_s\,ds, \qquad t \in [\sigma,\alpha].$$

Therefore similarly to (4.34) we have

$$\begin{aligned}
|z^k(t) - z(t)| \;\leq\;& |\sigma_k - \sigma| L_1 N_0 (|v| + |h|_{L^\infty}) \\
&+ \Big| \int_{\nu_k}^t L(s,x)(z^k_s - z_s)\,ds \Big| + \int_{\nu_k}^t \Big| (L(s,x^k) - L(s,x)) z^k_s \Big|\,ds, \quad t \in [\nu_k, \alpha],
\end{aligned}$$

where $\nu_k := \max\{\sigma, \sigma_k\}$. An obvious modification of Lemma 4.5 and

$$\begin{aligned}
&|z(u^k(s)) - z(u(s))| \\
&\quad\leq |\bar h(u^k(s)) - \bar h(u(s))| + |y(u^k(s)) - y(u(s))| \\
&\quad\leq |\bar h(u^k(s)) - \bar h(u(s))| + \operatorname*{ess\,sup}_{t\in[\sigma,\alpha]} |\dot y(t)| |u^k(s) - u(s)| \\
&\quad\leq |\bar h(u^k(s)) - \bar h(u(s))| \\
&\qquad + L_1 N_0 N_3 (|v| + |h|_{L^\infty}) L_2 L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}), \quad s \in [\sigma, \alpha]
\end{aligned}$$

yield

$$|z^k(t) - z(t)| \leq B_k(|v| + |h|_{L^\infty}) + C^h_k + \Big| \int_{\nu_k}^t L(s,x)(z^k_s - z_s)\,ds \Big|, \qquad t \in [\nu_k, \alpha], \quad (5.14)$$

where

$$\begin{aligned}
B_k :=\;& |\sigma_k - \sigma| L_1 N_0 + \alpha N_0 \Omega_f \Big(N_0 L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}) \Big) N_3 \\
&+ \alpha L_1 N \Omega_\tau \Big(L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}) \Big) N_3 + \alpha L_1 L_2 L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}) N_3 \\
&+ L_1 L_2 N_3 \int_{\nu_k}^\alpha \Big| \dot x(u^k(s)) - \dot x(u(s)) \Big| ds + \alpha L_1^2 N_0 N_3 L_2 L(|\sigma_k - \sigma| + |\varphi^k - \varphi|_{W^{1,\infty}}),
\end{aligned}$$

and

$$C^h_k := L_1 \int_{\nu_k}^\alpha |\bar h(u^k(s)) - \bar h(u(s))|\,ds.$$

Let $\varepsilon > 0$ be fixed. Then for large enough k we have $\nu_k < \sigma + \varepsilon$, so for such k

$$\begin{aligned}
C^h_k \;=\;& L_1 \Big(\int_{\nu_k}^{\sigma+\varepsilon} |\bar h(u^k(s)) - \bar h(u(s))|\,ds + \int_{\sigma+\varepsilon}^\alpha |\bar h(u^k(s)) - \bar h(u(s))|\,ds \Big) \\
\leq\;& L_1 \Big(\varepsilon 2 |h|_{L^\infty} + \int_{\sigma+\varepsilon}^\alpha |\bar h(u^k(s)) - \bar h(u(s))|\,ds \Big).
\end{aligned}$$

Therefore Lemma 2.2 yields that $C_k^h \to 0$ as $k \to \infty$, since ε can be arbitrarily close to 0. A similar argument shows that $B_k \to 0$ as $k \to \infty$.

Now we consider the last term of (5.14). We have

$$\begin{aligned}
&\Big|\int_{\nu_k}^t L(s,x)(z_s^k - z_s)\,ds\Big| \\
&\quad \le \sum_{i=1}^m \int_{\nu_k}^t |A^i(s)||z^k(s-\lambda^i(s)) - z(s-\lambda^i(s))|\,ds \\
&\qquad + \int_{\nu_k}^t \int_{-r}^0 |A(s,\theta)||z^k(s+\theta) - z(s+\theta)|\,d\theta\,ds + L_1 \int_{\nu_k}^t |z^k(u(s)) - z(u(s))|\,ds \\
&\qquad + L_1 N \sum_{j=1}^\ell \int_{\nu_k}^t |b^j(s)||z^k(s-\xi^j(s)) - z(s-\xi^j(s))|\,ds \\
&\qquad + \int_{\nu_k}^t \int_{-r}^0 |b(s,\theta)||z^k(s+\theta) - z(s+\theta)|\,d\theta\,ds, \qquad t \in [\nu_k, \alpha]. \qquad (5.15)
\end{aligned}$$

