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FUNDAMENTAL SOLUTION AND ASYMPTOTIC STABILITY OF LINEAR DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this paper we formulate sufficient conditions for the asymptotic stablity of linear delay systems of the form

$$\dot{x}_k(t) = -\sum_{\ell=0}^{m}\sum_{j=1}^{n} a_{kj}^{(\ell)} x_j(t-\tau_{kj}^{(\ell)}), \qquad k=1,\dots,n, \quad t\ge 0,$$

where $a_{kj}^{(0)}, a_{kj}^{(\ell)} \in \mathbb{R}$, $\tau_{kj}^{(0)}=0$, $\tau_{kj}^{(\ell)} \ge 0$, $k,j=1,\dots,n$, $\ell=1,\dots,m$. In order to apply our results, we give estimates for the integral $\int_0^\infty |v(t)|\,dt$, where v is the fundamental solution of certain associated scalar linear delay differential equations with multiple delays.

Keywords. linear delay differential equations, fundamental solution, asymptotic stability
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1 Introduction

Consider the delay system

$$\dot{x}_k(t) = -\sum_{j=1}^{n} a_{kj}x_j(t) - \sum_{j=1}^{n} b_{kj}x_j(t-\tau_{kj}), \qquad k=1,\dots,n, \quad t\ge 0, \quad (1.1)$$

where $a_{kj}, b_{kj} \in \mathbb{R}$, $\tau_{kj} \ge 0$, $k,j=1,\dots,n$. The stability of the trivial (zero) solution of special classes of (1.1) has been studied, e.g., [3]–[19]. In this paper we extend and improve these results for (1.1). Moreover, we formulate our results for the more general linear delay system

$$\dot{x}_k(t) = -\sum_{\ell=0}^{m}\sum_{j=1}^{n} a_{kj}^{(\ell)} x_j(t-\tau_{kj}^{(\ell)}), \qquad k=1,\dots,n, \quad t\ge 0, \qquad (1.2)$$

where $a_{kj}^{(0)}, a_{kj}^{(\ell)} \in \mathbb{R}$, $\tau_{kj}^{(0)}=0$, $\tau_{kj}^{(\ell)} \ge 0$, $k,j=1,\dots,n$, $\ell=1,\dots,m$.

First we recall some known results for the stability of (1.1). All these results rely on the notion of an M-matix. A square matrix is called non-singular M-matrix, if all its off-diagonal elements are non-positive, and all its principal minors are positive. We refer, e.g., to [2] for many equivalent form of this definition. We recall one equivalent property: the inverse of the matrix is a positive matrix.

Necessary and sufficient condition of the asymptotic stability of the trivial solution of (1.1) independently of the selection of the delays $\tau_{kj} \geq 0$ $(k, j = 1, \ldots, n)$ was given by Hofbauer and So in [15] for the case when the coefficient matrix (a_{kj}) is diagonal, i.e., $a_{kj} = 0$ $(k \neq j,\ k, j = 1, \ldots, n)$ and there is no delay in the main diagonal terms, i.e., $b_{kk} = 0$ $(k = 1, \ldots, n)$. Later this result was extended by Campbell ([3]) for the case when $a_{kj} = 0$ $(k \neq j,$ $k, j = 1, \ldots, n)$ but b_{kk} is not necessary zero. It was shown in [3] for this latter case that the trivial solution of (1.1) is asymptotically stable independently of the delays, if and only if the $n \times n$ matrix C with components

$$c_{kj} = \begin{cases} a_{kk} - |b_{kk}|, & k = j, \\ -|b_{kj}|, & k \neq j \end{cases} \tag{1.3}$$

is a nonsingular M-matrix.

The sufficient part of the above result was extended in [10] to the general case of (1.1). It was shown that if the $n \times n$ matrix D with components

$$d_{kj} = \begin{cases} a_{kk} - |b_{kk}|, & k = j, \\ -|a_{kk}| - |b_{kj}|, & k \neq j \end{cases} \tag{1.4}$$

is a nonsingular M-matrix, then the trivial solution of (1.1) is asymptotically stable independently of the selection of the delays $\tau_{kj} \geq 0$ $(k, j = 1, \ldots, n)$.

For the case when the stability depends on the delays there are not many known conditions. Next we recall two results in this direction, which are given for the special case of (1.1), when there is no instantaneous negative feedback in the system, i.e., consider

$$\dot{x}_k(t) = -\sum_{j=1}^{n} b_{kj} x_j(t - \tau_{kj}), \qquad k = 1, \ldots, n, \quad t \geq 0, \tag{1.5}$$

where $b_{kj} \in \mathbb{R}$, $\tau_{kj} \geq 0$, $k, j = 1, \ldots, n$.

In [5] the following result was proved.

