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EXPONENTIAL STABILITY OF A STATE-DEPENDENT DELAY SYSTEM

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Abstract. In this paper we study exponential stability of the trivial solution of the state-dependent delay system $\dot{x}(t) = \sum_{i=1}^{m} A_i(t)x(t - \tau_i(t, x_t))$. We show that under mild assumptions, the trivial solution of the state-dependent system is exponentially stable, if and only if the trivial solution of the corresponding linear time-dependent delay system $\dot{y}(t) = \sum_{i=1}^{m} A_i(t)y(t - \tau_i(t, \mathbf{0}))$ is exponentially stable. We also compare the order of the exponential stability of the nonlinear equation to that of its linearized equation. We show, that in some cases, the two orders are equal. As an application of our main result, we formulate a necessary and sufficient condition for the exponential stability of the trivial solution of a threshold-type delay system.

1. **Introduction.** In this paper we study exponential stability of the trivial solution of delay systems with state-dependent delays (SD-DDEs) of the form

$$\dot{x}(t) = \sum_{i=1}^{m} A_i(t)x(t - \tau_i(t, x_t)). \tag{1}$$

We obtain sharp linearized stability results for SD-DDEs of the form (1) by comparing the stability of the trivial solution to that of an associated linear delay equation. Cooke obtained sufficient stability conditions for the scalar and autonomous version of (1) with $m = 1$ first comparing the stability the trivial solution to that of an associated ODE in [3], and later in [4], to that of an associated linear differential equation with time-dependent delay.

In [11] we studied scalar SD-DDEs of the form $\dot{x}(t) = a(t)x(t-\tau(t, x(t)))$, where in addition to a sufficient condition, a necessary condition was obtained to guarantee exponential stability of the trivial solution of the SD-DDE. To obtain the necessary part, it was assumed that τ is differentiable with respect to both variables, and

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the partial derivatives are bounded and satisfy a certain smallness condition to guarantee monotonicity of the function $t \mapsto t - \tau(t, x(t))$. In this paper we extend these results to the more general equation (1), and using a different technique, we significantly relax the condition used for the necessary part of the main result: instead of the differentiability and the monotonicity conditions cited above a much weaker condition (H5) (see Section 2) is assumed. This weaker condition is satisfied for treshold-type delays and for delays of the form

$$\tau_i(\psi) = \begin{cases} c_i|\psi(0)|, & |\psi(0)| < \gamma_i, \\ c_i\gamma_i, & |\psi(0)| \geq \gamma_i, \end{cases}$$

where $c_i \geq 0$ and $\gamma_i > 0$ are constants.

Stability of more general SD-DDEs was investigated, e.g., in [5], [23], [30]. Stability conditions for general nonlinear differential equations with state-dependent delay using linearization techniques were obtained for different classes of SD-DDEs in [6], [18]–[20]. For a recent review on basic theory of state-dependent delay equations and related applications we refer to [21].

The organization of the paper is as follows. In Section 2 we give our assumptions and formulate our main results (Theorem 2.4 below). We also investigate the relation between the order of stability of the trivial solution of the SD-DDE and that of its linearized equation, and also give estimate of the domain of attraction of the nonlinear equation. Several explicit exponential stability conditions are formulated as corollaries of our main theorem applying known stability results for the comparison equation. In Section 3 we show how our main theorem can be applied for threshold-type delay equations. Section 4 introduces some notations and lemmas and contains the proofs of the main results.

2. **Formulation of the Main Results.** Throughout this paper a fixed norm on $\mathbb{R}^n$ and its induced matrix norm on $\mathbb{R}^{n\times n}$ is denoted by $|\cdot|$. The Banach space of continuous functions $\psi\colon [-r, 0] \to \mathbb{R}^n$ equipped with the norm $\|\psi\| = \sup\{|\psi(s)| : s \in [-r, 0]\}$ is denoted by C. The solution segment function $x_t : [-r, 0] \to \mathbb{R}^n$ is defined by $x_t(s) = x(t + s)$.

Consider the delay system

$$\dot{x}(t) = \sum_{i=1}^{m} A_i(t)x(t - \tau_i(t, x_t)), \qquad t \geq t_0. \tag{1}$$

We assume that $r > 0$ is fixed, $t_0 \geq 0$, $A_i\colon [0,\infty) \to \mathbb{R}^{n\times n}$ and $\tau_i\colon [0,\infty)\times C \to [0, r]$ $(i = 1, \dots, m)$.

We compare the exponential stability of the trivial solution of (1) to that of the associated linear system

$$\dot{y}(t) = \sum_{i=1}^{m} A_i(t)y(t - \tau_i(t, \mathbf{0})), \qquad t \geq t_0, \tag{2}$$

where $\mathbf{0}$ is the constant 0 function in C.

We associate the initial condition

$$x(t) = \varphi(t - t_0), \qquad t \in [t_0 - r, t_0], \qquad \varphi \in C \tag{3}$$

to Equations (1) and (2).

Definition 2.1. We say that the trivial (zero) solution of the linear equation (2) is exponentially stable, if there exist constants $K_1 \geq 0$ and $\alpha > 0$ independent of t_0 such that

$$|y(t)| \leq K_1 e^{-\alpha(t-t_0)}\|\varphi\|, \qquad t \geq t_0 \geq 0. \tag{4}$$

If (4) holds, then we say that the order of exponential stability is α.

We note that this notion is also called in the literature as uniform exponential stability. It would be more precise to say the order of exponential stability is at least α, since (4) may hold with larger α, as well, but we use this terminology for simplicity.

We define the fundamental solution of (2) as the $n \times n$ matrix solution of the initial value problem

$$\frac{\partial}{\partial t}V(t,s) = \sum_{i=1}^{m} A_i(t)V(t-\tau_i(t,\mathbf{0}),s), \qquad t \geq s, \tag{5}$$

$$V(t,s) = \begin{cases} I, & t = s, \\ 0 & t < s. \end{cases} \tag{6}$$

Here I and 0 denote the identity and the zero matrices, respectively.

If the trivial solution of (2) is exponentially stable with order α, then it is known (see, e.g., [15]), that there exists $K_2 \geq 1$ such that

$$|V(t,s)| \leq K_2 e^{-\alpha(t-s)}, \qquad t \geq 0, \quad s \in \mathbb{R}. \tag{7}$$

Definition 2.2. We say the trivial solution of (1) is exponentially stable, if there exist $K_3 \geq 0$, $\beta > 0$ and $\sigma > 0$ independent of t_0 such that

$$|x(t)| \leq K_3 e^{-\beta(t-t_0)}\|\varphi\|, \qquad t \geq t_0 \geq 0, \quad \|\varphi\| \leq \sigma.$$

Definition 2.3. We say the trivial solution of (1) is exponentially stable in the large with order β, if for every $\sigma > 0$ there exists $K_4 = K_4(\sigma) \geq 0$ such that

$$|x(t)| \leq K_4 e^{-\beta(t-t_0)}\|\varphi\|, \qquad t \geq t_0 \geq 0, \quad \|\varphi\| \leq \sigma.$$

We assume throughout the paper

(H1) $A_i\colon [0,\infty) \to \mathbb{R}^{n\times n}$ is continuous, and $|A_i(t)| \leq b_i$, $t \in [0,\infty)$ for $i = 1,\dots,m$;
(H2) $\varphi \in C$;
(H3) the delay functions $\tau_i\colon [0,\infty)\times C \to [0,r]$ are continuous for $i = 1,\dots,m$;
(H4) there exist a constant $0 < \gamma \leq \infty$ and continuous functions $\omega_i\colon [0,\gamma) \to [0,\infty)$, such that

$$|\tau_i(t,\psi) - \tau_i(t,\mathbf{0})| \leq \omega_i(\|\psi\|), \qquad t \geq 0, \quad \|\psi\| < \gamma, \quad i = 1,\dots,m,$$

where $\omega_i(0) = 0$ $(i = 1,\dots,m)$.
(H5) the sets $\{s \in [0,r] : \ s - \tau_i(s+t_0,\mathbf{0}) = 0\}$ have Lebesgue measure 0 for $i = 1,\dots,m$ and $t_0 \geq 0$.

