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# STATE DEPENDENT REGENERATIVE DELAY IN MILLING PROCESSES

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## ABSTRACT

*Traditional models of regenerative machine tool chatter use constant time delays assuming that the period between two subsequent cuts is a constant determined definitely by the spindle speed. These models result in delay-differential equations with constant time delay.*

*If the vibrations of the tool relative to the workpiece are also included in the surface regeneration model, then the resulted time delay is not constant, but it depends on the actual and a delayed position of the tool. In this case, the governing equation is a delay-differential equation with state dependent time delay. Equations with state dependent delays can not be linearized in the traditional sense, but there exists linear equations that can be associated to them. This way, the local behavior of the system with state dependent delays can be investigated.*

*In this study, a two degree of freedom model is presented for milling process. A thorough modeling of the regeneration effect results in the governing delay-differential equation with state dependent time delay. It is shown through the linearization of the nonlinear equation that an additional term arises in the linearized equation of motion due to the state-dependency of the time delay.*

## 1 INTRODUCTION

The rapid development of machining technology during the past decade and the commercialization of reliable high-speed machining systems has driven the need for thorough dynamical investigation of high-speed cutting processes. One important phenomena that limits the productivity of machining is the development of self-excited vibrations, also known as machine tool chatter. The work of Tlusty [1] and Tobias [2] led to the development of the regenerative machine tool chatter theory. The basis of regenerative cutting model is that either the tool, or the workpiece or both are flexible and the chip thickness varies due to the relative vibrations of the tool and the workpiece. The tool cuts the surface that was formed in the precious cut, and the chip thickness is determined by the current and a previous position of the tool/workpiece. In standard models, the time delay between two succeeding cuts is considered to be constant, it is equal to the period of the workpiece rotation in turning, or to the tooth passing period in milling. Due to the regenerative effect, cutting models are described by delayed differential equations (DDEs). DDEs have infinite dimensional state spaces, therefore their stability analysis is not trivial and closed form stability criteria often can not be given.

In milling processes, regenerative delay is determined by the

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rotation of the tool. For a tool with equally spaced teeth rotating with constant spindle speed, the time delay can be approximated by the rotation period of the tool over the number of teeth. The corresponding mathematical model is a DDE with periodic co-efficients and with a single, constant time delay, where both the period of the coefficients and the time delay are equal to the tooth passing period. Stability analysis of these systems is usually performed by numerical simulations [3], [4], and by different analytical techniques [5]– [12]. Models with constant time delay provide good capture of the regenerative dynamics, and can be used to obtain linear stability properties.

More realistic models include the feed motion and the consequent trochoidal path of the cutter tooth. In this case, the time delay between the succeeding cuts is not constant, it changes periodically in time. As one of the first efforts in this direction, Balachandran and Zhao [3], [13] used two different time delays in their models, one along the feed direction, the other along the perpendicular direction. Due to the feed, the delay along the feed direction is a bit smaller than the delay along the perpendicular direction. This model resulted in a periodic DDE with two constant time delay.

In their recent paper, Long and Balachandran [14] analyzed the feed rate effect on the regenerative delay. They pointed out that the delay is time dependent, since the cutting teeth are not exactly at the same angular position at the present and at the preceding cut. The resulted model is a periodic DDE with time periodic time delay. Long and Balachandran approximated this system by a DDE with constant time delay, and derived stability charts using the semi-discretization method [15], [16].

Time periodic delays also arise in models of varying spindle speed machining [17]– [19]. Stability analysis of systems with periodic delay is more complicated than that of systems with constant time delay, still there are numerical algorithms that can be used to perform these computations. An efficient technique to analyze DDEs with time periodic delay is the semi-discretization method, as it was shown in [19] for varying spindle speed turning.

If the regeneration process is modeled more accurately, then the vibration of the tool is also included in the regeneration model. The vibration of the tool superimposes on the trochoidal path of the teeth, and affects the time delay in the regeneration process. This results in a DDE with state-dependent delay (SD-DDE), i.e., the delay itself depends on the present and on the delayed states. The analysis of SD-DDEs is a special and recently developing research area in mathematics [20]– [24]. However, in engineering practice, SD-DDEs are rarely used since the appropriate mathematical tools, like linearization techniques, have just been developed recently by mathematicians, and these new results were not adopted by engineering yet.

SD-DDEs are nonlinear systems, since the delay in the argument of the state depends on the state itself. Linearization of SD-DDEs corresponds to a kind of perturbation technique: we are looking for a linear DDE for that the original system can be considered as a perturbation. The linearized system is a DDE with constant (or time dependent) delay. For example, consider the autonomous SD-DDE

$$\dot{x}(t) = x(t - (\tau_0 + x(t))). \qquad (1)$$

This is a nonlinear equation due to the state-dependent time delay $\tau(x(t)) = \tau_0 + x(t)$. The DDE

$$\dot{y}(t) = y(t - \tau_0) \qquad (2)$$

with constant time delay is a linear system that can be considered as a variational system of Eq. (1) around the equilibrium $x = 0$. In this sense, linearization means that if the $y \equiv 0$ solution of Eq. 2 is stable, then it follows that the $x \equiv 0$ solution of Eq. 1 is stable too. Linearization technique for general autonomous SD-DDEs was given by Hartung and Turi [21] and for time periodic SD-DDEs by Hartung [24].

Recently, Insperger et al. [25] showed that the accurate model of turning process results in an autonomous SD-DDE. They used the linearization technique of autonomous SD-DDEs according to Hartung and Turi [21]. They showed that the associated linear equation is not identical to the linear DDE with constant time delay used in standard turning models, but an additional term (with a relatively small multiplier) arises in the equation. This means that the incorporation of the state-dependency of the regenerative delay in the autonomous turning model slightly affects the linear stability properties.