Using the monotonicity assumptions (A1) (iv) (b), (A2) (iv) (b) and Lemma 4.3, we define the constants $\eta, \gamma_i, \mu_j \in [\sigma,\alpha]$ for $i = 1,\dots,m$ and $j = 1,\dots,\ell$ as follows. If $u(\alpha) \le \sigma$, then let $\eta := \alpha$, otherwise let η be the unique solution of $u(\eta) = \sigma$. If $\alpha - \lambda^i(\alpha) \le \sigma$, then let $\gamma_i := \alpha$, otherwise let γ_i be the unique solution of $\gamma_i - \lambda^i(\gamma_i) = \sigma$. If $\alpha - \xi^j(\alpha) \le \sigma$, then let $\mu_j := \alpha$, otherwise let μ_j be the unique solution of $\mu_j - \xi^j(\mu_j) = \sigma$. Then we have

$$u(s) \le \sigma \quad \text{for } s \in [\sigma,\eta], \qquad u(s) > \sigma \quad \text{for } s \in (\eta,\alpha], \qquad (5.16)$$

$$s - \lambda^i(s) \le \sigma \quad \text{for } s \in [\sigma,\gamma_i], \qquad s - \lambda^i(s) > \sigma \quad \text{for } s \in (\gamma_i,\alpha] \quad \text{for } i = 1,\dots,m, \quad (5.17)$$

and

$$s - \xi^j(s) \le \sigma \quad \text{for } s \in [\sigma,\mu_j], \qquad s - \xi^j(s) > \sigma \quad \text{for } s \in (\mu_j,\alpha] \quad \text{for } j = 1,\dots,\ell. \quad (5.18)$$

Similarly, we define the constants $\gamma_{k,i}$ and $\mu_{k,j}$ as the solutions of

$$\gamma_{k,i} - \lambda^i(\gamma_{k,i}) = \sigma_k, \qquad \mu_{k,j} - \xi^j(\mu_{k,j}) = \sigma_k, \qquad \text{and} \qquad u(\eta_k) = \sigma_k,$$

or if $s - \lambda^i(s) > \sigma_k$ for all $s \in [\sigma,\alpha]$, then $\gamma_{k,i} := \sigma$; if $s - \lambda^i(s) < \sigma_k$ for all $s \in [\sigma,\alpha]$, then $\gamma_{k,i} := \alpha$; if $s - \xi^j(s) > \sigma_k$ for all $s \in [\sigma,\alpha]$, then $\mu_{k,j} := \sigma$; if $s - \xi^j(s) < \sigma_k$ for all $s \in [\sigma,\alpha]$, then $\mu_{k,j} := \alpha$; if $u(s) > \sigma_k$ for all $s \in [\sigma,\alpha]$, then $\eta_k := \sigma$; and if $u(s) < \sigma_k$ for all $s \in [\sigma, \alpha]$, then $\eta_k := \alpha$. Assumptions (A1) (iv) (b) and (A2) (iv) (b) and Lemma 4.3 yield that there exists a constant $\varepsilon_0 > 0$ such that

$$\operatorname*{ess\,inf}_{t\in[\sigma,\alpha^*_\sigma]} \frac{d}{dt}(t-\lambda^i(t)) \geq \varepsilon_0, \qquad \operatorname*{ess\,inf}_{t\in[\sigma,\alpha^*_\sigma]} \frac{d}{dt}(t-\xi^j(t)) \geq \varepsilon_0 \qquad \text{and} \qquad \operatorname*{ess\,inf}_{t\in[\sigma,\alpha^*_\sigma]} \dot u(t) \geq \varepsilon_0$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, \ell$. Therefore the Mean Value Theorem yields

$$\varepsilon_0|\gamma_{k,i} - \gamma_i| \leq |\sigma - \sigma_k|, \qquad \varepsilon_0|\mu_{k,j} - \mu_j| \leq |\sigma - \sigma_k| \qquad \text{and} \qquad \varepsilon_0|\eta_k - \eta| \leq |\sigma - \sigma_k|$$

for all $k \in \mathbb{N}$, $i = 1, \ldots, m$ and $j = 1, \ldots, \ell$.

Fix an $\varepsilon > 0$, and let $k_0 \in \mathbb{N}$ be such that $|\sigma - \sigma_k| < \varepsilon$ for $k \geq k_0$. Let $\delta := \varepsilon/\varepsilon_0$. Then we have that

$$s-\lambda^i(s) \leq \min\{\sigma, \sigma_k\},\ s \in [\nu_k, \max\{\nu_k, \gamma_i-\delta\}) \text{ and } s-\lambda^i(s) > \nu_k,\ s \in (\min\{\gamma_i+\delta, \alpha\}, \alpha]$$

$$s-\mu_j(s) \leq \min\{\sigma, \sigma_k\},\ s \in [\nu_k, \max\{\nu_k, \mu_j-\delta\}) \text{ and } s-\mu_j(s) > \nu_k,\ s \in (\min\{\mu_j+\delta, \alpha\}, \alpha]$$

and

$$u(s) \leq \min\{\sigma, \sigma_k\}, \quad s \in [\max\{\nu_k, \nu_k, \eta - \delta\}) \quad \text{and} \quad u(s) > \nu_k, \quad s \in (\min\{\eta + \delta, \alpha\}, \alpha].$$