We remark that if the value of the function $s \mapsto s - \tau_i(s+t_0,\mathbf{0})$ is equal to 0 at most for countably many s, then (H5) holds.

We note that conditions (H1)–(H3) guarantee the existence, but not the uniqueness of the solution (see, e.g., [7], [14], [19]).

Our main result is formulated in the following theorem.

Theorem 2.4. *Suppose (H1)–(H5). Then the trivial solution of* (1) *is exponentially stable, if and only if the trivial solution of* (2) *is exponentially stable.*

In the proof of the necessary part it was important the we "linearize" around the trivial solution. It would be interesting to extend this result to more general solutions, e.g., to periodic solutions (see [18] where a sufficient condition was formulated in this case). It is also an open problem to relax or omit condition (H5) in Theorem 2.4.

We note that assumption (H5) is used only in the necessary part of the proof of the previous theorem. Therefore, we can formulate the sufficient part of Theorem 2.4 as follows.

Theorem 2.5. *Suppose (H1)–(H4). If the trivial solution of* (2) *is exponentially stable, then the trivial solution of* (1) *is exponentially stable, as well.*

The proof of Theorem 2.5 yields the next corollary.

Corollary 1. *Suppose (H1)–(H4). If the trivial solution of* (2) *is exponentially stable with order* α*, then for every* $0<\beta<\alpha$ *the trivial solution of* (1) *is exponentially stable with order* β*, as well.*

Note that the proof of the necessary part of Theorem 2.4 shows that the exponential stability of the trivial solution of (1) with order α implies the exponential stability for the trivial solution of (3) with the same order.

Corollary 2. *Suppose (H1)–(H5). If the trivial solution of* (1) *is exponentially stable with order* α*, then the trivial solution of* (2) *is exponentially stable with order* α*, as well.*

Under some more restriction on the functions ω_i, we can prove that the order of the exponential stability of the trivial solution of the linear equation (2) is preserved for that of the SD-DDE (1). We also give an explicit estimate for the domain of attraction of the trivial solution.

Theorem 2.6. *Assume (H1)–(H4), moreover there exists* $c_0>0$ *such that*

$$\int_0^{c_0} \frac{\omega_i(u)}{u}\,du<\infty, \tag{8}$$

ω_i *is monotone increasing on* $[0,c_0]$ $(i=1,\ldots,m)$*, and suppose the trivial solution of* (2) *is exponentially stable with order* α*. Let* K_1 *and* K_2 *be defined by* (4) *and* (7)*, respectively,* $b=\sum_{i=1}^m b_i$*,* $M_2=\max\{1, K_1+2K_2be^{(\alpha+b)r}/\alpha\}$*. Let*

$$\sigma_0=\sup_{c\in\mathcal{U}}\left\{\frac{c}{M_2}\Big(1-\frac{K_2be^{2\alpha r}}{\alpha}\sum_{i=1}^m b_i\int_0^c\frac{\omega_i(u)}{u}\,du\Big)\right\}, \tag{9}$$

where

$$\mathcal{U}=\left\{c>0\colon\ \frac{K_2be^{2\alpha r}}{\alpha}\sum_{i=1}^m b_i\int_0^c\frac{\omega_i(u)}{u}\,du<1\right\}.$$

Then for every $0<\sigma<\sigma_0$ *there exists* $K\ge 0$ *such that the solution of* (1) *satisfies*

$$|x(t)|\le Ke^{-\alpha(t-t_0)}\|\varphi\|,\qquad t\ge t_0\ge 0,\quad \|\varphi\|\le\sigma,$$

i.e., the trivial solution of (1) *is exponentially stable with order* α*, as well.*

This theorem implies immediately the next corollary.

Corollary 3. *Assume (H1)–(H4), moreover*

$$\frac{K_2 b e^{2\alpha r}}{\alpha}\sum_{i=1}^{m} b_i \int_0^\infty \frac{\omega_i(u)}{u}\,du < \infty, \tag{10}$$

ω_i is monotone increasing on $[0, c_0]$ ($i = 1, \dots, m$), and suppose the trivial solution of (2) *is exponentially stable with order α. Then the trivial solution of* (1) *is exponentially stable in the large with order α.*

As an application of the previous corollary, consider

$$\dot{x}(t) = -bx(t - \tau(x(t))), \qquad t \geq 0, \tag{11}$$

where $b > 0$, $\tau(u) = re|u|e^{-|u|}$. It is easy to check that $\omega(u) = reue^{-u}$ satisfies (H4), $K_1 = K_2 = 1$, $\alpha = b$ and

$$\frac{K_2 b e^{2\alpha r}}{\alpha}\sum_{i=1}^{m} b_i \int_0^\infty \frac{\omega_i(u)}{u}\,du = ebre^{2br}.$$

Therefore, if $ebre^{2br} < 1$, then the trivial solution of (11) is exponentially stable in the large with order b.

Next we apply known conditions in the scalar case to check the exponential stability of the trivial solution of (2), and we obtain several corollaries of our main results. Note that in the single delay case similar results are formulated in [11].

Next result follows from conditions of Krisztin [22].

Corollary 4. *Suppose (H1)–(H4) hold with $n = 1$, $A_i(t) \leq 0$ for $i = 1, \dots, m$. Then the trivial solution of* (1) *is exponentially stable, if*

$$\sum_{i=1}^{m} b_i \sup_{t\geq 0} \tau_i(t, 0) < 1.$$

Moreover, if $A_i(t) = -b_i$ ($t \geq 0$), then

$$\sum_{i=1}^{m} b_i \sup_{t\geq 0} \tau_i(t, 0) < \frac{3}{2}$$

yields the exponential stability of the trivial solution of (1)*. If, in addition, $\tau_i(t, \mathbf{0}) = r_i$ are constant, then*

$$\sum_{i=1}^{m} b_i r_i < \frac{\pi}{2} \tag{12}$$

implies the exponential stability of the trivial solution of (1)*. If $m = 1$, then* (12) *is also necessary for the exponential stability of the trivial solution* (1)*.*

Next result is based on Theorem 4.1 of [12], which generalizes a condition of Yoneyama [30] for the multiple delay case.

Corollary 5. *Suppose (H1)–(H4) hold with $n = 1$, $A_i(t) \leq 0$ for $i = 1, \dots, m$. Moreover, suppose*

$$\sum_{i,k=1}^{m} \limsup_{t\to\infty} \int_{t-\tau_i(t,\mathbf{0})}^{t} -A_k(s)\,ds < 1,$$

and there exists $\alpha > 0$ such that

$$\frac{1}{t - t_0}\int_{t_0}^{t} -A_k(s)\,ds \geq \alpha, \qquad t > t_0 \geq 0, \qquad k = 1, \dots, m.$$

Then the trivial solution of (1) *is exponentially stable.*

Corollary 6. *Suppose (H1)–(H4) hold with $n = 1$, and $\tau_i(t, 0) = 0$ for all $t \geq 0$ and $i = 1, \ldots, m$. Then the trivial solution of* (1) *is exponentially stable if and only if there exists $\alpha > 0$ such that*

$$\sum_{i=1}^{m} \int_{t_0}^{t} A_i(s)\, ds \leq -\alpha(t - t_0), \qquad t > t_0 \geq 0.$$

Note this result for $m = 1$ was obtained in [11] assuming additional smoothness on the delay.