In this paper, it is shown that an accurate modeling of the regenerative effect in milling results in a time periodic and state-dependent delay in the model equation. The corresponding time periodic SD-DDE is linearized according to Hartung [24]. It is shown, that an additional term arises in the linearized equation due to the state-dependency of the delay, that is, the state-dependent time delay affects linear stability properties.

## 2 MECHANICAL MODEL

The 2 DOF model of milling process is considered according to Figure 1. The tool is assumed to be flexible that experiences bending oscillations in directions $x$ and $y$, while the workpiece is assumed to be stiff. The 2 DOF oscillator is excited by the cutting force $\mathbf{F}(t)$. For the sake of simplicity, we assume a symmetric tool, that is, the mass $m$, damping $c$ and stiffness $k$ parameters are equal for both $x$ and $y$ directions. Note, however, that the forthcoming analysis is the same for the general, non-symmetric case. The governing equation reads

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_x(t), \qquad (3)$$
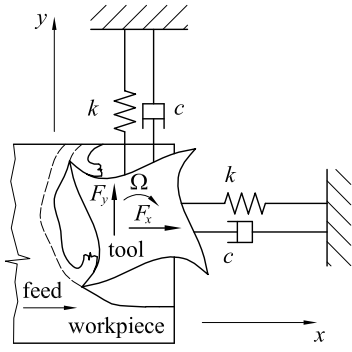$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = F_y(t). \qquad (4)$$

Figure 1.   Mechanical model.

According to the standard 2 DOF model of milling, the force components $F_x(t)$ and $F_y(t)$ can be given as

$$F_x(t) = \sum_{j=1}^{N} \left( wg(\varphi_j(t))h_j^q \right.$$
$$\left. \times \left( K_t \cos(\varphi_j(t)) + K_n \sin(\varphi_j(t)) \right) \right), \quad (5)$$

$$F_y(t) = \sum_{j=1}^{N} \left( wg(\varphi_j(t))h_j^q \right.$$
$$\left. \times \left( -K_t \sin(\varphi_j(t)) + K_n \cos(\varphi_j(t)) \right) \right), \quad (6)$$

where, $N$ is the number of the teeth, $w$ is the depth of cut,

$$\varphi_j(t) = -\Omega t + (j-1)\vartheta \quad (7)$$

is the angular position of the $j^{\text{th}}$ tooth at time $t$, $\Omega$ is the spindle speed of the tool in [rad/s], $\vartheta = 2\pi/N$ is the pitch angle of the tool, $K_t$ and $K_n$ are the tangential and the normal cutting coefficients, $h_j$ is the chip thickness cut by the $j^{\text{th}}$ tooth and exponent $q$ is a constant ($q = 0.8$ is a typical value for this parameter). The function $g$ is a screen function, it is equal to 1, if the $j^{\text{th}}$ tooth is active, and it is 0 if not:

$$g(\varphi_j(t)) = \begin{cases} 1 \text{ if } & \varphi_{\text{enter}} < \varphi_j(t) < \varphi_{\text{exit}} \\ 0 & \text{otherwise} \end{cases}, \quad (8)$$

where $\varphi_{\text{enter}}$ and $\varphi_{\text{exit}}$ are the angles where the teeth enter and exit the cut, respectively (see Figure 2).

In this model, it is assumed that the contact between the cutting teeth and the workpiece is determined solely by the rotation
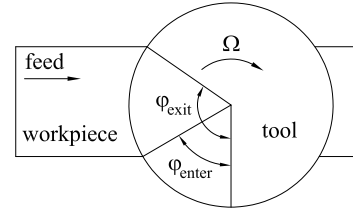


Figure 2.   Exit and enter angles in milling

of the tool. Only the local non-linearity of the cutting force and the nonlinearity due to the state-dependent delay is included into the model, and the so-called fly-over effect [26] is not modeled.

Introduce the time period $\tilde{\tau} = 2\pi/(N\Omega) = \vartheta/\Omega$. Note, that $\tilde{\tau}$ is equal to the constant tooth passing period used in standard milling models. It can easily be seen by substitution into Eq. (7) that $\varphi_j(t)$ satisfies

$$\varphi_j(t + \tilde{\tau}) = \varphi_{j-1}(t). \quad (9)$$

## 2.1  Delay model

The chip thickness $h_j$ cut by the $j^{\text{th}}$ tooth is determined by the current position of the $j^{\text{th}}$ and by an earlier position of the previous, $(j-1)^{\text{th}}$ tooth. In standard milling models, the time delay between two succeeding cuts are considered to be constant determined by the tool rotation and the number of the teeth: $\tilde{\tau} = 2\pi/(N\Omega)$. In this section, an accurate model of the regeneration of the succeeding cuts is presented in order to show that the time delay is actually time- and state-dependent.