Let $w^k(t) := \max\{|z^k(s) - z(s)| \colon\ s \in [\nu_k, t]\}$. Then the first integral on the right hand side of (5.15) can be estimated as follows for $t \in [\min\{\gamma_i + \delta, \alpha\}, \alpha]$

$$\begin{aligned}
&\int_{\nu_k}^{t} |A^i(s)||z^k(s-\lambda^i(s)) - z(s-\lambda^i(s))|\,ds \\
&\quad= \int_{\nu_k}^{\max\{\nu_k,\gamma_i-\delta\}} |A^i(s)||z^k(s-\lambda^i(s)) - z(s-\lambda^i(s))|\,ds \\
&\qquad+ \int_{\max\{\nu_k,\gamma_i-\delta\}}^{\min\{\gamma_i+\delta,\alpha\}} |A^i(s)||z^k(s-\lambda^i(s)) - z(s-\lambda^i(s))|\,ds \\
&\qquad+ \int_{\min\{\gamma_i+\delta,\alpha\}}^{t} |A^i(s)||z^k(s-\lambda^i(s)) - z(s-\lambda^i(s))|\,ds \\
&\quad\leq \int_{\nu_k}^{\max\{\nu_k,\gamma_i-\delta\}} |A^i(s)||h(s-\lambda^i(s)-\sigma_k) - h(s-\lambda^i(s)-\sigma)|\,ds \\
&\qquad+2\delta N_3(|v| + |h|_{L^\infty}) \int_{\max\{\nu_k,\gamma_i-\delta\}}^{\min\{\gamma_i+\delta,\alpha\}} |A^i(s)|\,ds + \int_{\min\{\gamma_i+\delta,\alpha\}}^{t} |A^i(s)|w^k(s)\,ds \\
&\quad\leq \int_{\nu_k}^{\max\{\nu_k,\gamma_i-\delta\}} |A^i(s)||h(s-\lambda^i(s)-\sigma_k) - h(s-\lambda^i(s)-\sigma)|\,ds \\
&\qquad+2\delta N_3(|v| + |h|_{L^\infty}) \int_{\sigma}^{\alpha} |A^i(s)|\,ds + \int_{\nu_k}^{t} |A^i(s)|w^k(s)\,ds.
\end{aligned}$$

Note that the final estimate holds for all $t \in [\nu_k, \alpha]$. Splitting all the integrals on the right hand side of (5.15) in a similar way, and using assumptions (A1) (iv) (c) and (A2) (iv) (c), we get the following estimate from (5.15):

$$\Big| \int_{\nu_k}^{t} L(s,x)(z_s^k - z_s)\, ds \Big| \le D_k^{h,\delta} + 2\max\{\delta,\varepsilon\}\alpha L_1 N_0 N_3(|v| + |h|_{L^\infty}) + L_1 N_0 \int_{\nu_k}^{t} w^k(s)\, ds, \tag{5.19}$$

for $t \in [\nu_k, \alpha]$, where

$$\begin{aligned}
D_k^{h,\delta} \;:=\; & \sum_{i=1}^{m} \int_{\nu_k}^{\max\{\nu_k,\gamma_i-\delta\}} |A^i(s)||h(s-\lambda^i(s)-\sigma_k) - h(s-\lambda^i(s)-\delta)|\, ds \\
& + \int_{\nu_k}^{\alpha_\sigma^*} \int_{-r}^{\min\{\sigma,\sigma_k\}-s} |A(s,\theta)||h(s+\theta-\sigma_k) - h(s+\theta-\sigma)|\, d\theta\, ds \\
& + L_1 \int_{\nu_k}^{\max\{\nu_k,\eta-\delta\}} |h(u(s)-\sigma_k) - h(u(s)-\sigma)|\, ds \\
& + L_1 N \sum_{j=1}^{\ell} \int_{\nu_k}^{\max\{\nu_k,\mu_j-\delta\}} |b^j(s)||h(s-\xi^j(s)-\sigma_k) - h(s-\xi^j(s)-\sigma)|\, ds \\
& + \int_{\nu_k}^{\alpha_\sigma^*} \int_{-r}^{\min\{\sigma,\sigma_k\}-s} |b(s,\theta)||h(s+\theta-\sigma_k) - h(s+\theta-\sigma)|\, d\theta\, ds.
\end{aligned}$$

Then it is easy to see that the Dominated Convergence Theorem yields that for each fixed $h \in L^\infty$ and $\delta > 0$ we have $D_k^{h,\delta} \to 0$, as $k \to \infty$.

Combining (5.14) and (5.19) we get

$$w^k(t) \le (B_k + 2\max\{\delta,\varepsilon\}\alpha L_1 N_0 N_3)(|v| + |h|_{L^\infty}) + C_k^h + D_k^{h,\delta} + L_1 N_0 \int_{\nu_k}^{t} w^k(s)\, ds,\ t \in [\nu_k, \alpha],$$

therefore the Gronwall's inequality implies

$$|z^k(t) - z(t)| \le w^k(t) \le \Big((B_k + 2\max\{\delta,\varepsilon\}\alpha L_1 N_0 N_3)(|v| + |h|_{L^\infty}) + C_k^h + D_k^{h,\delta} \Big) N_1,\ t \in [\nu_k, \alpha].$$

This proves the continuity of z wrt σ and φ, since δ and ε can be arbitrary close to 0.