For the special case when the linearized system (2) is two-dimensional ($n = 2$), autonomous and has a single constant delay, we can use a condition of Hara and Sugie [16] to check uniform asymptotic stability, i.e., exponential stability of the trivial solution.

Corollary 7. *Suppose (H1)–(H4) hold with $n = 2$, $\tau_i(t, 0) = \tau$ for all $t \geq 0$ and $i = 1, \ldots, m$ and $\sum_{i=1}^{m} A_i(t) = A$ for $t \geq 0$. Then the trivial solution of* (1) *is exponentially stable if and only if*

$$2\sqrt{\det A}\,\sin\!\Big(\tau\sqrt{\det A}\Big) < -\operatorname{tr} A < \frac{\pi}{2\tau} + \frac{2\tau \det A}{\pi}$$

and

$$0 < \det A < \Big(\frac{\pi}{2\tau}\Big)^2.$$

Similar sufficient or sufficient and necessary conditions can be formulated for SD-DDEs by combining Theorem 2.4 or 2.5 and conditions from, e.g., [13], [17], [25], [26], [28].

3. **Applications for threshold-type delay equations.** In this section we consider the delay system

$$\dot{x}(t) = A_0(t)x(t) + \sum_{i=1}^{m} A_i(t)x(t - \tilde{\sigma}_i(t)), \qquad t \geq t_0, \tag{1}$$

where the delay functions $\tilde{\sigma}_i$ are defined by the threshold relations

$$\int_{t-\tilde{\sigma}_i(t)}^{t} f_i(t, u - t, x(t), x(u))\, du = 1 \tag{2}$$

for $i = 1, \ldots, m$, where f_i are given nonnegative scalar functions for $i = 1, \ldots, m$. Similar equations, so-called threshold-type delay equations, were frequently used in biological models ([2], [21], [29]), and were investigated, e.g., in [8], [9], [10], [12], [24], [27].

We show that under the following assumptions the delay functions $\tilde{\sigma}_i$ are well-defined and can be rewritten in the form of τ_i in (1), i.e., $\tilde{\sigma}_i$ depends on x_t, as well.

We assume $F_0 > 0$, $t_0 \geq 0$ are given, and let $r = 1/F_0$; moreover,

(A1) $A_i : [0, \infty) \to \mathbb{R}^n$ are continuous, and $|A_i(t)| \leq b_i$, $t \in [0, \infty)$, for $i = 0, 1, \ldots, m$;

(A2) $\varphi \in C$;

(A3) $f_i\colon [0,\infty)\times[-r,0]\times\mathbb{R}^n\times\mathbb{R}^n\to(0,\infty)$ continuous functions for $i=1,\dots,m$, and

$$f_i(t,s,u,v)\ge F_0,\qquad t\ge 0,\quad s\in[-r,0],\quad u,v\in\mathbb{R}^n;$$

(A4) there exist a constant $\gamma>0$ and a function $\tilde\omega:\ [0,\gamma)\to[0,\infty)$, such that $\tilde\omega(0)=0$ and

$$\int_{-r}^{0}\Big|f_i(t,s,\psi(0),\psi(s))-f_i(t,s,\mathbf{0},\mathbf{0})\Big|\,ds\le\tilde\omega(\|\psi\|),\qquad t\ge 0,\ \ \|\psi\|<\gamma,\ \ i=1,\dots,m;$$

(A5) the sets $\Big\{t\in[0,r]\colon \displaystyle\int_{-t}^{0} f_i(t+t_0,s,\mathbf{0},\mathbf{0})\,ds=1\Big\}$ have Lebesgue-measure 0 for $i=1,\dots,m$ and $t_0\ge 0$.

Introducing the new variable $s=u-t$ we can rewrite (2) as

$$\int_{t-\tilde\sigma_i(t)}^{t} f_i(t,u-t,x(t),x(u))\,du=\int_{-\tilde\sigma_i(t)}^{0} f_i(t,s,x(t),x(t+s))\,ds=1.$$

Note that such unique $\tilde\sigma_i(t)\in[0,r]$ exists, since by (A3)

$$\int_{-r}^{0} f_i(t,s,x(t),x(t+s))\,ds\ge F_0r=1,\qquad i=1,\dots,m.$$

Now we can reformulate the problem: We rewrite (1)-(2) in the form

$$\dot x(t)=A_0(t)x(t)+\sum_{i=1}^{m}A_i(t)x(t-\sigma_i(t,x_t)),\qquad t\ge t_0,\tag{3}$$

where the delay functions $\sigma_i\colon [0,\infty)\times C\to[0,r]$ are defined by the treshold relation

$$\int_{-\sigma_i(t,\psi)}^{0} f_i(t,s,\psi(0),\psi(s))\,ds=1.\tag{4}$$

The solution of (3) corresponds to an initial condition of the form (3).

In the case when $f_i(t,s,u,v)$ does not depend on s and v, i.e., has the form $f_i(t,s,u,v)=g_i(t,u)$, relation (4) reduces to

$$\sigma_i(t,\psi)=\frac{1}{g_i(t,\psi(0))}.$$

Therefore such formulation of threshold delays includes a large class of "usual" state-dependent delays.

As in Section 2, we associate the linear equation

$$\dot x(t)=A_0(t)x(t)+\sum_{i=1}^{m}A_i(t)x(t-\sigma_i(t,\mathbf{0})),\qquad t\ge t_0\tag{5}$$

to (3).

Theorem 2.4 has the folowing corollary.

Theorem 3.1. *Assume (A1)–(A5). Then the trivial solution of* (3) *is exponentially stable, if and only if the trivial solution of* (5) *is exponentially stable.*

Proof. It is enough to show that assumptions (A1)–(A5) imply assumptions (H1)–(H5) of Theorem 2.4.

Clearly, (A1) and (A2) are identical to (H1) and (H2), respectively.

We show that σ_i is a continuous function. Fix an arbitrary $\bar{t} \in [t_0, \infty)$ and $\bar{\psi} \in C$. Then

$$1 = \int_{-\sigma_i(\bar{t},\bar{\psi})}^{0} f_i(\bar{t}, s, \bar{\psi}(0), \bar{\psi}(s))\, ds = \int_{-\sigma_i(t,\psi)}^{0} f_i(t, s, \psi(0), \psi(s))\, ds,$$

and so

$$\int_{-\sigma_i(\bar{t},\bar{\psi})}^{-\sigma_i(t,\psi)} f_i(\bar{t}, s, \bar{\psi}(0), \bar{\psi}(s))\, ds = \int_{-\sigma_i(t,\psi)}^{0} [f_i(t, s, \psi(0), \psi(s)) - f_i(\bar{t}, s, \bar{\psi}(0), \bar{\psi}(s))]\, ds.$$

Consequently,

$$\begin{aligned} \left| \int_{-\sigma_i(\bar{t},\bar{\psi})}^{-\sigma_i(t,\psi)} f_i(\bar{t}, s, \bar{\psi}(0), \bar{\psi}(s))\, ds \right| &\leq \int_{-r}^{0} \Big| f_i(t, s, \psi(0), \psi(s)) - f_i(\bar{t}, s, \bar{\psi}(0), \bar{\psi}(s)) \Big|\, ds \\ &\to 0, \end{aligned}$$

as $|t - \bar{t}| + \|\psi - \bar{\psi}\| \to 0$. On the other hand, using (A3)

$$F_0 |\sigma_i(t,\psi) - \sigma_i(\bar{t},\bar{\psi})| \leq \left| \int_{-\sigma_i(\bar{t},\bar{\psi})}^{-\sigma_i(t,\psi)} f_i(\bar{t}, s, \bar{\psi}(0), \bar{\psi}(s))\, ds \right|.$$

Therefore

$$|\sigma_i(t,\psi) - \sigma_i(\bar{t},\bar{\psi})| \to 0 \qquad \text{as} \quad |t - \bar{t}| + \|\psi - \bar{\psi}\| \to 0,$$

and consequently, (H3) holds with $\tau_i = \sigma_i$.