Theorem 1.1 (Theorem 3.2 in [5]) *If*

$$0 < b_{kk}\tau_{kk} < \frac{\pi}{2}, \qquad k = 1, \ldots, n, \tag{1.6}$$

and the $n \times n$ matrix E with elements

$$e_{kj} = \begin{cases} \frac{\alpha_{0k}^2}{\alpha_{0k}^2 + \beta_{0k}^2} b_{kk}, & k = j, \\ -|b_{kj}|, & k \neq j \end{cases} \tag{1.7}$$

is a nonsingular M-matrix, where $\alpha_{0k} + i\beta_{0k}$ is the leading characteristic root of the equation

$$\dot{y}_k(t) = -b_{kk} y_k(t - \tau_{kk}), \qquad t \geq 0, \tag{1.8}$$

then the trivial solution of (1.5) *is asymptotically stable.*

We say that $\lambda_{0k} = \alpha_{0k} + i\beta_{0k}$ is a leading characteristic root of (1.8), if it is a solution of the characteristic equation

$$\lambda = -b_{kk} e^{-\lambda \tau_{kk}}, \qquad \lambda \in \mathbb{C}, \tag{1.9}$$

and for any other root $\lambda = \alpha + i\beta$ of (1.9), $\alpha < \alpha_{0k}$ holds, assuming $\lambda \neq \alpha_{0k} \pm i\beta_{0k}$. Note that the above condition gives back the optimal condition of stability for small delays, since if

$$0 < b_{kk}\tau_{kk} < \frac{1}{e}, \qquad k = 1, \ldots, n, \tag{1.10}$$

then the leading characteristic root of (1.8) is real for all $k = 1, \ldots, n$, so the matrix E in (1.7) simplifies to $\tilde{E}$ defined by

$$\tilde{e}_{kj} = \begin{cases} b_{kk}, & k = j, \\ -|b_{kj}|, & k \neq j. \end{cases} \tag{1.11}$$

If $\tilde{E}$ is a nonsingular M-matrix, the the corresponding ODE

$$\dot{x}_k(t) = -\sum_{k=1}^{n} b_{kj} x_j(t), \qquad k = 1, \ldots, n$$

is asymptotically stable (see, e.g., [1]). It was shown in [6] that if $b_{kj} \geq 0$ for all $k \neq j$, $(k, j = 1, \ldots, n)$ and (1.10) holds, then $\tilde{E}$ beeing a nonsingular M-matrix is the necessary and sufficient condition of the asymptotic stability of (1.5).

The main idea of the proof of Theorem 1.1 is to consider (1.5) as the perturbation of (1.8). Let v_k denote the fundamental solution of (1.5), i.e., the solution of the initial value problem

$$\dot{v}_k(t) = -b_{kk} v_k(t - \tau_{kk}), \qquad t \geq 0, \tag{1.12}$$

$$v_k(t) = \begin{cases} 1, & t = 0, \\ 0, & t < 0. \end{cases} \tag{1.13}$$

Then knowing an estimate of the form

$$\int_0^\infty |v_k(t)|\, dt \leq \frac{1}{b_{kk}} \gamma_k, \tag{1.14}$$

one can repeat the proof of Theorem 1.1 and show

Theorem 1.2 *Assume* (1.6)*. If* (1.14) *holds and the matrix F with elements*

$$f_{kj} = \begin{cases} \frac{1}{\gamma_k} b_{kk}, & k = j, \\ -|b_{kj}|, & k \neq j. \end{cases}$$

is an M-matrix, then (1.5) *is asymptotically stable.*

Recently So, Tang and Zou [19] gave the following sufficient condition for the asymptotic stability of (1.5):

Theorem 1.3 (Theorem 1.3 in [19]) *Suppose*

$$0 < b_{kk}\tau_{kk} < \frac{3}{2}, \qquad k = 1, \ldots, n, \tag{1.15}$$

and the matrix G with elements

$$g_{kj} = \begin{cases} b_{kk}, & k = j, \\ -\dfrac{1 + \frac{1}{9} b_{kk}\tau_{kk}(3 + 2b_{kk}\tau_{kk})}{1 - \frac{1}{9} b_{kk}\tau_{kk}(3 + 2b_{kk}\tau_{kk})} |b_{kj}|, & k \neq j \end{cases} \tag{1.16}$$

is a nonsingular M-matrix, then the trivial solution of (1.5) *is asymptotically stable.*

Clearly, matrix G is a nonsingular M-matrix, if and only if the matrix $\tilde{G}$ defined by

$$\tilde{g}_{kj} = \begin{cases} \dfrac{1 - \frac{1}{9} b_{kk}\tau_{kk}(3 + 2b_{kk}\tau_{kk})}{1 + \frac{1}{9} b_{kk}\tau_{kk}(3 + 2b_{kk}\tau_{kk})} b_{kk}, & k = j, \\ -|b_{kj}|, & k \neq j \end{cases} \tag{1.17}$$

is a nonsingular M-matrix.