The continuity of $z(t,\sigma,\varphi,\cdot,\cdot)$ in t, and therefore as a function of t, σ and φ, as well, can be argued as in the proof of Lemma 4.6, using that

$$|\dot z(t,\sigma,\varphi,v,h)| \le L_1 N_0 N_3(|v| + |h|_{L^\infty}), \qquad t \in [\sigma, \alpha]. \tag{5.20}$$

□

For $(t,\sigma,\varphi)\in H_1$ we define the bounded linear operator

$$T(t,\sigma,\varphi)\colon \mathbb{R}^n\times L^\infty\to\mathbb{R}^n,\qquad T(t,\sigma,\varphi)(v,h):=z(t,\sigma,\varphi,v,h),\tag{5.21}$$

where $z(t,\sigma,\varphi,v,h)$ is the solution of the IVP (5.9)-(5.11). Lemma 5.2 yields that for each $(v,h)\in\mathbb{R}^n\times L^\infty$ the function $\mathbb{R}^2\times W^{1,\infty}\supset H_1\to\mathbb{R}^n$, $(t,\sigma,\varphi)\mapsto T(t,\sigma,\varphi)(v,h)$ is continuous.

We have the following result concerning the differentiability of solutions wrt σ in this special case.

Theorem 5.3 *Assume (A1) (i)–(iv), (A2) (i)–(iv). Suppose $(\hat\sigma,\hat\varphi)\in\Pi$ is such that $x(\cdot,\hat\sigma,\hat\varphi)\in X(\hat\sigma,\alpha)$. Let P_1 and H_1 be the sets defined by Lemma 4.3 and* (4.33)*, respectively. Let $x(t,\sigma,\varphi)$ denote the solutions of the IVP* (3.1)-(3.2)*. Then the function*

$$\mathbb{R}^2\times W^{1,\infty}\supset H_1\to\mathbb{R}^n,\qquad (t,\sigma,\varphi)\mapsto x(t,\sigma,\varphi)$$

is continuously differentiable wrt σ, and

$$D_2x(t,\sigma,\varphi)=T(t,\sigma,\varphi)(-f(\sigma,\varphi,\varphi(-\tau(\sigma,\varphi))),-\dot\varphi),$$

where $T(t,\sigma,\varphi)$ is defined by (5.21)*.*

Proof Let $(\sigma,\varphi)\in P_1$, $t\in(\sigma,\alpha]$. Let $h_k\in\mathbb{R}$ $(k\in\mathbb{N})$ be a sequence such that $h_k\to 0$ as $k\to\infty$, $(\sigma+h_k,\varphi)\in P_1$ and $\sigma+h_k<t$ for $k\in\mathbb{N}$. To simplify notation, let $x^k(t):=x(t,\sigma+h_k,\varphi)$, $x(t):=x(t,\sigma,\varphi)$, $u(s):=s-\tau(s,x_s)$, $u^k(s):=s-\tau(s,x^k_s)$, $v:=-f(\sigma,\varphi,\varphi(-\tau(\sigma,\varphi)))$, $z(t):=T(t,\sigma,\varphi)(v,-\dot\varphi)$, and $\alpha^*_\sigma:=\min\{\sigma+r,\alpha\}$. We distinguish two cases.

(i) First suppose $h_k<0$ for all $k\in\mathbb{N}$. Then, as in the proof of Theorem 5.1, we get for $t\in[\sigma,\alpha]$

$$\begin{aligned}x^k(t)-x(t)-z(t)h_k &= x^k(\sigma)-x(\sigma)-z(\sigma)h_k\\ &\quad+\int_\sigma^t\Big(f(s,x^k_s,x^k(u^k(s)))-f(s,x_s,x(u(s)))-L(s,x)z_sh_k\Big)ds.\end{aligned}$$

Let $q^k(s) := x^k(s) - x(s) - z(s)h_k$, $s \in [\sigma - r, \alpha]$. Then using (A1) (iv) and (A2) (iv) we have the relation analogously to (4.40) for $s \in [\sigma, \alpha]$