We get by the computation above with $\bar{t} = t$ and $\bar{\psi} = \mathbf{0}$ that

$$|\sigma_i(t,\psi) - \sigma_i(t,\mathbf{0})| \leq \frac{1}{F_0} \int_{-r}^{0} \Big| f_i(t, s, \psi(0), \psi(s)) - f_i(t, s, \mathbf{0}, \mathbf{0}) \Big|\, ds.$$

Therefore (A4) immediately implies (H4).

To show (H5) first note that $t - \sigma_i(t + t_0, \mathbf{0}) = 0$ implies

$$\int_{-\sigma_i(t+t_0,\mathbf{0})}^{0} f_i(t + t_0, s, \mathbf{0}, \mathbf{0})\, ds = \int_{-t}^{0} f_i(t + t_0, s, \mathbf{0}, \mathbf{0})\, ds = 1.$$

Therefore (A5) implies (H5). □

Corollary 8. *Assume (A1)–(A4). Then the exponential stability of the trivial solution of* (5) *implies that for the trivial solution of* (3)*, as well.*

We note that Theorems 2.4 and 3.1 can be trivially combined for equations of the form

$$\dot{x}(t) = A_0(t)x(t) + \sum_{i=1}^{m} A_i(t)x(t - \tau_i(t, x_t)) + \sum_{i=1}^{k} B_i(t)x(t - \sigma_i(t, x_t)), \qquad t \geq t_0,$$

where the "explicit" delay functions τ_i satisfy (H3)-(H5), and the treshold delays σ_i defined by (4) satisfy (A3)-(A5).

4. **Proof of the Main Results.** The next lemma can be proved using Gronwall's inequality (see [11] for details in case of a similar equation).

Lemma 4.1. *Assume (H1)–(H3). Then any solution of* (1)–(3) *satisfies*

$$|x(t)| \le e^{b(t-t_0)}\|\varphi\|, \qquad t \ge t_0 \ge 0,$$

where b is defined by

$$b = \sum_{i=1}^{m} b_i. \tag{1}$$

We can rewrite (1) as

$$\dot{x}(t) = \sum_{i=1}^{m} A_i(t)x(t-\tau_i(t,\mathbf{0})) + f(t), \qquad t \ge t_0, \quad i = 1,\dots,n,$$

where

$$f(t) = \sum_{i=1}^{m} A_i(t)\Big(x(t-\tau_i(t,x_t)) - x(t-\tau_i(t,\mathbf{0}))\Big). \tag{2}$$

Using the variation of constants formula we get

$$x(t) = y(t) + \int_{t_0}^{t} V(t,s)f(s)\,ds, \qquad t \ge t_0, \tag{3}$$

where y is the solution of (2) corresponding to the initial condition (3).

We will need an estimate of f.

Lemma 4.2. *Assume (H1)–(H4), $t_0 \ge 0$, and suppose x is a solution of* (1)-(3) *satisfying $|x(t)| < \gamma$ for $t \ge t_0 - r$, where $\gamma > 0$ is defined in (H4). Then*

$$|f(t)| \le \begin{cases} b \max\limits_{u\in[t-2r,t]} |x(u)| \sum\limits_{i=1}^{m} b_i\omega_i(\|x_t\|), & t_0 + r < t, \\ 2be^{br}\|\varphi\|, & t \in [t_0, t_0 + r], \end{cases}$$

where b is defined by (1).

Proof. We introduce the notations

$$\eta_i(t) = \min\{t-\tau_i(t,\mathbf{0}), t-\tau_i(t,x_t)\} \quad \text{and} \quad \xi_i(t) = \max\{t-\tau_i(t,\mathbf{0}), t-\tau_i(t,x_t)\}. \tag{4}$$

Then

$$|x(t-\tau_i(t,\mathbf{0})) - x(t-\tau_i(t,x_t))| = |x(\xi_i(t)) - x(\eta_i(t))|.$$

First suppose $t > t_0 + r$. Then $t_0 \le \eta_i(t) \le \xi_i(t)$, and so

$$
\begin{aligned}
|f(t)| &= \Big|\sum_{i=1}^m A_i(t)\int_{\eta_i(t)}^{\xi_i(t)} \dot x(u)\,du\Big| \\
&\le \sum_{i=1}^m b_i \int_{\eta_i(t)}^{\xi_i(t)} \Big|\dot x(u)\Big|\,du \\
&= \sum_{i=1}^m b_i \int_{\eta_i(t)}^{\xi_i(t)} \Big|\sum_{k=1}^m A_k(u)x(u-\tau_k(u,x_u))\Big|\,du \\
&\le \sum_{i=1}^m b_i \sum_{k=1}^m b_k \int_{\eta_i(t)}^{\xi_i(t)} |x(u-\tau_k(u,x_u))|\,du \qquad (5)\\
&\le \sum_{i=1}^m b_i \sum_{k=1}^m b_k(\xi_i(t)-\eta_i(t)) \max_{u\in[t-2r,t]} |x(u)| \\
&= \sum_{i=1}^m b_i \sum_{k=1}^m b_k|\tau_i(t,x_t)-\tau_i(t,\mathbf{0})| \max_{u\in[t-2r,t]} |x(u)| \\
&\le b \max_{u\in[t-2r,t]} |x(u)| \sum_{i=1}^m b_i\omega_i(\|x_t\|).
\end{aligned}
$$

Now consider the case when $t \in [t_0, t_0 + r]$. We use Lemma 4.1 to estimate

$$
\begin{aligned}
|f(t)| &\le \sum_{i=1}^m b_i\Big(|x(\xi_i(t))| + |x(\eta_i(t))|\Big) \\
&\le \sum_{i=1}^m b_i\Big(e^{b\max\{\xi_i(t)-t_0,0\}} + e^{b\max\{\eta_i(t)-t_0,0\}}\Big)\|\varphi\| \\
&\le 2be^{br}\|\varphi\|.
\end{aligned}
$$

□

Proof of the sufficient part of Theorem 2.4 (proof of Theorem 2.5). First we assume that the trivial solution of (2) is exponentially stable with order α. Let K_1, m, K_2 and b be defined by (4), (7) and (1), respectively.