Figure 3 shows the milling tool in the position of two succeeding cuts. The center point of the tool (denoted by point $O$) is given by vector $\mathbf{o}$, the cutting edge of the $j^{\text{th}}$ tooth (denoted by point $P_j$) is given by vector $\mathbf{p}_j$.
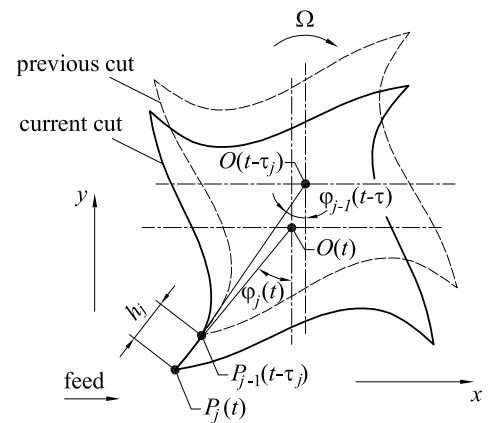


Figure 3.   Accurate model of chip removal in milling

The bending vibration of the tool is described by the $x$ and $y$ coordinates. The position of the tool's center point relative to the workpiece can be given as a combination of the feed motion and the bending vibrations of the tool:

$$\mathbf{o}(t) = \begin{pmatrix} -vt + x(t) \\ y(t) \end{pmatrix}, \qquad (10)$$

where, $v$ is the feed speed. Here, the coordinate system is fixed to the workpiece, therefore the feed motion is negative. The position of the $j^{\text{th}}$ tooth can be given as

$$\mathbf{p}_j(t) = \begin{pmatrix} -R\sin(\varphi_j(t)) - vt + x(t) \\ R\cos(\varphi_j(t)) + y(t) \end{pmatrix}, \qquad (11)$$

where $R$ is the radius of the tool. The chip thickness is determined by the positions of two succeeding teeth $p_j(t)$ and $p_{j-1}(t - \tau_j)$, where $\tau_j$ is the delay between the $(j-1)^{\text{th}}$ and the $j^{\text{th}}$ teeth.

In most mechanical models in the literature, the time delay is considered to be constant for all the teeth using the assumption that the vibration amplitude and the feed rate are relatively small compared to the diameter of the tool. However, an accurate model of the cutting process results in a time and state dependent delay. The exact condition for the delay is that point $P_{j-1}(t - \tau_j)$ is located in the section determined by points $P_j(t)$ and $O(t)$:

$$\mathbf{p}_j(t)(1 - s_j) + \mathbf{o}(t)s_j = \mathbf{p}_{j-1}(t - \tau_j), \qquad (12)$$

where $s_j \in [0, 1]$ is a running parameter characterizing the chip thickness $h_j = s_j R$ cut by the $j^{\text{th}}$ tooth.

Substitution of Eqs. (10) and (11) into Eq. (12) results in the system of equations

$$-R(1 - s_j)\sin\varphi_j(t) = -R\sin\varphi_{j-1}(t - \tau_j) \\ + v\tau_j + x(t - \tau_j) - x(t), \quad (13)$$

$$-R(1 - s_j)\cos\varphi_j(t) = -R\cos\varphi_{j-1}(t - \tau_j) \\ + y(t - \tau_j) - y(t) \quad (14)$$

for the two unknowns, $s_j$ and $\tau_j$.

The difference of Eq. (13) multiplied by $\cos\varphi_j(t)$ and Eq. (14) multiplied by $\sin\varphi_j(t)$ and substitution of Eq. (7) gives the following implicit equation for the time delay:

$$\begin{aligned} \big(v\tau_j + x(t - \tau_j) - x(t)\big)\cos\big(-\Omega t + (j-1)\vartheta\big) \\ - \big(y(t - \tau_j) - y(t)\big)\sin\big(-\Omega t + (j-1)\vartheta\big) \\ = R\sin\big(\Omega\tau_j - \vartheta\big). \quad (15) \end{aligned}$$

This equation shows that the time delay $\tau_j$ depends on time $t$, on current position $x(t)$ and $y(t)$ and on the delayed position $x(t - \tau_j)$ and $y(t - \tau_j)$ of the tool, that is, the time delay is time and state dependent: $\tau_j = \tau_j(t, x_t, y_t)$, where $x_t(s) = x(t + s)$, $y_t(s) = y(t + s)$, $s \in [-r, 0]$, $r \in \mathbb{R}^+$. It can also be seen that in this model, the time delays associated to different cutting teeth are different as opposed to the models using an overall constant time delay.

Note, that the only explicit time dependent functions in Eq. (15) are

$$\begin{aligned} \cos\big(-\Omega t + (j-1)\vartheta\big) = \cos\varphi_j(t), \\ \sin\big(-\Omega t + (j-1)\vartheta\big) = \sin\varphi_j(t). \end{aligned}$$

These are periodic functions of period $T = 2\pi/\Omega$. Here, $T$ is the rotation period of the tool. Consequently, the explicit time dependence of $\tau_j$ is also $T$-periodic:

$$\tau_j(t, x_t, y_t) = \tau_j(t + T, x_t, y_t). \qquad (16)$$

Consider the constant time period $\tilde{\tau} = T/N = 2\pi/(N\Omega) = \vartheta/\Omega$ used in standard milling models. Due to Eq. (9), the time periodic terms in Eq. (15) satisfy

$$\begin{aligned} \cos\big(-\Omega(t + \tilde{\tau}) + (j-1)\vartheta\big) = \cos\varphi_j(t + \tilde{\tau}) = \cos\varphi_{j-1}(t), \\ \sin\big(-\Omega(t + \tilde{\tau}) + (j-1)\vartheta\big) = \sin\varphi_j(t + \tilde{\tau}) = \sin\varphi_{j-1}(t). \end{aligned}$$

Consequently, $\tau_j(t, x_t, y_t)$ satisfies

$$\tau_j(t + \tilde{\tau}, x_t, y_t) = \tau_{j-1}(t, x_t, y_t). \qquad (17)$$

This gives the connection between time delays associated to two succeeding cuts.

It should also be mentioned that the time delay variation is not strong, since the terms $\big(v\tau_j + x(t - \tau_j) - x(t)\big)$ and $\big(y(t - \tau_j) - y(t)\big)$ in Eq. (15) are usually much smaller than the radius $R$ of the tool. Practically, the time delay slightly varies around the mean value $\tilde{\tau} = 2\pi/(N\Omega)$.