$$
\begin{aligned}
f(s, x_s^k, x^k(u^k(s))) &- f(s, x_s, x(u(s))) - L(s,x)z_s \\
= \;& \sum_{i=0}^{m} A^i(s) q^k(s - \lambda^i(s)) + \int_{-r}^{0} A(s,\theta) q^k(s+\theta)\, d\theta \\
& + D_3 f(s, x_s, x(u(s))) q^k(u^k(s)) \\
& + D_3 f(s, x_s, x(u(s))) \Big(x(u^k(s)) - x(u(s)) - \dot{x}(u(s))(u^k(s) - u(s)) \Big) \\
& + D_3 f(s, x_s, x(u(s))) \dot{x}(u(s)) \Big(u^k(s) - u(s) + D_2\tau(s, x_s)(x_s^k - x_s) \Big) \\
& - D_3 f(s, x_s, x(u(s))) \dot{x}(u(s)) \\
& \quad \times \Big(\sum_{j=0}^{\ell} b^j(s) q^k(s - \xi^j(s)) + \int_{-r}^{0} b(s,\theta) q^k(s+\theta)\, d\theta \Big) \\
& + D_3 f(s, x_s, x(u(s)))(z(u^k(s)) - z(u(s))) \\
& + \omega_f(s, x_s, x(u(s)), x_s^k, x^k(u^k(s))).
\end{aligned}
\tag{5.22}
$$

Then we get for $t \in [\sigma, \alpha]$

$$
\begin{aligned}
|q^k(t)| \;\le\; & a_k + b_k + c_k + d_k|h_k| + |q^k(\sigma)| + \int_{\sigma}^{t} \Big(\sum_{i=0}^{m} |A^i(s)||q^k(s - \lambda^i(s))| \\
& + \int_{-r}^{0} |A(s,\theta)||q^k(s+\theta)|\, d\theta + L_1 |q^k(u^k(s))| \\
& + L_1 N \sum_{j=0}^{\ell} |b^j(s)||q^k(s - \xi^j(s))| + L_1 N \int_{-r}^{0} |b(s,\theta)||q^k(s+\theta)|\, d\theta \Big) ds,
\end{aligned}
\tag{5.23}
$$

where a_k, b_k and c_k, d_k are defined by formulas (4.42)–(4.45), respectively, and N is defined by (3.3).

Assuming the monotonicity assumptions (A1) (iv) (b), (A2) (iv) (b) and Lemma 4.3, there exist constants $\eta, \gamma_i, \mu_j \in [\sigma, \alpha]$ for $i = 1, \ldots, m$ and $j = 1, \ldots, \ell$ such that (5.16)–(5.18) hold. Let $\bar{\eta}_k$ be the unique solution of $u^k(\bar{\eta}_k) = \sigma$ if such a solution exists, otherwise let $\bar{\eta}_k := \alpha$. Introduce the function $w^k(t) := \max\{|q^k(s)| \colon\; s \in [\sigma, t]\}$. Then (5.16)–(5.18) and (5.23) yield

$$
\begin{aligned}
|q^k(t)| \;\leq\;& a_k + b_k + c_k + d_k|h_k| + |q^k(\sigma)| + \sum_{i=1}^{m} \int_{\sigma}^{\gamma_i} |A^i(s)||q^k(s-\lambda^i(s))|\,ds \\
&+ \int_{\sigma}^{\min\{\sigma+r,t\}} \int_{-r}^{\sigma-s} |A(s,\theta)||q^k(s+\theta)|\,d\theta\,ds \\
&+ L_1 \int_{\sigma}^{\bar{\eta}_k} q^k(u^k(s))\,ds + L_1 N \sum_{j=1}^{\ell} \int_{\sigma}^{\mu_j} |q^k(s-\xi^j(s))|\,ds \\
&+ L_1 N \int_{\sigma}^{\min\{\sigma+r,t\}} \int_{-r}^{\sigma-s} |b(s,\theta)||q^k(s+\theta)|\,d\theta\,ds \\
&+ \sum_{i=1}^{m} \int_{\gamma_i}^{t} |A^i(s)| w^k(s)\,ds + \int_{\sigma}^{\min\{\sigma+r,t\}} \int_{\sigma-s}^{0} |A(s,\theta)| w^k(s)\,d\theta\,ds \\
&+ \int_{\min\{\sigma+r,t\}}^{t} \int_{-r}^{0} |A(s,\theta)| w^k(s)\,d\theta\,ds + L_1 \int_{\bar{\eta}_k}^{t} w^k(s)\,ds \\
&+ L_1 N \sum_{j=1}^{\ell} \int_{\mu_j}^{t} |b^j(s)| w^k(s)\,ds + \int_{\sigma}^{\min\{\sigma+r,t\}} \int_{\sigma-s}^{0} |b(s,\theta)| w^k(s)\,d\theta\,ds \\
&+ \int_{\min\{\sigma+r,t\}}^{t} \int_{-r}^{0} |b(s,\theta)| w^k(s)\,d\theta\,ds, \qquad \in [\sigma,\alpha].
\end{aligned}
$$