In Corollary 4.3 (see Section 4 below) we get, as a special case of our main result, Theorem 4.1, that if

$$0 < b_{kk}\tau_{kk} < 1 + \frac{1}{e}, \qquad k = 1, \ldots, n, \tag{1.18}$$

and the matrix H with components

$$h_{kj} = \begin{cases} \dfrac{1 - (b_{kk}\tau_{kk} - \frac{1}{e})_+}{1 + (b_{kk}\tau_{kk} - \frac{1}{e})_+} b_{kk}, & k = j, \\ -|b_{kj}|, & k \neq j. \end{cases} \tag{1.19}$$

is a nonsingular M-matrix, then the trivial solution of (1.5) is asymptotically stable. Here a_+ denotes the positive part of the number a, i.e., $a_+ = \max(a, 0)$.

The perturbation technique used to prove Theorems 1.1 and 1.2 is well-known for obtaining stability results for different classes of scalar and non-scalar differential and difference equations. See, e.g., [5] and [8]–[18] for applications of this method. The applicability of these and similar theorems depends on if we can compute or estimate the absolute integral of the fundamental solution in (1.14), which is a difficult task in the case when the fundamental solution changes sign (see [5] and [8] for more details). To the best of our knowledge, Theorem 2.1 in [5] is the only known estimate of this integral in this case, and it is formulated only for the simple single delay equation (1.8):

Theorem 1.4 (Theorem 2.1 in [5]) *Assume* (1.6), *and let* $\lambda_{0k} = \alpha_{0k} + i\beta_{0k}$ *be the leading characteristic root of* (1.8). *Then*

$$\int_0^\infty |v_k(t)|\,dt \le \frac{1}{b_{kk}} \frac{\alpha_{0k}^2 + \beta_{0k}^2}{\alpha_{0k}^2}. \tag{1.20}$$

The main goal of this paper is to extend and improve Theorem 1.4 for much larger classes of linear delay equations, and apply the above perturbation technique for the general linear delay system (1.2). In Section 2 we study scalar linear differential equations with multiple delay, and give estimates (1.14) for its fundamental solution using a characteristic root (see Theorem 2.2) and also in a special case, in terms of the parameters of the equation (see Theorem 2.4). In Section 3 we investigate in details scalar linear single delay equations, and give explicit necessary and sufficient conditions (see Theorem 3.1) for that estimate given in Theorem 2.2 be applicable. In Section 4 we formulate sufficient conditions for the asymptotic stability of (1.2) using estimates of absolute integral of the fundamental solutions of certain associated scalar multiple delay equations. We give applications of our general stability condition on simpler examples, and compare the conditions of Theorems 1.1, 1.2, 1.3 and Corollary 4.3 for Equation (1.5).

2 Multiple Delay Case

In this section we study the scalar delay equation with multiple delays

$$\dot{x}(t) = -\sum_{k=0}^{m} a_k x(t - \tau_k), \qquad t \ge 0. \tag{2.1}$$

We shall assume that

(H1) $a_0 \in \mathbb{R}$, $a_k \ge 0$ $(k = 1, \dots, m)$, $\sum_{k=0}^m a_k \ne 0$, and $0 = \tau_0 < \tau_1 < \dots < \tau_m$.

(H2) The trivial solution of (2.1) is asymptotically stable.

The characteristic equation of (2.1) is

$$\lambda = -\sum_{k=0}^{m} a_k e^{-\lambda \tau_k}. \tag{2.2}$$

Let α and β denote the real and imaginary part of λ, i.e., $\lambda = \alpha + i\beta$. In terms of α and β Equation (2.2) is equivalent to the system

$$\alpha + a_0 = -\sum_{k=1}^{m} a_k e^{-\alpha\tau_k} \cos \beta\tau_k, \tag{2.3}$$

$$\beta = \sum_{k=1}^{m} a_k e^{-\alpha\tau_k} \sin \beta\tau_k. \tag{2.4}$$

Since a_k are real numbers, we can assume that $\beta \geq 0$, since if λ is a solution of (2.2), then its conjugate is also a solution.

Let v denote the fundamental solution of (2.1), i.e., the solution of the initial value problem

$$\dot{v}(t) = -\sum_{k=0}^{m} a_k v(t-\tau_k), \qquad t \geq 0, \tag{2.5}$$

$$v(t) = \begin{cases} 1, & t = 0, \\ 0, & t \in [-\tau_m, 0). \end{cases} \tag{2.6}$$

We have collected some well-known results from the literature in the next proposition related to the fundamental solution. For the proof of part (i) see, e.g., [14], part (ii) is a simple generalization of a result found, e.g., in [6], and part (iii) is proved, e.g., in [12], and (iv) and (v) can be found, e.g., in [13].

Proposition 2.1 *Assume (H1).*

(i) Assumption (H2) is equivalent to any of the following conditions

(1) All solutions $\lambda = \alpha + i\beta$ *of* (2.2) *satisfy* $\alpha < 0$.

(2) The fundamental solution $v(t)$ *of* (2.1) *tends to 0 exponentially as* $t \to \infty$.