If the vibrations of the tool is not included into the delay model, i.e., $x(t) \equiv 0$ and $y(t) \equiv 0$, then the implicit equation (15) of the time delay is simplified to

$$v\tau_j\cos\big(-\Omega t + (j-1)\vartheta\big) = R\sin\big(\Omega\tau_j - \vartheta\big). \qquad (18)$$

Here, the time delay depends only on time. This is the case of time periodic delay that was investigated by Long and Balachandran [14].

If the feed is negligible relative to the tool diameter (i.e. $v\tau_j \ll R$), and $v\tau_j = 0$ is substituted into Eq. (18), then the resulted equation reads

$$\sin\left(\Omega\tau_j - \vartheta\right) = 0. \qquad (19)$$

This gives the constant delay $\tau_j = \tilde{\tau} = \vartheta/\Omega = 2\pi/(N\Omega)$ that is usually used in standard milling models.

## 2.2 Chip thickness

As it was shown at Eq. (12) the chip thickness can be written as $h_j = s_j R$. The sum of Eq. (13) multiplied by $\sin\varphi_j(t)$ and Eq. (14) multiplied by $\cos\varphi_j(t)$ and the substitution of Eq. (7) gives

$$
\begin{aligned}
h_j = s_j R = R\Big( & 1 - \cos(\Omega\tau_j(t,x_t,y_t) - \vartheta)\Big) \\
& + \Big(v\tau_j(t,x_t,y_t) + x(t - \tau_j(t,x_t,y_t)) - x(t)\Big)\sin\varphi_j(t) \\
& + \Big(y(t - \tau_j(t,x_t,y_t)) - y(t)\Big)\cos\varphi_j(t). \quad (20)
\end{aligned}
$$

As it can be seen, the chip thickness depends on time $t$, on current position $x(t)$ and $y(t)$ and on the delayed position $x(t - \tau_j(t,x_t,y_t))$ and $y(t - \tau_j(t,x_t,y_t))$ of the tool and on the state-dependent delay $\tau_j(t,x_t,y_t)$ itself, too.

If the constant time delay $\tau_j(t,x_t,y_t) \equiv \tilde{\tau} = \vartheta/\Omega$ is substituted into Eq. (20), then we get

$$
\begin{aligned}
h_j = \Big(v\tilde{\tau} + x(t - \tilde{\tau}) - x(t)\Big)\sin\varphi_j(t) \\
+ \Big(y(t - \tilde{\tau}) - y(t)\Big)\cos\varphi_j(t). \quad (21)
\end{aligned}
$$

This expression for the chip thickness is used by the standard models of the milling process with constant delay (see, e.g., [5], [12]).

## 2.3 The system with state dependent delay

Equations (3), (4), (5), (6) and (20) defines the system of SD-DDEs

$$
\begin{aligned}
m\ddot{x}(t) + c\dot{x}(t) + kx(t) \\
= \sum_{j=1}^{N}\alpha_{x,j}(t)\Big( R(1 - \cos(\Omega\tau_j(t,x_t,y_t) - \vartheta)) \\
+ (v\tau_j(t,x_t,y_t) + x(t - \tau_j(t,x_t,y_t)) - x(t))\sin\varphi_j(t) \\
+ (y(t - \tau_j(t,x_t,y_t)) - y(t))\cos\varphi_j(t)\Big)^q, \quad (22)
\end{aligned}
$$

$$
\begin{aligned}
m\ddot{y}(t) + c\dot{y}(t) + ky(t) \\
= \sum_{j=1}^{N}\alpha_{y,j}(t)\Big( R(1 - \cos(\Omega\tau_j(t,x_t,y_t) - \vartheta)) \\
+ (v\tau_j(t,x_t,y_t) + x(t - \tau_j(t,x_t,y_t)) - x(t))\sin\varphi_j(t) \\
+ (y(t - \tau_j(t,x_t,y_t)) - y(t))\cos\varphi_j(t)\Big)^q, \quad (23)
\end{aligned}
$$

where

$$\alpha_{x,j}(t) = wg(\varphi_j(t))\Big(K_t\cos(\varphi_j(t)) + K_n\sin(\varphi_j(t))\Big) \qquad (24)$$

and

$$\alpha_{y,j}(t) = wg(\varphi_j(t))\Big(K_n\cos(\varphi_j(t)) - K_t\sin(\varphi_j(t))\Big), \qquad (25)$$

and the time delay is given by the implicit equation (15). Note, that $\alpha_{x,j}(t)$ and $\alpha_{y,j}(t)$ satisfy:

$$\alpha_{l,j}(t) = \alpha_{l,j}(t + T), \qquad (26)$$

and, due to Eq. (9),

$$\alpha_{l,j}(t + \tilde{\tau}) = \alpha_{l,j-1}(t), \qquad (27)$$

where $l = x,y$, and $T = 2\pi/\Omega$, $\tilde{\tau} = 2\pi/(N\Omega)$.

It follows from Eqs. (16), (17) and (26), (27) that system (22)–(23) are $\tilde{\tau}$-periodic in time.

## 3 THE LINEARIZED EQUATION OF MOTION

For nonlinear systems, a standard way for stability analysis consists of two steps: (1) linearization and (2) investigation of the characteristic roots or characteristic multipliers of the linear system. Linearization of SD-DDEs is not so straightforward as it is for ordinary differential equations. An SD-DDE is always nonlinear, since the delay itself depends on the state, while, the linearized system is a DDE with constant or time dependent delay. Usually, there is no direct method for the construction of the linearized system.