Therefore using assumptions (A1) (iv) (c), (A2) (iv) (c) and (3.7) we get

$$
q^k(t) \leq A_k + L_1 N_0 \int_{\sigma}^{t} w^k(s)\,ds, \qquad t \in [\sigma,\alpha], \tag{5.24}
$$

where

$$
\begin{aligned}
A_k \;:=\;& a_k + b_k + c_k + d_k|h_k| + |q^k(\sigma)| + \sum_{i=1}^{m} \int_{\sigma}^{\gamma_i} |A^i(s)||q^k(s-\lambda^i(s))|\,ds \\
&+ \int_{\sigma}^{\alpha_\sigma^*} \int_{-r}^{\sigma-s} |A(s,\theta)||q^k(s+\theta)|\,d\theta\,ds + L_1 \int_{\sigma}^{\bar{\eta}_k} |q^k(u^k(s))|\,ds \\
&+ L_1 N \sum_{j=1}^{\ell} \int_{\sigma}^{\mu_j} |b^j(s)||q^k(s-\xi^j(s))|\,ds \\
&+ L_1 N \int_{\sigma}^{\alpha_\sigma^*} \int_{-r}^{\sigma-s} |b(s,\theta)||q^k(s+\theta)|\,d\theta\,ds.
\end{aligned}
$$

The monotonicity of the right-hand-side of (5.24) in t implies

$$
w^k(t) \leq A_k + L_1 N_0 \int_{\sigma}^{t} w^k(s)\,ds, \qquad t \in [\sigma,\alpha], \tag{5.25}
$$

and hence Gronwall's inequality yields

$$|x^k(t) - x(t) - z(t)h_k| \le w^k(t) \le A_k N_1, \qquad t \in [\sigma, \alpha], \tag{5.26}$$

where N_1 is defined by (3.7). It is enough to show that $\frac{A_k}{|h_k|} \to 0$ as $k \to \infty$.

We have

$$\begin{aligned}
\frac{|q^k(\sigma)|}{|h_k|} &= \frac{1}{|h_k|}\Big|\varphi(0) + \int_{\sigma+h_k}^{\sigma} f(s, x^k_s, x^k(u^k(s)))\, ds - \varphi(0) - vh_k\Big| \\
&= \frac{1}{|h_k|}\Big|\int_{\sigma+h_k}^{\sigma} \Big(f(s, x^k_s, x^k(u^k(s))) - f(\sigma, \varphi, \varphi(-\tau(\sigma,\varphi)))\Big)\, ds\Big| \\
&\to 0, \qquad \text{as } k \to \infty,
\end{aligned} \tag{5.27}$$

as it was shown in the proof of Theorem 5.1.

For $s \in [\sigma - r, \sigma)$ such that $\dot\varphi(s-\sigma)$ exists, i.e., for a.e. $s \in [\sigma - r, \sigma)$ and for large enough k such that $s < \sigma_k$ we have

$$\frac{|q^k(s)|}{|h_k|} = \Big|\frac{\varphi(s-\sigma-h_k) - \varphi(s-\sigma)}{-h_k} - \dot\varphi(s-\sigma)\Big| \to 0, \qquad \text{as } k \to \infty. \tag{5.28}$$

Therefore (5.16)-(5.18) and the Dominated Convergence Theorem imply

$$\int_\sigma^{\gamma_i} |A^i(s)| \frac{|q^k(s - \lambda^i(s))|}{|h_k|}\, ds \to 0 \quad \text{and} \quad \int_\sigma^{\mu_j} |b^j(s)| \frac{|q^k(s - \xi^j(s))|}{|h_k|}\, ds \to 0 \tag{5.29}$$

as $k \to \infty$ for $i = 1, \ldots, m$ and $j = 1, \ldots, \ell$,

$$\int_\sigma^{\alpha^*_\sigma} \int_{-r}^{\sigma-s} |A(s,\theta)| \frac{|q^k(s+\theta)|}{|h_k|}\, d\theta\, ds \to 0, \qquad \text{as } k \to \infty, \tag{5.30}$$

and

$$\int_\sigma^{\alpha^*_\sigma} \int_{-r}^{\sigma-s} |b(s,\theta)| \frac{|q^k(s+\theta)|}{|h_k|}\, d\theta\, ds \to 0, \qquad \text{as } k \to \infty. \tag{5.31}$$

Relation (3.4) and Lemma 5.2 yield

$$\begin{aligned}
\frac{q^k(u^k(s))}{|h_k|} &= \frac{1}{|h_k|}|x^k(u^k(s)) - x(u^k(s)) - z(u^k(s))h_k| \\
&\le \frac{1}{|h_k|}|x^k(u^k(s)) - x(u^k(s))| + |z(u^k(s))| \\
&\le L + K_0, \qquad \text{a.e. } s \in [\sigma, \alpha],
\end{aligned} \tag{5.32}$$

where $K_0 := N_3\Big(|f(\sigma, \varphi, \varphi(-\tau(\sigma,\varphi)))| + |\dot\varphi|_{L^\infty}\Big)$.