(3) The fundamental solution $v(t)$ *of* (2.1) *is in* $L^1[0,\infty)$, *i.e.,*

$$\int_0^\infty |v(t)|\, dt < \infty.$$

(ii) The fundamental solution v *of* (2.1) *is positive on* $[0,\infty)$, *if and only if the characteristic equation* (2.2) *has a real root.*

(iii) If (H2) holds, then

$$\int_0^\infty v(t)\, dt = \frac{1}{\sum_{k=0}^m a_k}.$$

(iv) If $a_0 \geq 0$ *and*

$$\sum_{k=1}^{m} a_k \tau_k > \frac{1}{e},$$

then all solutions of (2.1) *(including the fundamental solution) are oscillatory, i.e., have arbitrary large zeros.*

(v) If $a_0 \ge 0$ *and*

$$\left(\sum_{k=0}^{m} a_k\right)\tau_m \le \frac{1}{e},$$

then there exists a nonoscillatory solution of (2.1)*, in particular, the fundamental solution* $v(t)$ *of* (2.1) *is positive for* $t > 0$.

The next result extends Theorem 1.4 for the multiple delay equation (2.1).

Theorem 2.2 *Assume (H1), (H2), and suppose* $\alpha_0 + i\beta_0$ *is a solution of* (2.2) *such that*

$$0 < \beta_0\tau_m < \frac{\pi}{2}. \tag{2.7}$$

Then

$$\int_0^\infty |v(t)|\,dt \le \frac{1}{\sum_{k=0}^m a_k}\left(1 - \frac{2\beta_0 e^{\frac{\alpha_0\pi}{2\beta_0}}}{\alpha_0\left(1 - e^{\frac{\alpha_0\pi}{\beta_0}}\right)}\right). \tag{2.8}$$

Proof Since $\lambda_0 = \alpha_0 + i\beta_0$ is a solution of (2.2), the function

$$x(t) = e^{\alpha_0 t}\cos\beta_0 t$$

is a solution of (2.1). On the other hand, the variation of constants formula (see, e.g., [14]) implies

$$x(t) = v(t)x(0) - \sum_{k=1}^{m} a_k \int_{-\tau_k}^{0} v(t-s-\tau_k)x(s)\,ds, \qquad t \ge 0.$$

Therefore,

$$v(t) = e^{\alpha_0 t}\cos\beta_0 t + \sum_{k=1}^{m} a_k \int_{-\tau_k}^{0} v(t-s-\tau_k)e^{\alpha_0 s}\cos\beta_0 s\,ds, \qquad t \ge 0.$$

Introduce the notation $A := \int_0^\infty |v(t)|\,dt < \infty$. Integrating from 0 to ∞ and using initial condition (2.6) we get

$$\begin{aligned}
A &= \int_0^\infty |v(t)|\,dt \\
&\le \int_0^\infty |e^{\alpha_0 t}\cos\beta_0 t|\,dt + \sum_{k=1}^{m} a_k \int_0^\infty \int_{-\tau_k}^{0} |v(t-s-\tau_k)||e^{\alpha_0 s}|\cos\beta_0 s|\,ds\,dt \\
&= \int_0^\infty |e^{\alpha_0 t}\cos\beta_0 t|\,dt + \sum_{k=1}^{m} a_k \int_{-\tau_k}^{0} e^{\alpha_0 s}|\cos\beta_0 s| \int_0^\infty |v(t-s-\tau_k)|\,dt\,ds \\
&= \int_0^\infty |e^{\alpha_0 t}\cos\beta_0 t|\,dt + A\sum_{k=1}^{m} a_k \int_{-\tau_k}^{0} e^{\alpha_0 s}|\cos\beta_0 s|\,ds \\
&= \int_0^\infty |e^{\alpha_0 t}\cos\beta_0 t|\,dt + A\sum_{k=1}^{m} a_k \int_{0}^{\tau_k} e^{-\alpha_0 s}|\cos\beta_0 s|\,ds.
\end{aligned}$$

Assumption (2.7) yields

$$\cos\beta_0 s \geq 0, \qquad s \in [0, \tau_m],$$

therefore

$$A \leq \int_0^\infty |e^{\alpha_0 t}\cos\beta_0 t|\,dt + A\sum_{k=1}^m a_k \int_0^{\tau_k} e^{-\alpha_0 s}\cos\beta_0 s\,ds. \tag{2.9}$$

From relations (2.3) and (2.4), it readily follows

$$\begin{aligned}
&\sum_{k=1}^m a_k \int_0^{\tau_k} e^{-\alpha_0 s}\cos\beta_0 s\,ds \\
&\qquad = \sum_{k=1}^m a_k \frac{-e^{-\alpha_0\,\tau_k}\alpha_0\,\cos(\beta_0\,\tau_k) + e^{-\alpha_0\,\tau_k}\beta_0\,\sin(\beta_0\,\tau_k) + \alpha_0}{{\alpha_0}^2 + {\beta_0}^2} \\
&\qquad = \frac{(\alpha_0 + a_0)\alpha_0 + \beta_0^2}{{\alpha_0}^2 + {\beta_0}^2} + \frac{\alpha_0}{{\alpha_0}^2 + {\beta_0}^2}\sum_{k=1}^m a_k, \\
&\qquad = 1 + \frac{\alpha_0}{{\alpha_0}^2 + {\beta_0}^2}\sum_{k=0}^m a_k,
\end{aligned}$$