We first show that the trivial solution of (1) is stable. By assumption (H4), there exists $0 < \varepsilon_0 < \gamma$ such that

$$\frac{K_2 b}{\alpha}\sum_{i=1}^m b_i \max_{0\le u\le\varepsilon_0} \omega_i(u) < \frac{1}{3}.$$

Let $0 < \varepsilon < \varepsilon_0$ be arbitrary, and $\delta > 0$ is such that

$$\delta = \min\left\{\varepsilon, \frac{\varepsilon\alpha}{3(K_1\alpha + 2K_2 be^{br})}\right\}.$$

Fix an initial function satisfying $\|\varphi\| < \delta$, and let x be any solution of (1) corresponding to this initial function. Then $|\varphi(0)| < \delta \le \varepsilon$, therefore there exists $T > t_0$ such that $|x(t)| < \varepsilon$ for $t \in [t_0, T)$. Suppose $|x(T)| = \varepsilon$. First consider the case when $T \le t_0 + r$. Then it follows from the variation of constants formula (3) and

Lemma 4.2 that

$$\begin{aligned}|x(T)| &\leq |y(T)| + \int_{t_0}^{T} |V(T,s)||f(s)|\,ds \\ &\leq K_1 e^{-\alpha(T-t_0)}\|\varphi\| + \int_{t_0}^{T} K_2 e^{-\alpha(T-s)} 2be^{br}\|\varphi\|\,ds.\end{aligned}$$

Therefore,

$$\varepsilon \leq K_1 e^{-\alpha(T-t_0)}\delta + K_2 e^{-\alpha T} 2be^{br}\delta \int_{t_0}^{T} e^{\alpha s}\,ds \leq \Big(K_1 + \frac{2K_2 be^{br}}{\alpha}\Big)\delta \leq \frac{\varepsilon}{3},$$

which is a contradiction. Now suppose $T > t_0 + r$. Then Lemma 4.2 yields

$$\begin{aligned}|x(T)| &\leq |y(T)| + \int_{t_0}^{t_0+r} |V(T,s)||f(s)|\,ds + \int_{t_0+r}^{T} |V(T,s)||f(s)|\,ds \\ &\leq K_1 e^{-\alpha(T-t_0)}\|\varphi\| + \int_{t_0}^{t_0+r} K_2 e^{-\alpha(T-s)} 2be^{br}\|\varphi\|\,ds \\ &\quad + \int_{t_0+r}^{T} K_2 e^{-\alpha(T-s)} b \max_{u\in[s-2r,s]} |x(u)| \sum_{i=1}^{m} b_i \omega_i(\|x_s\|)\,ds,\end{aligned}$$

and so

$$\begin{aligned}\varepsilon &\leq K_1 e^{-\alpha(T-t_0)}\delta + K_2 e^{-\alpha T} 2be^{br}\delta \int_{t_0}^{t_0+r} e^{\alpha s}\,ds \\ &\quad + K_2 e^{-\alpha T} b\varepsilon \sum_{i=1}^{m} b_i \max_{0\leq u\leq \varepsilon} \omega_i(u) \int_{t_0+r}^{T} e^{\alpha s}\,ds \\ &\leq \Big(K_1 + \frac{2K_2 be^{br}}{\alpha}\Big)\delta + \frac{K_2 b\varepsilon}{\alpha} \sum_{i=1}^{m} b_i \max_{0\leq u\leq \varepsilon_0} \omega_i(u) \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3},\end{aligned}$$

which is a contradiction, again. Therefore $|x(t)| < \varepsilon$ is satisfied for all $t > t_0$, i.e., the trivial solution of (1) is stable, moreover, it is uniformly stable, since δ does not depend on t_0.

Let $0 < \beta < \alpha$ be arbitrary. Next we show that the trivial solution of (1) is exponentially stable with order β. Let $0 < \varepsilon < \gamma$ be such that the constant

$$M_1 := \frac{K_2 be^{2\beta r}}{\alpha - \beta} \sum_{i=1}^{m} b_i \max_{0\leq u\leq \varepsilon} \omega_i(u)$$

satisfies $M_1 < 1$, and $0 < \sigma \leq \varepsilon$ be such that $|x(t)| < \varepsilon$ for $t \geq t_0$ and for $\|\varphi\| < \sigma$. Fix any initial function satisfying $\|\varphi\| < \sigma$, and let x be any solution of (1) corresponding to the initial function φ.

First consider the case when $t \in [t_0, t_0 + r]$. Then multiplying the variation of constants formula by $e^{\beta(t-t_0)}$ we get

$$e^{\beta(t-t_0)}|x(t)| \leq K_1 e^{-(\alpha-\beta)(t-t_0)}\|\varphi\| + K_2 e^{\beta(t-t_0)} \int_{t_0}^{t} e^{-\alpha(t-s)}|f(s)|\,ds.$$

Introduce the function $z(t) := e^{\beta(t-t_0)}|x(t)|$, $t \ge t_0 - r$. With this notation and using Lemma 4.2 we have

$$z(t) \le K_1 e^{-(\alpha-\beta)(t-t_0)}\|\varphi\| + 2K_2 b e^{br} e^{\beta(t-t_0)-\alpha t}\|\varphi\| \int_{t_0}^{t_0+r} e^{\alpha s}\, ds, \qquad t \in [t_0, t_0+r].$$

Define $M_2 = \max\{1, K_1 + 2K_2 b e^{(\alpha+b)r}/\alpha\}$. Then it follows

$$z(t) \le M_2\|\varphi\|, \qquad t \in [t_0, t_0+r].$$

Next suppose $t > t_0 + r$. Then

$$\begin{aligned} z(t) \quad &\le \quad K_1 e^{-(\alpha-\beta)(t-t_0)}\|\varphi\| \\ &\quad + K_2 e^{\beta(t-t_0)}\left(\int_{t_0}^{t_0+r} e^{-\alpha(t-s)}|f(s)|\, ds + \int_{t_0+r}^{t} e^{-\alpha(t-s)}|f(s)|\, ds\right). \end{aligned}$$

The first two terms can be estimated as before, in the last term we apply inequality (5):

$$\begin{aligned} z(t) \quad &\le \quad M_2\|\varphi\| + K_2 e^{\beta(t-t_0)-\alpha t}\int_{t_0+r}^{t} e^{\alpha s}\sum_{i=1}^{m} b_i \sum_{k=1}^{m} b_k \int_{\eta_i(s)}^{\xi_i(s)} |x(u-\tau_k(u,x_u))|\, du\, ds \\ &\le \quad M_2\|\varphi\| + K_2 e^{-(\alpha-\beta)t} \\ &\quad \times \int_{t_0+r}^{t} e^{\alpha s}\sum_{i=1}^{m} b_i \sum_{k=1}^{m} b_k \int_{\eta_i(s)}^{\xi_i(s)} e^{-\beta(u-\tau_k(u,x_u))} z(u-\tau_k(u,x_u))\, du\, ds \\ &\le \quad M_2\|\varphi\| + K_2 b e^{\beta r} e^{-(\alpha-\beta)t} \max_{u\in[t_0-r,t]} z(u) \int_{t_0+r}^{t} e^{\alpha s}\sum_{i=1}^{m} b_i \int_{\eta_i(s)}^{\xi_i(s)} e^{-\beta u}\, du\, ds. \end{aligned}$$

Using that $\eta_i(s) \ge s - r$, we get

$$\begin{aligned} z(t) \quad &\le \quad M_2\|\varphi\| + K_2 b e^{\beta r} e^{-(\alpha-\beta)t} \max_{u\in[t_0-r,t]} z(u) \\ &\quad \times \int_{t_0+r}^{t} e^{\alpha s}\sum_{i=1}^{m} b_i e^{-\beta(s-r)}(\xi_i(s)-\eta_i(s))\, ds \\ &\le \quad M_2\|\varphi\| \\ &\quad + K_2 b e^{2\beta r} e^{-(\alpha-\beta)t} \max_{u\in[t_0-r,t]} z(u) \int_{t_0+r}^{t} e^{(\alpha-\beta)s}\sum_{i=1}^{m} b_i \omega_i(\|x_s\|)\, ds. \end{aligned} \tag{6}$$

Using the definition of M_1 and that $|x(t)| < \varepsilon$ for all $t \ge 0$, we obtain

$$\begin{aligned} z(t) \quad &\le \quad M_2\|\varphi\| + K_2 b e^{2\beta r} e^{-(\alpha-\beta)t} \max_{u\in[t_0-r,t]} z(u) \sum_{i=1}^{m} b_i \max_{0\le u\le\varepsilon} \omega_i(u) \int_{t_0+r}^{t} e^{(\alpha-\beta)s}\, ds \\ &\le \quad M_2\|\varphi\| + M_1 \max_{u\in[t_0-r,t]} z(u). \end{aligned} \tag{7}$$