Let $\eta \in [\sigma, \alpha]$ be such that $u(s) \le \sigma$ for $s \in [\sigma, \eta]$ and $u(s) > \sigma$ for $s \in (\eta, \alpha]$. Then, clearly, $\bar{\eta}_k \to \eta$ as $k \to \infty$. Suppose first that $\bar{\eta}_k > \eta$. Then for an arbitrarily fixed $\delta_0 > 0$ we have

$$\int_\sigma^{\bar{\eta}_k} |q^k(u^k(s))|\, ds = \int_\sigma^{\eta-\delta_0} |q^k(u^k(s))|\, ds + \int_{\eta-\delta_0}^{\bar{\eta}_k} |q^k(u^k(s))|\, ds.$$

If $\bar{\eta}_k \le \eta$, then

$$\int_\sigma^{\bar{\eta}_k} |q^k(u^k(s))|\, ds \le \int_\sigma^{\eta-\delta_0} |q^k(u^k(s))|\, ds + \int_{\eta-\delta_0}^{\eta} |q^k(u^k(s))|\, ds.$$

Therefore in both cases using (5.32) we obtain

$$\int_\sigma^{\bar{\eta}_k} |q^k(u^k(s))|\, ds \le \int_\sigma^{\eta-\delta_0} |q^k(u^k(s))|\, ds + (L + K_0)(|\bar{\eta}_k - \eta| + \delta_0)|h_k|. \tag{5.33}$$

Let $\varepsilon_0 := (\sigma - u(\eta - \delta_0))/2$. Then $|u^k(s) - u(s)| < \varepsilon_0$ and $|h_k| < \varepsilon_0$ for sufficiently large k. Therefore, for such k we have $u^k(s) < \sigma$ and $u^k(s) - h_k < \sigma$ for $s \in [\sigma, \eta - \delta_0]$, and so Lemma 2.2 and the Dominated Convergence Theorem show that

$$\begin{aligned}
&\int_\sigma^{\eta-\delta_0} \frac{|q^k(u^k(s))|}{|h_k|}\, ds \\
&= \int_\sigma^{\eta-\delta_0} \frac{1}{|h_k|}\Big|\varphi(u^k(s) - \sigma - h_k) - \varphi(u^k(s) - \sigma) + \dot{\varphi}(u^k(s) - \sigma)h_k\Big|\, ds \\
&= \int_\sigma^{\eta-\delta_0} \int_0^1 |\dot{\varphi}(u^k(s) - \sigma - \nu h_k) - \dot{\varphi}(u^k(s) - \sigma)|\, d\nu\, ds \\
&= \int_0^1 \int_\sigma^{\eta-\delta_0} |\dot{\varphi}(u^k(s) - \sigma - \nu h_k) - \dot{\varphi}(u(s) - \sigma)|\, ds\, d\nu \\
&\quad + \int_0^1 \int_\sigma^{\eta-\delta_0} |\dot{\varphi}(u^k(s) - \sigma) - \dot{\varphi}(u(s) - \sigma)|\, ds\, d\nu \\
&\to 0, \qquad k \to \infty.
\end{aligned}$$

Therefore (5.33) implies

$$\int_\sigma^{\bar{\eta}_k} \frac{|q^k(u^k(s))|}{|h_k|}\, ds \to 0, \qquad k \to \infty, \tag{5.34}$$

since δ_0 was arbitrarily close to 0.

Combining (5.3), $d_k \to 0$ as $k \to \infty$ by Lemma 2.2, (5.27), (5.29)–(5.31) and (5.34) we get that $A_k/|h_k| \to 0$ as $k \to \infty$, which concludes the proof in case (i).

(ii) Assume now that $h_k > 0$ for all $k \in \mathbb{N}$. Then (5.6) and (5.22) yield

$$\begin{aligned} |q^k(t)| \;\leq\;& \bar{a}_k + \bar{b}_k + \bar{c}_k + \bar{d}_k|h_k| + |q^k(\sigma + h_k)| \\ &+ \int_{\sigma+h_k}^{t} \Big(\sum_{i=0}^{m} |A^i(s)||q^k(s - \lambda^i(s))| + \int_{-r}^{0} |A(s,\theta)||q^k(s+\theta)|\, d\theta \\ &\quad + L_1|q^k(u^k(s))| + L_1 N \sum_{j=0}^{\ell} |b^j(s)||q^k(s - \xi^j(s))| \\ &\quad + L_1 N \int_{-r}^{0} |b(s,\theta)||q^k(s+\theta)|\, d\theta \Big) ds, \qquad t \in [\sigma + h^k, \alpha] \end{aligned} \tag{5.35}$$

where $\bar{a}_k$, $\bar{b}_k$, $\bar{c}_k$ and $\bar{d}_k$ are the constants defined in the proof of Theorem 5.1.