hence

$$A \leq \int_0^\infty |e^{\alpha_0 t}\cos\beta_0 t|\,dt + A\left(1 + \frac{\alpha_0}{{\alpha_0}^2 + {\beta_0}^2}\sum_{k=0}^m a_k\right),$$

and so

$$A \leq \frac{{\alpha_0}^2 + {\beta_0}^2}{-\alpha_0\sum_{k=0}^m a_k}\int_0^\infty |e^{\alpha_0 t}\cos\beta_0 t|\,dt, \tag{2.10}$$

since $\alpha_0 < 0$. Now we compute the integral $B := \int_0^\infty |e^{\alpha_0 t}\cos\beta_0 t|\,dt$. Let

$$t_j = \frac{\frac{\pi}{2} + j\pi}{\beta_0}, \qquad (j = 0, 1, \ldots),$$

then

$$\begin{aligned}
B \;=\;& \int_0^{t_0} e^{\alpha_0 t}\cos\beta_0 t\,dt - \sum_{j=0}^\infty \int_{t_{2j}}^{t_{2j+1}} e^{\alpha_0 t}\cos\beta_0 t\,dt \\
&+ \sum_{j=0}^\infty \int_{t_{2j+1}}^{t_{2j+2}} e^{\alpha_0 t}\cos\beta_0 t\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\beta_0 e^{\frac{\alpha_0\pi}{2\beta_0}} - \alpha_0}{\alpha_0^2+\beta_0^2} + \frac{\beta_0}{\alpha_0^2+\beta_0^2}\sum_{j=0}^{\infty}\left(e^{\frac{(3+4j)\alpha_0\pi}{2\beta_0}} + e^{\frac{(1+4j)\alpha_0\pi}{2\beta_0}}\right) \\
&\quad + \frac{\beta_0}{\alpha_0^2+\beta_0^2}\sum_{j=0}^{\infty}\left(e^{\frac{(5+4j)\alpha_0\pi}{2\beta_0}} + e^{\frac{(3+4j)\alpha_0\pi}{2\beta_0}}\right) \\
&= \frac{\beta_0 e^{\frac{\alpha_0\pi}{2\beta_0}} - \alpha_0}{\alpha_0^2+\beta_0^2} + \frac{\beta_0}{\alpha_0^2+\beta_0^2}\left(\frac{e^{\frac{5\alpha_0\pi}{2\beta_0}}}{1-e^{\frac{2\alpha_0\pi}{\beta_0}}} + 2\frac{e^{\frac{3\alpha_0\pi}{2\beta_0}}}{1-e^{\frac{2\alpha_0\pi}{\beta_0}}} + \frac{e^{\frac{\alpha_0\pi}{2\beta_0}}}{1-e^{\frac{2\alpha_0\pi}{\beta_0}}}\right) \\
&= \frac{\beta_0 e^{\frac{\alpha_0\pi}{2\beta_0}} - \alpha_0}{\alpha_0^2+\beta_0^2} + \frac{\beta_0 e^{\frac{\alpha_0\pi}{2\beta_0}}}{(\alpha_0^2+\beta_0^2)\left(1-e^{\frac{2\alpha_0\pi}{\beta_0}}\right)}\left(e^{\frac{2\alpha_0\pi}{\beta_0}} + 2e^{\frac{\alpha_0\pi}{\beta_0}} + 1\right) \\
&= \frac{\beta_0 e^{\frac{\alpha_0\pi}{2\beta_0}} - \alpha_0}{\alpha_0^2+\beta_0^2} + \frac{\beta_0 e^{\frac{\alpha_0\pi}{2\beta_0}}\left(1+e^{\frac{\alpha_0\pi}{\beta_0}}\right)}{(\alpha_0^2+\beta_0^2)\left(1-e^{\frac{\alpha_0\pi}{\beta_0}}\right)} \\
&= \frac{-\alpha_0}{\alpha_0^2+\beta_0^2} + \frac{2\beta_0 e^{\frac{\alpha_0\pi}{2\beta_0}}}{(\alpha_0^2+\beta_0^2)\left(1-e^{\frac{\alpha_0\pi}{\beta_0}}\right)}.
\end{aligned}
$$

This relation combined with (2.10) implies (2.8).

□

We note that if we use estimate

$$\int_0^\infty |e^{\alpha_0 t}\cos\beta_0 t|\,dt \le \int_0^\infty e^{\alpha_0 t}\,dt = -\frac{1}{\alpha_0}.$$

in (2.10) instead of the exact value of the integral $\int_0^\infty |e^{\alpha_0 t}\cos\beta_0 t|\,dt$, then we get

$$\int_0^\infty |v(t)|\,dt \le \frac{1}{\sum_{k=0}^m a_k}\frac{\alpha_0^2+\beta_0^2}{\alpha_0^2}, \tag{2.11}$$

which is the multiple delay analogue of (1.20). Of course, the proof of Theorem 2.2 shows that (2.8) is a better estimate than (2.11) if $\beta_0 \neq 0$.