The right-hand-side of (7) is monotone in t, $M_2 \ge 1$ and $z(t) \le |\varphi(t-t_0)| \le \|\varphi\|$ for $t \in [t_0 - r, t_0]$, therefore (7) yields

$$\max_{u\in[t_0-r,t]} z(u) \le M_2\|\varphi\| + M_1 \max_{u\in[t_0-r,t]} z(u). \tag{8}$$

Hence $z(t) \le \frac{M_2}{1-M_1}\|\varphi\|$, and consequently,

$$|x(t)| \le \frac{M_2}{1-M_1} e^{-\beta(t-t_0)}\|\varphi\|, \qquad t \ge t_0, \quad \|\varphi\| < \sigma,$$

which completes the proof. □

For the rest of the paper $\widetilde{\varphi}\colon [-r,\infty) \to \mathbb{R}^n$ denotes the extension of $\varphi\colon [-r,0] \to \mathbb{R}^n$ to the right by the zero vector:

$$\widetilde{\varphi}(s) = \begin{cases} \varphi(s), & s \in [-r,0], \\ 0, & t > 0. \end{cases}$$

The proof of the necessary part of Theorem 2.4 will be based on the next lemma, where we prove the existence of a certain auxiliary function under condition (H5).

Lemma 4.3. *Suppose (H3)–(H5). Then for any $\varepsilon > 0$ and any $t_0 \in \mathbb{R}$ there exists a continuously differentiable scalar function $h_{\varepsilon,t_0}\colon [-r,0] \to [0,1]$ such that*

$$\int_0^r \widetilde{h}_{\varepsilon,t_0}(s - \tau_i(s+t_0,\mathbf{0}))\,ds \le \varepsilon, \qquad i = 1,\dots,m,$$

and $\|h_{\varepsilon,t_0}\| = h_{\varepsilon,t_0}(0) = 1$.

Proof. The sets

$$E_i = \{s \in [0,r]\colon\ s - \tau_i(s+t_0,\mathbf{0}) = 0\}, \qquad F_i = \{s \in [0,r]\colon\ s - \tau_i(s+t_0,\mathbf{0}) < 0\}$$

and

$$G_{i,k} = \Big\{s \in [0,r]\colon\ s - \tau_i(s+t_0,\mathbf{0}) < -\frac{1}{k}\Big\}, \qquad k = 1,2,\dots$$

are Lebesgue measurable for $i = 1,\dots,m$ and $k = 1,2,\dots$, moreover

$$G_{i,1} \subset G_{i,2} \subset G_{i,3} \subset \cdots \quad \text{and} \quad \bigcup_{k=1}^{\infty} G_{i,k} = F_i.$$

Therefore

$$\lim_{k\to\infty} \mu(G_{i,k}) = \mu(F_i), \qquad i = 1,\dots,m,$$

where μ denotes the Lebesgue-measure. Fix an arbitrary $0 < \varepsilon < 2r$, and let $k_0 > 1/r$ be such that

$$0 \le \mu(F_i) - \mu(G_{i,k_0}) < \frac{\varepsilon}{2}, \qquad i = 1,\dots,m.$$

Let $h_{\varepsilon,t_0}\colon [-r,0] \to \mathbb{R}$ be a strictly monotone increasing continuously differentiable scalar function satisfying

$$h_{\varepsilon,t_0}(-r) = 0, \qquad h_{\varepsilon,t_0}\left(-\frac{1}{k_0}\right) = \frac{\varepsilon}{2r} \quad \text{and} \quad h_{\varepsilon,t_0}(0) = 1.$$

Since the set E_i has measure 0, we get

$$\begin{aligned}
&\int_0^r \widetilde{h}_{\varepsilon,t_0}(s - \tau_i(s+t_0,\mathbf{0}))\,ds \\
&\quad = \int_{E_i} h_{\varepsilon,t_0}(s - \tau_i(s+t_0,\mathbf{0}))\,ds + \int_{F_i} h_{\varepsilon,t_0}(s - \tau_i(s+t_0,\mathbf{0}))\,ds \\
&\quad = \int_{G_{i,k_0}} h_{\varepsilon,t_0}(s - \tau_i(s+t_0,\mathbf{0}))\,ds + \int_{F_i\setminus G_{i,k_0}} h_{\varepsilon,t_0}(s - \tau_i(s+t_0,\mathbf{0}))\,ds.
\end{aligned}$$

Since

$$h_{\varepsilon,t_0}(s - \tau_i(s+t_0,\mathbf{0})) < h_{\varepsilon,t_0}\left(-\frac{1}{k_0}\right) = \frac{\varepsilon}{2r} \qquad \text{for } s \in G_{i,k_0},$$

and $0 \le h_{\varepsilon,t_0}(t) \le 1$ for $t \in [-r, 0]$, we get

$$\int_0^r h_{\varepsilon,t_0}(s - \tau_i(s + t_0, \mathbf{0}))\, ds \le \frac{\varepsilon}{2r}\mu(G_{i,k_0}) + \mu(F_i \setminus G_{i,k_0}) \le \frac{\varepsilon}{2r} r + \frac{\varepsilon}{2} = \varepsilon.$$

□

The next lemma gives an estimate of the derivative of an exponentially decaying solution of (1).

Lemma 4.4. *Suppose (H1)–(H3), $|x(t)| \le K e^{-\alpha(t-t_0)}\|\varphi\|$ for $t \ge t_0$, assuming $\|\varphi\| < \sigma$ where $K \ge 1$, and b be defined by (1). Then*

$$|\dot{x}(t)| \le bK\|\varphi\|, \qquad t \ge t_0.$$

Proof. It follows from the assumptions and from (1)

$$|\dot{x}(t)| \le \sum_{i=1}^m b_i |x(t - \tau_i(t, x_t))| \le \sum_{i=1}^m b_i K e^{-\alpha \max\{t - \tau_i(t,x_t) - t_0, 0\}}\|\varphi\| \le bK\|\varphi\|,$$

for $t \ge t_0$. □

Proof of the necessary part of Theorem 2.4. It is known (see, e.g., [15]) that the trivial solution of (2) is exponentially stable with order α if and only if the fundamental solution of (2) (i.e., the solution of the initial value problem (5)-(6)) satisfies an estimate of the form (7) for some positive constants K_2. Now suppose the trivial solution of (1) is exponentially stable with order α, i.e., there exist K and $\sigma > 0$ independent of t_0 such that any solution of (1) satisfies $|x(t)| \le K e^{-\alpha(t-t_0)}\|\varphi\|$ for $t \ge t_0$, assuming $\|\varphi\| < \sigma$. We show, in two steps, that the fundamental solution of (2) satisfies (7) with $K_2 = K$, therefore the trivial solution of (2) is exponentially stable with order α.

Step 1. First we show that for any fixed $t_0 \ge 0$ and $0 < \varepsilon < 1$

$$|V(t, t_0)| \le K e^{-\alpha(t-t_0)} + \frac{3}{4}\varepsilon \max_{t_0 \le s \le t} |V(t, s)|, \qquad t \ge t_0 \tag{9}$$

holds. Let $\delta_0 > 0$ be such that

$$\frac{bKe^{\alpha r}}{\alpha} \sum_{i=1}^m b_i \max_{0 \le u \le \delta_0} \omega_i(u) < \frac{\varepsilon}{4} \qquad \text{and} \qquad \delta_0 < \left\{\frac{\varepsilon}{4b}, \gamma\right\}. \tag{10}$$

Fix a continuously differentiable initial function $h_{\delta_0,t_0} \colon [-r, 0] \to \mathbb{R}$ defined by Lemma 4.3, and let $M = \|\dot{h}_{\delta_0,t_0}\|$. Let $\delta_1 > 0$ be such that

$$2(M + bK) r \sum_{i=1}^m b_i \max_{0 \le u \le \delta_1} \omega_i(u) < \frac{\varepsilon}{4}, \qquad \delta_1 < \delta_0. \tag{11}$$

Finally, let $\delta > 0$ be such that

$$\delta < \min\left\{\sigma, \frac{\delta_1}{K}\right\}. \tag{12}$$

Let $a \in \mathbb{R}^n$ with $|a| = \delta$ be fixed, and let $\varphi(t) = h_{\delta_0,t_0}(t)a$. Then $\varphi(0) = a$ and $\|\varphi\| = |a| = \delta$. Let x and y be a solution of (1) and (2), respectively, both corresponding to this initial function, φ.