We have

$$\begin{aligned} \frac{|q^k(\sigma + h_k)|}{h_k} &= \frac{1}{h_k}\Big| x^k(\sigma + h_k) - x(\sigma + h_k) - z(\sigma + h_k)h_k \Big| \\ &= \frac{1}{h_k}\Big| \varphi(0) - \varphi(0) - \int_{\sigma}^{\sigma+h_k} f(s, x_s, x(u(s)))\, ds - v h_k \\ &\qquad - \int_{\sigma}^{\sigma+h_k} L(s,x) z_s h_k\, ds \Big| \\ &\leq \frac{1}{h_k}\Big| \int_{\sigma}^{\sigma+h_k} \Big(f(s, x_s, x(u(s))) - f(\sigma, \varphi, \varphi(u(\sigma))) \Big) ds \Big| \\ &\quad + h_k L_1 N_0 N_3 (|v| + |\dot{\varphi}|_{L^\infty}) \\ &\to 0, \qquad k \to \infty. \end{aligned}$$

Then using the result from the proof of Theorem 5.1 that $\bar{a}_k \to 0$, $\bar{b}_k \to 0$, $\bar{c}_k \to 0$ and $\bar{d}_k \to 0$, an argument similar to the proof of part (i) and Theorem 5.1 shows the differentiability of $x(t, \sigma, \varphi)$ wrt σ.

To show the continuity of $D_2 x(t, \sigma, \varphi)$, consider $(t, \sigma, \varphi), (\bar{t}, \bar{\sigma}, \bar{\varphi}) \in H_1$, and let $v := f(\sigma, \varphi, \varphi(-\tau(\sigma, \varphi)))$ and $\bar{v} := f(\bar{\sigma}, \bar{\varphi}, \bar{\varphi}(-\tau(\bar{\sigma}, \bar{\varphi})))$. Then by Lemma 5.2 and (5.20) we have

$$
\begin{aligned}
&|D_2x(t,\sigma,\varphi)-D_2x(\bar t,\bar\sigma,\bar\varphi)| \\
&\qquad = \ |T(t,\sigma,\varphi)(-v,-\dot\varphi)-T(\bar t,\bar\sigma,\bar\varphi)(-\bar v,-\dot{\bar\varphi})| \\
&\qquad \le \ |T(t,\sigma,\varphi)[(-v,-\dot\varphi)-(-\bar v,-\dot{\bar\varphi})]|+|[T(t,\sigma,\varphi)-T(\bar t,\sigma,\varphi)](-\bar v,-\dot{\bar\varphi})| \\
&\qquad\quad\ +|[T(\bar t,\sigma,\varphi)-T(\bar t,\bar\sigma,\bar\varphi)](-\bar v,-\dot{\bar\varphi})| \\
&\qquad \le \ N_3(|v-\bar v|+|\dot\varphi-\dot{\bar\varphi}|_{L^\infty})+|z(t,\sigma,\varphi,-\bar v,-\dot{\bar\varphi})-z(\bar t,\sigma,\varphi,-\bar v,-\dot{\bar\varphi})| \\
&\qquad\quad\ +|z(\bar t,\sigma,\varphi,-\bar v,-\dot{\bar\varphi})-z(\bar t,\bar\sigma,\bar\varphi,-\bar v,-\dot{\bar\varphi})| \\
&\qquad \le \ N_3(|v-\bar v|+|\varphi-\bar\varphi|_{W^{1,\infty}})+L_1N_0N_3(|\bar v|+|\dot{\bar\varphi}|_{L^\infty})|t-\bar t| \\
&\qquad\quad\ +|z(\bar t,\sigma,\varphi,-\bar v,-\dot{\bar\varphi})-z(\bar t,\bar\sigma,\bar\varphi,-\bar v,-\dot{\bar\varphi})|,
\end{aligned}
$$

which proves the continuity of $D_2x(t,\sigma,\varphi)$. □

Remark 5.4 We close this paper by noting that if we fix $(\sigma,\varphi)\in P_1$, and assume that φ is differentiable at 0 from the left, then for $h>0$ we have

$$
\lim_{h\to 0+}\frac{x(\sigma,\sigma+h,\varphi)-x(\sigma,\sigma,\varphi)}{h}=\lim_{h\to 0+}\frac{\varphi(-h)-\varphi(0)}{h}=-\dot\varphi(0-),
$$

and

$$
\begin{aligned}
\lim_{h\to 0+}\frac{x(\sigma,\sigma-h,\varphi)-x(\sigma,\sigma,\varphi)}{-h} &= \lim_{h\to 0+}\frac{\varphi(0)+\int_{\sigma-h}^{\sigma}f(s,x_s,x(u(s)))\,ds-\varphi(0)}{-h} \\
&= -f(\sigma,\varphi,\varphi(\sigma-\tau(\sigma,\varphi))).
\end{aligned}
$$

Therefore, if we consider $x(t,\sigma,\varphi)$ on the set $\{(t,\sigma,\varphi)\colon (\sigma,\varphi)\in P_1,\ t\in[\sigma-r,\alpha]\}$, then $D_2x(t,\sigma,\varphi)$ exists at $t=\sigma$ if and only if the compatibility condition

$$
\dot\varphi(0-)=f(\sigma,\varphi,\varphi(\sigma-\tau(\sigma,\varphi)))
$$

holds.

We note that in Theorem 5.3 differentiability of $x(t,\sigma,\varphi)$ wrt σ at $t=\sigma$ was considered only as the right derivative, since we restricted the function to the set H_1, i.e., for $(\sigma,\varphi)\in P_1$, $t\in[\sigma,\alpha]$.

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