Remark 2.3 *It is easy to see that for any fixed* $\alpha_0 < 0$

$$\lim_{\beta_0\to 0+}\frac{2\beta_0 e^{\frac{\alpha_0\pi}{2\beta_0}}}{\alpha_0\left(1-e^{\frac{\alpha_0\pi}{\beta_0}}\right)} = 0. \tag{2.12}$$

If in the case when the fundamental solution is positive, i.e., $\beta_0 = 0$*, we interpret the constant on the right-hand side of* (2.8) *using the limit* (2.12)*, then in this case* (2.8) *holds, as well. Moreover, by Proposition 2.1 (ii) and (iii), in this case* (2.8) *is satisfied with equality.*

It follows from Proposition 2.1 (v) that if $(\sum_{k=1}^m a_k)\,\tau_m \le \frac{1}{e}$, then the fundamental solution of (2.1) is positive. Therefore the only interesting case is to estimate $\int_0^\infty |v(t)|\,dt$ when $(\sum_{k=1}^m a_k)\,\tau_m > \frac{1}{e}$. (Note that in this case the fundamental solution can be positive, as well, see the gap between the conditions of part (iv) and (v) of Proposition 2.1). In the next Theorem we formulate an explicit estimate in terms of the parameters of the equation for this case.

Theorem 2.4 *Suppose (H1)-(H2) and $a_0 \ge 0$, and let v be the fundamental solution of* (2.1)*. Then if τ_m is such that*

$$\frac{1}{e} < A\tau_m < 1 + \frac{1}{e}, \tag{2.13}$$

where

$$A = \sum_{k=0}^m a_k,$$

then

$$\int_0^\infty |v(t)|\,dt \le \frac{1}{A} \cdot \frac{1 - \frac{1}{e} + A\tau_m}{1 + \frac{1}{e} - A\tau_m}. \tag{2.14}$$

Proof We define

$$\sigma_m = \frac{1}{eA} \quad \text{and} \quad \sigma_0 = 0.$$

Then (2.13) implies

$$0 < A(\tau_m - \sigma_m) < 1.$$

Fix arbitrary $\sigma_k \ge 0$ $(k = 1, \ldots, m-1)$ satisfying

$$\tau_k - (\tau_m - \sigma_m) \le \sigma_k \le \min(\tau_k, \sigma_m).$$

Then (2.13) yields

$$0 \le \sum_{k=0}^m a_k(\tau_k - \sigma_k) \le A(\tau_m - \sigma_m) < 1.$$

Let w be the fundamental solution of Equation (2.5) where we replace all τ_k by σ_k, i.e., the solution of

$$\dot w(t) = -\sum_{k=0}^m a_k w(t - \sigma_k), \qquad t \ge 0 \tag{2.15}$$

and the associated initial condition (2.6). Then the definition of σ_m and Proposition 2.1 (v) yield $w(t) > 0$ for $t > 0$. We rewrite (2.5) as

$$\dot v(t) = -\sum_{k=0}^m a_k v(t - \sigma_k) - \sum_{k=0}^m a_k \Big(v(t - \tau_k) - v(t - \sigma_k) \Big).$$

Then the variation of constants formula implies

$$v(t) = w(t) - \sum_{k=0}^{m} a_k \int_0^t w(t-s)\Big(v(s-\tau_k) - v(s-\sigma_k)\Big)\,ds,$$

therefore, using Proposition 2.1 (iii),

$$\begin{aligned}
&\int_0^\infty |v(t)|\,dt \\
&\qquad\le \int_0^\infty w(t)\,dt + \sum_{k=0}^{m} a_k \int_0^\infty \int_0^t w(t-s)\Big|v(s-\tau_k) - v(s-\sigma_k)\Big|\,ds\,dt \\
&\qquad= \int_0^\infty w(t)\,dt + \sum_{k=0}^{m} a_k \int_0^\infty w(t)\,dt \int_0^\infty \Big|v(s-\tau_k) - v(s-\sigma_k)\Big|\,ds \\
&\qquad= \frac{1}{A}\left(1 + \sum_{k=0}^{m} a_k \int_0^\infty \Big|v(s-\tau_k) - v(s-\sigma_k)\Big|\,ds\right). \qquad (2.16)
\end{aligned}$$