Then the variation-of-constants formula yields

$$x(t) = y(t) + \int_{t_0}^{t} V(t,s) f(s)\, ds, \qquad t \geq t_0,$$

where f is defined by (2). Theorem 1.2 from Section 6.1 of [15] (see also [1]) yields the following relation

$$y(t) = V(t,t_0)\varphi(0) + \int_{t_0}^{t_0+r} V(t,s) \sum_{i=1}^{m} A_i(s)\widetilde{\varphi}(s - \tau_i(s,\mathbf{0}) - t_0)\, ds, \qquad t \geq t_0. \quad (13)$$

Then

$$\begin{aligned} V(t,t_0)\varphi(0) &= x(t) - \int_0^r V(t,s+t_0) \sum_{i=1}^{m} A_i(s+t_0)\widetilde{\varphi}(s - \tau_i(s+t_0,\mathbf{0}))\, ds \\ &\quad - \int_{t_0}^{t} V(t,s) f(s)\, ds, \end{aligned}$$

and so for $t > t_0 + r$ it follows

$$\begin{aligned} |V(t,t_0)a| &\leq Ke^{-\alpha(t-t_0)}\|\varphi\| + \int_0^r |V(t,s+t_0)| \sum_{i=1}^{m} b_i |\widetilde{\varphi}(s - \tau_i(s+t_0,\mathbf{0}))|\, ds \\ &\quad + \int_{t_0}^{t_0+r} |V(t,s)||f(s)|\, ds + \int_{t_0+r}^{t} |V(t,s)||f(s)|\, ds. \end{aligned} \quad (14)$$

We denote the last three integrals by I_1, I_2 and I_3, respectively, and we estimate them separately. Using Lemma 4.3 and (10) we get

$$\begin{aligned} I_1 &\leq \max_{t_0 \leq s \leq t_0+r} |V(t,s)| \sum_{i=1}^{m} b_i \int_0^r \widetilde{h}_{\delta_0,t_0}(s - \tau_i(s+t_0,\mathbf{0}))|a|\, ds \\ &\leq b\delta_0 |a| \max_{t_0 \leq s \leq t_0+r} |V(t,s)| \\ &\leq \frac{\varepsilon}{4}|a| \max_{t_0 \leq s \leq t_0+r} |V(t,s)|. \end{aligned} \quad (15)$$

Relations (10), (11) and (12), $|x(t)| \leq K\|\varphi\| = K\delta < \delta_1 < \delta_0$ for $t \geq t_0$, $\|\varphi\| = |a|$ and Lemma 4.2 yield

$$\begin{aligned} I_3 &\leq \max_{t_0+r \leq s \leq t} |V(t,s)| b \int_{t_0+r}^{t} \max_{s-2r \leq u \leq s} |x(u)| \sum_{i=1}^{m} b_i \omega_i(\|x_s\|)\, ds \\ &\leq \max_{t_0+r \leq s \leq t} |V(t,s)| b \int_{t_0+r}^{t} Ke^{-\alpha(s-2r-t_0)}\|\varphi\| \sum_{i=1}^{m} b_i \max_{0 \leq u \leq \delta_0} \omega_i(u)\, ds \\ &= \max_{t_0+r \leq s \leq t} |V(t,s)| bKe^{\alpha(2r+t_0)}|a| \sum_{i=1}^{m} b_i \max_{0 \leq u \leq \delta_0} \omega_i(u) \int_{t_0+r}^{t} e^{-\alpha s}\, ds \\ &\leq \max_{t_0+r \leq s \leq t} |V(t,s)||a| \frac{bKe^{\alpha r}}{\alpha} \sum_{i=1}^{m} b_i \max_{0 \leq u \leq \delta_0} \omega_i(u) \\ &\leq \frac{\varepsilon}{4}|a| \max_{t_0+r \leq s \leq t} |V(t,s)|. \end{aligned} \quad (16)$$

To estimate I_2 first consider

$$I_2 \leq \max_{t_0 \leq s \leq t_0+r} |V(t,s)| \sum_{i=1}^{m} b_i \int_{t_0}^{t_0+r} |x(\xi_i(s)) - x(\eta_i(s))|\, ds,$$

where we used (2), and ξ_i and η_i are defined by (4). We divide the interval $[t_0, t_0+r]$ into three disjoint sets:

$$A_i = \{s \in [t_0, t_0+r]\colon\ \xi_i(s) < 0\}, \qquad B_i = \{s \in [t_0, t_0+r]\colon\ 0 < \eta_i(s)\},$$

and

$$C_i = \{s \in [t_0, t_0+r]\colon\ \eta_i(s) \le 0 \le \xi_i(s)\}.$$

We estimate the integral separately on the sets A_i, B_i and C_i. Using the Mean Value Theorem, (H4) and the definitions of M and δ_1 we get

$$\begin{aligned} |h_{\delta_0,t_0}(\xi_i(s)) - h_{\delta_0,t_0}(\eta_i(s))| &\le M(\xi_i(s)) - \eta_i(s)) \\ &= M|\tau_i(s, x_s) - \tau_i(s, \mathbf{0})| \\ &\le M \max_{0\le u\le\delta_1} \omega_i(u) \end{aligned}$$

for $s \in [t_0, t_0+r]$, therefore

$$\begin{aligned} \int_{A_i} |x(\xi_i(s)) - x(\eta_i(s))|\,ds &= \int_{A_i} |h_{\delta_0,t_0}(\xi_i(s)) - h_{\delta_0,t_0}(\eta_i(s))||a|\,ds \\ &\le M|a| \max_{0\le u\le\delta_1} \omega_i(u)\mu(A_i) \\ &\le M|a|r \max_{0\le u\le\delta_1} \omega_i(u). \end{aligned}$$

Lemma 4.4, the Mean Value Theorem, (H4) and $\|\varphi\| = |a|$ imply

$$\begin{aligned} \int_{B_i} |x(\xi_i(s)) - x(\eta_i(s))|\,ds &\le bK\|\varphi\| \int_{B_i} (\xi_i(s) - \eta_i(s))\,ds \\ &\le bK|a| \max_{0\le u\le\delta_1} \omega_i(u)\mu(B_i) \\ &\le bK|a|r \max_{0\le u\le\delta_1} \omega_i(u). \end{aligned}$$

Now the previous two cases yield

$$\begin{aligned} &\int_{C_i} |x(\xi_i(s)) - x(\eta_i(s))|\,ds \\ &\qquad \le \int_{C_i} |x(\xi_i(s)) - x(0)|\,ds + \int_{C_i} |h_{\delta_0,t_0}(0) - h_{\delta_0,t_0}(\eta_i(s))||a|\,ds \\ &\qquad \le bK|a| \int_{C_i} \xi_i(s)\,ds + M|a| \int_{C_i} -\eta_i(s)\,ds \\ &\qquad \le bK|a| \int_{C_i} (\xi_i(s) - \eta_i(s))\,ds + M|a| \int_{C_i} (\xi_i(s) - \eta_i(s))\,ds \\ &\qquad \le |a|(bK+M)r \max_{0\le u\le\delta_1} \omega_i(u). \end{aligned}$$