### 3.1 Linearization of periodic SD-DDEs

In [24], it was shown that periodic linear systems can be associated to time periodic SD-DDEs as variational system. Consider the periodic SD-DDE

$$\dot{z}(t) = f(t, z(t), z(t - \tau(t, z_t))), \qquad (28)$$

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with

$$f(t,z(t),z(t-\tau(t,z_t))) = f(t+T,z(t),z(t-\tau(t,z_t))), \quad (29)$$

and

$$\tau(t,z_t) = \tau(t+T,z_t), \quad (30)$$

where $z_t(s) = z(t+s)$, $s \in [-r,0]$, $r \in \mathbb{R}^+$.

If $\bar{z}(t)$ is a $T$-periodic solution of Eq. (28), then the associated linear system is

$$\dot{u} = D_2 f(t,\bar{z}(t),\bar{z}(t-\tau(t,\bar{z})))u(t)$$
$$+ D_3 f(t,\bar{z}(t),\bar{z}(t-\tau(t,\bar{z})))u(t-\tau(t,\bar{z}))$$
$$- D_3 f(t,\bar{z}(t),\bar{z}(t-\tau(t,\bar{z})))\dot{\bar{z}}(t-\tau(t,\bar{z}))D_2\tau(t,\bar{z})u_t, \quad (31)$$

where $D_2$ and $D_3$ denotes the derivatives with respect to the 2nd and the 3rd argument, respectively. In this linearized system, the time delay is periodic in time, therefore, it is a periodic DDE with periodic delay.

According to [24], system (31) is the variational system of Eq. (28), if it preserves local stability properties. In other words, if the trivial solution $u \equiv 0$ of Eq. (31) is stable then it follows that the $T$-periodic solution $\bar{z}(t)$ of Eq. (28) is stable too.

## 3.2 Linearization of the milling model

Assume that $\bar{x}(t)$ and $\bar{y}(t)$ are periodic solution of system (22)–(23). It is not trivial if this periodic solutions exists, but we assume that the stable milling process is associated to a periodic motion of the tool with time period $\tilde{\tau} = 2\pi/(N\Omega)$, similarly to the milling models with constant time delay (see [27]).

The linear system associated to the periodic solution $\bar{x}(t)$ and $\bar{y}(t)$ can be given according to Eq. (31). In order to simplify notation, we introduce

$$X_j(t) \equiv x(t - \tau_j(t,x_t,y_t)), \quad (32)$$
$$\bar{X}_j(t) \equiv \bar{x}(t - \tau_j(t,\bar{x}_t,\bar{y}_t)), \quad (33)$$
$$Y_j(t) \equiv y(t - \tau_j(t,x_t,y_t)), \quad (34)$$
$$\bar{Y}_j(t) \equiv \bar{y}(t - \tau_j(t,\bar{x}_t,\bar{y}_t)) \quad (35)$$
$$\bar{\tau}_j(t) \equiv \tau_j(t,\bar{x}_t,\bar{y}_t). \quad (36)$$

Rewrite Eqs. (22) and (23) in the compact form

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t)$$
$$= \sum_{j=1}^{N} f_{x,j}(t,x(t),X_j(t),y(t),Y_j(t),\tau_j(t,x_t,y_t)), \quad (37)$$

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t)$$
$$= \sum_{j=1}^{N} f_{y,j}(t,x(t),X_j(t),y(t),Y_j(t),\tau_j(t,x_t,y_t)). \quad (38)$$

The associated linear system can be given as

$$m\ddot{\xi}(t) + c\dot{\xi}(t) + k\xi(t)$$
$$= \sum_{j=1}^{N} \Big\{ D_2 f_{x,j}(t,\bar{x}(t),\bar{X}_j(t),\bar{y}(t),\bar{Y}_j(t),\bar{\tau}_j(t))\xi(t)$$
$$+ D_3 f_{x,j}(t,\bar{x}(t),\bar{X}_j(t),\bar{y}(t),\bar{Y}_j(t),\bar{\tau}_j(t))\xi(t-\bar{\tau}_j(t))$$
$$+ D_4 f_{x,j}(t,\bar{x}(t),\bar{X}_j(t),\bar{y}(t),\bar{Y}_j(t),\bar{\tau}_j(t))\eta(t)$$
$$+ D_5 f_{x,j}(t,\bar{x}(t),\bar{X}_j(t),\bar{y}(t),\bar{Y}_j(t),\bar{\tau}_j(t))\eta(t-\bar{\tau}_j(t))$$
$$- D_3 f_{x,j}(t,\bar{x}(t),\bar{X}_j(t),\bar{y}(t),\bar{Y}_j(t),\bar{\tau}_j(t))$$
$$\times \dot{\bar{x}}(t-\bar{\tau}_j(t))\Big( D_2\tau_j(t,\bar{x}_t,\bar{y}_t))\xi_t + D_3\tau_j(t,\bar{x}_t,\bar{y}_t))\eta_t \Big)$$
$$- D_5 f_{x,j}(t,\bar{x}(t),\bar{X}_j(t),\bar{y}(t),\bar{Y}_j(t),\bar{\tau}_j(t))$$
$$\times \dot{\bar{y}}(t-\bar{\tau}_j(t))\Big( D_2\tau_j(t,\bar{x}_t,\bar{y}_t)\xi_t + D_3\tau_j(t,\bar{x}_t,\bar{y}_t)\eta_t \Big)$$
$$+ D_6 f_{x,j}(t,\bar{x}(t),\bar{X}_j(t),\bar{y}(t),\bar{Y}_j(t),\bar{\tau}_j(t))$$
$$\times \Big( D_2\tau_j(t,\bar{x}_t,\bar{y}_t)\xi_t + D_3\tau_j(t,\bar{x}_t,\bar{y}_t)\eta_t \Big) \Big\}, \quad (39)$$