Consider the integral on the right hand side. Simple manipulations and (2.6) yield

$$\begin{aligned}
&\int_0^\infty \Big|v(s-\tau_k) - v(s-\sigma_k)\Big|\,ds \\
&\qquad= \int_0^{\tau_k} \Big|v(s-\tau_k) - v(s-\sigma_k)\Big|\,ds + \int_{\tau_k}^\infty \Big|\int_{s-\tau_k}^{s-\sigma_k} \dot v(u)\,du\Big|\,ds \\
&\qquad= \int_0^{\tau_k} |v(s-\sigma_k)|\,ds + \int_{\tau_k}^\infty \Big|\sum_{j=0}^{m} a_j \int_{s-\tau_k}^{s-\sigma_k} v(u-\tau_j)\,du\Big|\,ds \\
&\qquad\le \int_0^{\tau_k-\sigma_k} |v(s)|\,ds + \sum_{j=0}^{m} a_j \int_{\tau_k}^\infty \int_{s-\tau_k}^{s-\sigma_k} |v(u-\tau_j)|\,du\,ds \\
&\qquad\le \int_0^{\tau_k-\sigma_k} |v(s)|\,ds + \sum_{j=0}^{m} a_j \int_0^\infty \int_{u+\sigma_k}^{u+\tau_k} |v(u-\tau_j)|\,ds\,du \\
&\qquad= \int_0^{\tau_k-\sigma_k} |v(s)|\,ds + (\tau_k - \sigma_k)\sum_{j=0}^{m} a_j \int_0^\infty |v(u)|\,du.
\end{aligned}$$

Therefore, combining this estimate with (2.16), we get

$$\int_0^\infty |v(u)|\,du \le \frac{1}{A}\left(1 + \sum_{k=0}^{m} a_k \int_0^{\tau_k-\sigma_k} |v(s)|\,ds\right) + \sum_{k=0}^{m} a_k(\tau_k - \sigma_k)\int_0^\infty |v(u)|\,du,$$

so

$$\int_0^\infty |v(u)|\,du \le \frac{1}{A}\,\frac{1 + \sum_{k=0}^{m} a_k \int_0^{\tau_k-\sigma_k} |v(s)|\,ds}{1 - \sum_{k=0}^{m} a_k(\tau_k - \sigma_k)}. \qquad (2.17)$$

Since $\dot{v}(0) = -a_0 \le 0$ and $v(0) = 1$, $v(t)$ is decreasing for small positive t. There are two cases: either $0 < v(t) < 1$ for all $t > 0$, or there exists $t_0 > 0$ such that $0 < v(t) < 1$ for $t \in [0, t_0)$ and $v(t_0) = 0$. Assume this latter case. Then integrating (2.5) from 0 to t_0 we get

$$1 = \sum_{k=0}^{m} a_k \int_0^{t_0} v(s - \tau_k)\, ds \le \left(\sum_{k=0}^{m} a_k \right) t_0.$$

Hence

$$\tau_k - \sigma_k \le \tau_m - \sigma_m \le \frac{1}{A} \le t_0, \qquad k = 0, \ldots, m,$$

therefore in both cases

$$\int_0^{\tau_k - \sigma_k} |v(s)|\, ds \le \tau_k - \sigma_k, \qquad k = 0, \ldots, m.$$

Then (2.14) follows from (2.17), using

$$\sum_{k=0}^{m} a_k(\tau_k - \sigma_k) \le A(\tau_m - \sigma_m) = A\tau_m - \frac{1}{e}.$$

□

Example 2.5 Consider the scalar equation with two delays

$$\dot{x}(t) = -0.1x(t) - 0.3x(t-1) - 0.5x(t-1.2), \qquad t \ge 0. \tag{2.18}$$

The graph of its fundamental solution is plotted in Figure 2.1. It can be seen from the graph that the trivial solution of (2.18) is asymptotically stable. In Figure 2.2 the two curves defined by (2.3) and (2.4) are plotted for this equation. The solid lines are the graphs of the curves corresponding to (2.3), the dotted lines are the curves defined by (2.4). One solution of (2.3)–(2.4) is $\alpha_0 = -0.3796769591$ and $\beta_0 = 1.186675690$. This root of (2.2) satisfies (2.7), therefore Theorem 2.2 can be applied. We get by (2.11) $\int_0^\infty |v(t)|\, dt \le 11.96519307$, and by applying (2.8) $\int_0^\infty |v(t)|\, dt \le 7.738495451$. Theorem 2.4 can also be applied, since $1/e < (a_0 + a_1 + a_2)\tau_2 = 1.08 < 1 + 1/e$. (2.14) yields $\int_0^\infty |v(t)|\, dt \le 4.337121867$.

3 Single Delay Case

In this section we consider the scalar delay equation

$$\dot{x}(t) = -ax(t) - bx(t - \tau), \qquad t \ge 0, \tag{3.1}$$

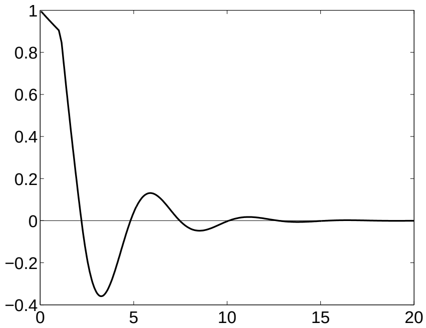

Figure 2.1: Fundamental solution of (2.18).