Therefore the three cases and (11) give

$$\begin{aligned} I_2 &\le \max_{t_0\le s\le t_0+r} |V(t,s)||a|2(M+bK)r \sum_{i=1}^{m} b_i \max_{0\le u\le\delta_1} \omega_i(u) \\ &\le \frac{\varepsilon}{4}|a| \max_{t_0\le s\le t_0+r} |V(t,s)|. \end{aligned} \tag{17}$$

Combining (14) with (15), (16) and (17) we get for $t > t_0 + r$ that

$$\begin{aligned} |V(t,t_0)a| &\leq Ke^{-\alpha(t-t_0)}|a| + \frac{\varepsilon}{2}|a| \max_{t_0 \leq s \leq t_0+r} |V(t,s)| + \frac{\varepsilon}{4}|a| \max_{t_0+r \leq s \leq t} |V(t,s)| \\ &\leq Ke^{-\alpha(t-t_0)}|a| + \frac{3\varepsilon}{4}|a| \max_{t_0 \leq s \leq t} |V(t,s)|. \end{aligned} \tag{18}$$

It is easy to see that (18) holds for $t \in [t_0, t_0 + r]$, as well. Since (18) holds for all $a \in \mathbb{R}^n$ with $|a| = \delta$, it implies (9).

Step 2: Now we show $|V(t,u)| \leq Ke^{-\alpha(t-u)}$ holds for $t \geq u \geq 0$.

Let $0 \leq u \leq \bar{s} \leq t$. Applying (9) with $t_0 = \bar{s}$ and using $0 < \varepsilon < 1$, we get

$$|V(t,\bar{s})| \leq K + \frac{3}{4} \max_{\bar{s} \leq s \leq t} |V(t,s)| \leq K + \frac{3}{4} \max_{u \leq s \leq t} |V(t,s)|.$$

Taking the maximum of the left-hand-side for $u \leq \bar{s} \leq t$ we get

$$\max_{u \leq s \leq t} |V(t,s)| \leq 4K.$$

Substituting this back to (9) with $t_0 = u$ we get

$$|V(t,u)| \leq Ke^{-\alpha(t-u)} + 3K\varepsilon.$$

Since ε was arbitrary small, we get that $|V(t,u)| \leq Ke^{-\alpha(t-u)}$ holds for $t \geq u \geq 0$, which yields that the trivial solution of (2) is exponentially stable with order α. $\square$

Proof of Theorem 2.6. We use all the notations introduced in the proof of Theorem 2.5 (the sufficient part of Theorem 2.4).

First note that condition (8) yields that $\mathcal{U}$ is not empty. Let $0 < \sigma < \sigma_0$ be fixed. Then there exists $0 < c$ such that

$$M_3 := \frac{K_2 b e^{2\alpha r}}{\alpha} \sum_{i=1}^m b_i \int_0^c \frac{\omega_i(u)}{u}\, du < 1$$

and

$$\sigma \leq \frac{c}{M_2} \left(1 - \frac{K_2 b e^{2\alpha r}}{\alpha} \sum_{i=1}^m b_i \int_0^c \frac{\omega_i(u)}{u}\, du \right) \leq c.$$

In the last estimate we used that $M_2 \geq 1$. Suppose $\|\varphi\| < \sigma$.

If we look at the proof of Theorem 2.5, we can easily see that for the derivation of (6) we have not used that $\beta < \alpha$, it holds with $\beta = \alpha$, as well, i.e.,

$$z(t) \leq M_2\|\varphi\| + K_2 b e^{2\alpha r} \max_{u \in [t_0-r,t]} z(u) \left| \int_{t_0+r}^t \sum_{i=1}^m b_i \omega_i(\|x_s\|)\, ds \right|, \qquad t \geq t_0 \geq 0, \tag{19}$$

where $z(t) = e^{\alpha(t-t_0)}|x(t)|$. Since

$$|x_s(u)| = |x(s+u)| = e^{-\alpha(s+u-t_0)} e^{\alpha(s+u-t_0)} |x(s+u)| = e^{-\alpha(s+u-t_0)} z(s+u)$$

for $u \in [-r, 0]$, it follows

$$\|x_s\| \leq e^{\alpha(t_0+r)} e^{-\alpha s} \|z_s\|.$$

Consequently, by the monotonicity of ω_i

$$z(t) \leq M_2\|\varphi\| + K_2 b e^{2\alpha r} \max_{u \in [t_0-r,t]} z(u) \left| \int_{t_0+r}^t \sum_{i=1}^m b_i \omega_i \left(e^{\alpha(t_0+r)} e^{-\alpha s} \|z_s\| \right) ds \right|.$$

Since $z(s) = e^{\alpha(s-t_0)}|\varphi(s-t_0)| \le \|\varphi\| < \sigma \le c$ for $s \in [t_0 - r, t_0]$, it follows $z(t) < c$ for t close enough to t_0. Suppose there exists $T > t_0$ such that $z(t) < c$ for $t \in [t_0 - r, T)$ and $z(T) = c$. Then

$$c \le M_2\|\varphi\| + K_2 b e^{2\alpha r} c \sum_{i=1}^{m} b_i \int_{t_0+r}^{T} \omega_i\Big(e^{\alpha(t_0+r)} e^{-\alpha s} c\Big)\, ds.$$

Introducing $N_1 = e^{\alpha(t_0+r)}$, the new variable $u = N_1 c e^{-\alpha s}$ and the definition of σ, we get

$$\begin{aligned} c &\le M_2\|\varphi\| + \frac{K_2 b e^{2\alpha r} c}{\alpha} \sum_{i=1}^{m} b_i \int_{N_1 c e^{-\alpha T}}^{N_1 c e^{-\alpha(t_0+r)}} \frac{\omega_i(u)}{u}\, du \\ &\le M_2\|\varphi\| + \frac{K_2 b e^{2\alpha r} c}{\alpha} \sum_{i=1}^{m} b_i \int_0^c \frac{\omega_i(u)}{u}\, du \\ &< M_2\sigma + \frac{K_2 b e^{2\alpha r} c}{\alpha} \sum_{i=1}^{m} b_i \int_0^c \frac{\omega_i(u)}{u}\, du \\ &\le c, \end{aligned}$$

where we used the definition of σ at the last estimate. Therefore $z(t) < c$ for all $t \ge t_0 \ge 0$. Then it follows from (19) repeating the above argument that

$$\begin{aligned} z(t) &\le M_2\|\varphi\| + K_2 b e^{2\alpha r} \max_{u\in[t_0-r,t]} z(u) \left| \int_{t_0+r}^{t} \sum_{i=1}^{m} b_i \omega_i(N_1 c e^{-\alpha s})\, ds \right| \\ &\le M_2\|\varphi\| + K_2 b e^{2\alpha r} \max_{u\in[t_0-r,t]} z(u) \sum_{i=1}^{m} b_i \int_0^c \frac{\omega_i(u)}{u}\, du, \qquad t \ge t_0 \ge 0, \end{aligned}$$

and therefore

$$\max_{u\in[t_0-r,t]} z(u) \le M_2\|\varphi\| + M_3 \max_{u\in[t_0-r,t]} z(u).$$

It implies

$$e^{\alpha(t-t_0)}|x(t)| \le \max_{u\in[t_0-r,t]} z(u) \le \frac{M_2}{1-M_3}\|\varphi\|, \qquad t \ge t_0 \ge 0,$$

i.e., the statement of the theorem holds with $K = \frac{M_2}{1-M_3}$. □

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