$$m\ddot{\eta}(t) + c\dot{\eta}(t) + k\eta(t)$$
$$= \sum_{j=1}^{N} \Big\{ D_2 f_{y,j}(t,\bar{x}(t),\bar{X}_j(t),\bar{y}(t),\bar{Y}_j(t),\bar{\tau}_j(t))\xi(t)$$
$$+ D_3 f_{y,j}(t,\bar{x}(t),\bar{X}_j(t),\bar{y}(t),\bar{Y}_j(t),\bar{\tau}_j(t))\xi(t-\bar{\tau}_j(t))$$
$$+ D_4 f_{y,j}(t,\bar{x}(t),\bar{X}_j(t),\bar{y}(t),\bar{Y}_j(t),\bar{\tau}_j(t))\eta(t)$$
$$+ D_5 f_{y,j}(t,\bar{x}(t),\bar{X}_j(t),\bar{y}(t),\bar{Y}_j(t),\bar{\tau}_j(t))\eta(t-\bar{\tau}_j(t))$$
$$- D_3 f_{y,j}(t,\bar{x}(t),\bar{X}_j(t),\bar{y}(t),\bar{Y}_j(t),\bar{\tau}_j(t))$$
$$\times \dot{\bar{x}}(t-\bar{\tau}_j(t))\Big( D_2\tau_j(t,\bar{x}_t,\bar{y}_t))\xi_t + D_3\tau_j(t,\bar{x}_t,\bar{y}_t))\eta_t \Big)$$
$$- D_5 f_{y,j}(t,\bar{x}(t),\bar{X}_j(t),\bar{y}(t),\bar{Y}_j(t),\bar{\tau}_j(t))$$
$$\times \dot{\bar{y}}(t-\bar{\tau}_j(t))\Big( D_2\tau_j(t,\bar{x}_t,\bar{y}_t)\xi_t + D_3\tau_j(t,\bar{x}_t,\bar{y}_t)\eta_t \Big)$$
$$+ D_6 f_{y,j}(t,\bar{x}(t),\bar{X}_j(t),\bar{y}(t),\bar{Y}_j(t),\bar{\tau}_j(t))$$
$$\times \Big( D_2\tau_j(t,\bar{x}_t,\bar{y}_t)\xi_t + D_3\tau_j(t,\bar{x}_t,\bar{y}_t)\eta_t \Big). \quad (40)$$

Here, again, $D_i$ denotes the derivative with respect to the $i^{\text{th}}$ argument.

The terms $D_i f_{x,j}$ and $D_i f_{y,j}$ ($i = 2, \ldots, 6$) can be given as

$$D_2 f_{l,j}(t, \bar{x}(t), \bar{X}_j(t), \bar{y}(t), \bar{Y}_j(t), \bar{\tau}_j(t))$$
$$= -\alpha_{l,j}(t) q \bar{h}_j^{q-1}(t) \sin \varphi_j(t), \quad (41)$$

$$D_3 f_{l,j}(t, \bar{x}(t), \bar{X}_j(t), \bar{y}(t), \bar{Y}_j(t), \bar{\tau}_j(t))$$
$$= \alpha_{l,j}(t) q \bar{h}_j^{q-1}(t) \sin \varphi_j(t), \quad (42)$$

$$D_4 f_{l,j}(t, \bar{x}(t), \bar{X}_j(t), \bar{y}(t), \bar{Y}_j(t), \bar{\tau}_j(t))$$
$$= -\alpha_{l,j}(t) q \bar{h}_j^{q-1}(t) \cos \varphi_j(t), \quad (43)$$

$$D_5 f_{l,j}(t, \bar{x}(t), \bar{X}_j(t), \bar{y}(t), \bar{Y}_j(t), \bar{\tau}_j(t))$$
$$= \alpha_{l,j}(t) q \bar{h}_j^{q-1}(t) \cos \varphi_j(t) \quad (44)$$

and

$$D_6 f_{l,j}(t, \bar{x}(t), \bar{X}_j(t), \bar{y}(t), \bar{Y}_j(t), \bar{\tau}_j(t))$$
$$= \alpha_{l,j}(t) q \bar{h}_j^{q-1}(t)(R\Omega \sin(\Omega \bar{\tau}_j(t) - \vartheta) + v \sin \varphi_j(t)), \quad (45)$$

where $l = x, y$, and

$$\bar{h}_j(t) = R(1 - \cos(\Omega \bar{\tau}_j(t) - \vartheta))$$
$$+ (v\bar{\tau}_j(t) + \bar{x}(t - \bar{\tau}_j(t)) - \bar{x}(t)) \sin \varphi_j(t)$$
$$+ (\bar{y}(t - \bar{\tau}_j(t)) - \bar{y}(t)) \cos \varphi_j(t) \quad (46)$$

is the chip thickness associated to the periodic tool motion described by $\bar{x}(t)$ and $\bar{y}(t)$.

The terms $D_2 \tau_j(t, \bar{x}, \bar{y})) \xi_t$ and $D_3 \tau_j(t, \bar{x}, \bar{y})) \eta_t$ can be given by implicit derivation of Eq. (15). Taking the Frechet derivative of both sides of Eq. (15) with respect to $x_t$ and using formula (31) for the derivative of the composite function $x(t - \tau(t, x_t, y_t))$ with respect to $x_t$ we get:

$$\left( v D_2 \tau_j(t, \bar{x}, \bar{y}) \xi_t + \xi(t - \bar{\tau}_j(t)) \right.$$
$$- \dot{\bar{x}}(t - \bar{\tau}_j(t)) D_2 \tau_j(t, \bar{x}, \bar{y}) \xi_t - \xi(t) \bigg) \cos \varphi_j(t)$$
$$+ \dot{\bar{y}}(t - \bar{\tau}_j(t)) D_2 \tau_j(t, \bar{x}, \bar{y}) \xi_t \sin \varphi_j(t)$$
$$= R\Omega \cos(\Omega \bar{\tau}_j(t) - \vartheta) D_2 \tau_j(t, \bar{x}, \bar{y}) \xi_t. \quad (47)$$

From here, we obtain

$$D_2 \tau_j(t, \bar{x}, \bar{y}) \xi_t =$$
$$\frac{-\cos \varphi_j(t)(\xi(t - \bar{\tau}_j(t)) - \xi(t))}{(v - \dot{\bar{x}}(t - \bar{\tau}_j(t))) \cos \varphi_j(t) + \dot{\bar{y}}(t - \bar{\tau}_j(t)) \sin \varphi_j(t) - R\Omega \cos(\Omega \bar{\tau}_j(t) - \vartheta)}.$$
$$(48)$$

Here, we assume that the denominator is not zero. This assumption is reasonable due to the following facts. As it is explained after Eq. (17), in a real milling processes, the time delay slightly deviates around the mean value: $\bar{\tau}_j(t) \approx \tilde{\tau} = 2\pi/(N\Omega)$. Consequently, $\cos(\Omega \bar{\tau}_j(t) - \vartheta) \approx 1$ in Eq. (48). Furthermore, if we assume that the vibration velocities $\dot{\bar{x}}(t)$ and $\dot{\bar{y}}(t)$ of the tool and the feed speed are relatively small compared to the speed of the cutting edges $R\Omega$, then the denominator in Eq. (48) is really not zero but a negative number.

Now, take the Frechet derivative of both sides of Eq. (15) with respect to $y_t$:

$$\left( v D_3 \tau_j(t, \bar{x}, \bar{y}) \eta_t - \dot{\bar{x}}(t - \bar{\tau}_j(t)) D_3 \tau_j(t, \bar{x}, \bar{y}) \eta_t \right) \cos \varphi_j(t)$$
$$- \left( \eta(t - \bar{\tau}_j(t)) - \dot{\bar{y}}(t - \bar{\tau}_j(t)) D_3 \tau_j(t, \bar{x}, \bar{y}) \eta_t - \eta(t) \right) \sin \varphi_j(t)$$
$$= R\Omega \cos(\Omega \bar{\tau}_j(t) - \vartheta) D_3 \tau_j(t, \bar{x}, \bar{y}) \eta_t. \quad (49)$$

From here, we obtain

$$D_3 \tau_j(t, \bar{x}, \bar{y}) \eta_t =$$
$$\frac{\sin \varphi_j(t)(\eta(t - \bar{\tau}_j(t)) - \eta(t))}{(v - \dot{\bar{x}}(t - \bar{\tau}_j(t))) \cos \varphi_j(t) + \dot{\bar{y}}(t - \bar{\tau}_j(t)) \sin \varphi_j(t) - R\Omega \cos(\Omega \bar{\tau}_j(t) - \vartheta)}.$$
$$(50)$$

Similarly to Eq. (48) the denominator in Eq. (50) is not zero but a negative number.

Substitution of all the derivative terms into Eqs. (39) and (40), the linearized equations of motion read

$$m\ddot{\xi}(t) + c\dot{\xi}(t) + k\xi(t)$$
$$= \sum_{j=1}^{N} \left\{ \alpha_{x,j}(t) q \bar{h}_j^{q-1}(t) \sin \varphi_j(t)(\xi(t - \bar{\tau}_j(t)) - \xi(t)) \right.$$
$$+ \alpha_{x,j}(t) q \bar{h}_j^{q-1}(t) \cos \varphi_j(t)(\eta(t - \bar{\tau}_j(t)) - \eta(t))$$
$$+ \alpha_{x,j}(t) q \bar{h}_j^{q-1}(t)$$
$$\times \left( - \sin \varphi_j(t) \dot{\bar{x}}(t - \bar{\tau}_j(t)) - \cos \varphi_j(t) \dot{\bar{y}}(t - \bar{\tau}_j(t)) \right.$$
$$+ R\Omega \sin(\Omega \bar{\tau}_j(t) - \vartheta) + v \sin \varphi_j(t) \bigg)$$
$$\times \left. \left( D_2 \tau_j(t, \bar{x}, \bar{y}) \xi_t + D_3 \tau_j(t, \bar{x}, \bar{y}) \eta_t \right) \right\} \quad (51)$$

and

$$
\begin{aligned}
m\ddot{\eta}(t) + c\dot{\eta}(t) + k\eta(t) \\
= \sum_{j=1}^{N} \Bigg\{ &\alpha_{y,j}(t)q\bar{h}_j^{q-1}(t)\sin\varphi_j(t)(\xi(t-\bar{\tau}_j(t))-\xi(t)) \\
&+ \alpha_{y,j}(t)q\bar{h}_j^{q-1}(t)\cos\varphi_j(t)(\eta(t-\bar{\tau}_j(t))-\eta(t)) \\
&+ \alpha_{y,j}(t)q\bar{h}_j^{q-1}(t) \\
&\times \Big( -\sin\varphi_j(t)\dot{\bar{x}}(t-\bar{\tau}_j(t)) - \cos\varphi_j(t)\dot{\bar{y}}(t-\bar{\tau}_j(t)) \\
&\qquad + R\Omega\sin(\Omega\bar{\tau}_j(t)-\vartheta) + v\sin\varphi_j(t) \Big) \\
&\times \Big( D_2\tau_j(t,\bar{x},\bar{y})\xi_t + D_3\tau_j(t,\bar{x},\bar{y})\eta_t \Big) \Bigg\}, \quad (52)
\end{aligned}
$$

where the terms $D_2\tau(t,\bar{x},\bar{y}))\xi_t$ and $D_3\tau(t,\bar{x},\bar{y}))\eta_t$ are given by Eqs. (48) and (50).

Note, that there are three terms in the right hand sides of Eqs. (51) and (52). The first two terms are identical to the terms in the models using constant or time-periodic time delay. The additional third term arises due to the state-dependent delay.

This means that the state-dependent delay affects the linear stability of the milling process, similarly to the autonomous turning model (see [25]). If the state-dependency of the time delay were neglected, then this third term would completely disappear from the equation, and we would get back the linear equation of motion of models without state-dependent delay.

Stability analysis of the linear DDE (51)–(52) with time periodic delay can be performed by numerical or semi-analytical techniques. The practical evaluation of stability is however quite problematic due to several reasons. First of all, the time delay is time periodic, and the computation tools for DDEs with time periodic delay is not so developed as it is for systems with constant time delay. Another problem is that the derivative of the periodic solution explicitly appears in the equations in the form $\dot{\bar{x}}(t-\bar{\tau}_j(t))$ and $\dot{\bar{y}}(t-\bar{\tau}_j(t))$.

Although it is shown clearly that an additional third term arises in the linearized equation due to the state dependency of the time delay, its influence is much smaller than that of the first and the second terms. If Eqs. (48) and (50) are substituted into the third terms in Eq. (51), then we get

$$
\begin{aligned}
\sum_{l=x,y} \Bigg\{ &\alpha_{l,j}(t)q\bar{h}_j^{q-1}(t)\cos\varphi_j(t)(\xi(t-\bar{\tau}_j(t))-\xi(t)) \\
&\times \frac{\sin\varphi_j(t)(\dot{\bar{x}}(t-\bar{\tau}_j(t))-v)+\cos\varphi_j(t)\dot{\bar{y}}(t-\bar{\tau}_j(t))-R\Omega\sin(\Omega\bar{\tau}_j(t)-\vartheta)}{(v-\dot{\bar{x}}(t-\bar{\tau}_j(t)))\cos\varphi_j(t)+\dot{\bar{y}}(t-\bar{\tau}_j(t))\sin\varphi_j(t)-R\Omega\cos(\Omega\bar{\tau}_j(t)-\vartheta)} \\
&- \alpha_{l,j}(t)q\bar{h}_j^{q-1}(t)\sin\varphi_j(t)(\eta(t-\bar{\tau}_j(t))-\eta(t)) \\
&\times \frac{\sin\varphi_j(t)\dot{\bar{x}}(t-\bar{\tau}_j(t))+\cos\varphi_j(t)(\dot{\bar{y}}(t-\bar{\tau}_j(t))-v)-R\Omega\sin(\Omega\bar{\tau}_j(t)-\vartheta)}{(v-\dot{\bar{x}}(t-\bar{\tau}_j(t)))\cos\varphi_j(t)+\dot{\bar{y}}(t-\bar{\tau}_j(t))\sin\varphi_j(t)-R\Omega\cos(\Omega\bar{\tau}_j(t)-\vartheta)} \Bigg\}.
\end{aligned}
$$
(53)

If a normal milling process is considered, then it can be seen that this term is significantly smaller than the first and the second terms in Eq. (51). As it was mentioned after Eq. (17), practically, the time delay slightly varies around the mean value $\tilde{\tau} = 2\pi/(N\Omega)$, therefore $\sin(\Omega\bar{\tau}_j(t)-\vartheta) \approx 0$ in the numerator of the fractions in Eq. (53), and $\cos(\Omega\bar{\tau}_j(t)-\vartheta) \approx 1$ in the denominator. If the vibration velocity $\dot{\bar{x}}(t)$ and $\dot{\bar{y}}(t)$ and the feed speed $v$ are small, then both fractions in Eq. (53) are small. In other words, the additional third terms in Eqs. (51) and (52) are small relative to the first and the second terms. This means that the omission of the state dependency of the time delay in milling models does not significantly affect the linear properties of the system. This, however, does not hold for the nonlinear behavior of the system, the nonlinear properties might strongly be affected by the state dependency of the delay.

## 4 CONCLUSIONS

It was shown that if the self excited vibrations of the milling tool superimpose on the trochoidal path of the cutting edges, then the surface regeneration results in a time- and state-dependent time delay in the model equations instead of the constant time delay that is usually used by standard milling models.

It was shown through the linearization of the nonlinear SD-DDE that an additional term arises in the linearized equation of motion due to the state-dependency of the time delay. Although, the state-dependent time delay affects the linear stability properties of the system, its influence is not significant for practical machining parameters, since the arising additional term is relatively slight for practical milling parameters. In this sense, the standard milling models using constant time delay describe linear stability properties well. However, nonlinear behavior of the system might strongly be affected by the state dependency of the delay.

The effect of state dependent time delay on cutting process dynamics might not be so significant in the present manufacturing practice, however, it might become more and more important with the development of the technology similarly to the history of modeling time periodic cutting forces in milling processes. In early milling models in the sixties, the time dependency of

the cutting forces was neglected, and an average constant cutting force was used for dynamic analysis [1], [2]. These models describe the system's dynamics approximately well for most cases, however, for highly interrupted cutting, they does not give reliable result. Continuous development of milling models and their analysis in the past decade showed that time periodicity of the cutting forces qualitatively affects the system's behavior: the chatter frequencies are multiplied and a new type of instability, the so-called period doubling or flip bifuraction arise [27]– [29]. Similarly, the incorporation of the state-dependent time delay into the milling model might become more important in the future.

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