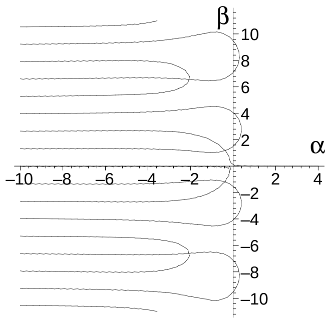

Figure 2.2: Characteristic curves of (2.18).

where $a, b \in \mathbb{R}$, $\tau > 0$. The characteristic equation of (2.1) is

$$\lambda = -a - be^{-\lambda\tau}. \tag{3.2}$$

We show that the leading characteristic root of (3.2) always satisfies condition (2.7) of Theorem 2.2 of the previous section. It is known (see, e.g., [14]) that (3.2) always has a leading root.

Before we formulate the main result of this section we introduce the following notation. For a fixed τ let $R(\tau) \subset \mathbb{R}^2$ be the set of points (a, b) bounded by the curves $b = \frac{\pi}{2\tau}e^{-a\tau}$, $b = -a$ and by the curve

$$a = -s\cot(\tau s), \quad b = \frac{s}{\sin(\tau s)}, \qquad s \in \left[0, \frac{\pi}{2\tau}\right]. \tag{3.3}$$

The points of the boundaries $b = -a$ and (3.3) do not belong to P, but the points of $b = \frac{\pi}{2\tau}e^{-a\tau}$ do. (See Figure 3.4.) The next theorem says that $R(\tau)$ is the set of parameters (a, b) for which estimates (2.8) and (2.11) can be applied.

Theorem 3.1 *Let $a, b \in \mathbb{R}$, $\tau > 0$.*

(i) A leading characteristic root $\lambda_0 = \alpha_0 + i\beta_0$ of (3.2) is

(1) a real number, if and only if

$$b\tau e^{a\tau} \le \frac{1}{e}; \tag{3.4}$$

(2) a complex number satisfying

$$0 < |\beta_0|\tau < \frac{\pi}{2}, \tag{3.5}$$

if and only if

$$\frac{1}{e} < b\tau e^{a\tau} < \frac{\pi}{2}. \tag{3.6}$$

(ii) Let v be the fundamental solution of (3.1)*.*

(1) Then $v(t) > 0$ for $t \geq 0$ and $v(t) \to 0$ as $t \to \infty$, if and only if

$$-a < b \leq \frac{1}{e\tau} e^{-a\tau}. \tag{3.7}$$

Moreover, in this case

$$\int_0^\infty v(t)\,dt = \frac{1}{a+b}. \tag{3.8}$$

(2) If $(a,b) \in R(\tau)$, then a leading characteristic root of (3.1) *satisfies* (2.7)*, and $v(t) \to 0$ as $t \to \infty$. Therefore in this case* (2.8) *and* (2.11) *hold.*

The stability region of (3.1) is well-known (see, e.g., [14]). To simplify notation we introduce the open set $S(\tau) \subset \mathbb{R}^2$ as the points bounded below by the line $b = -a$ and from above by the curve

$$a = -s\cot(\tau s), \quad b = \frac{s}{\sin(\tau s)}, \qquad s \in \left[0, \frac{\pi}{\tau}\right].$$

See Figure 3.4.

Lemma 3.2 (see, e.g., [14]) *The trivial solution of* (3.1) *is asymptotically stable, if and only if $(a,b) \in S(\tau)$.*

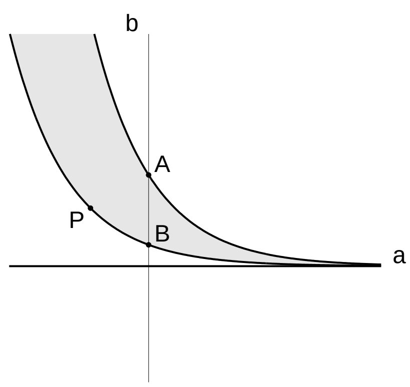

Figure 3.3: Parameters satisfying (3.6).

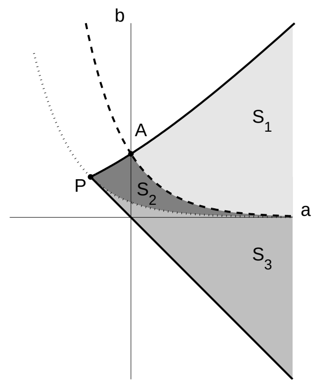

Figure 3.4: Partitions of the stability region of (3.1). $S(\tau) = S_1 \cup S_2 \cup S_3$, $R(\tau) = S_2 \cup S_3$

Figure 3.3 illustrates condition (3.6). The upper curve is $b = \frac{\pi}{2\tau}e^{-a\tau}$, and the lower curve is $b = \frac{1}{e\tau}e^{-a\tau}$. The point A in Figure 3.3 is $(0, \frac{\pi}{2\tau})$, $B = (0, \frac{1}{e\tau})$, and $P = (-\frac{1}{\tau}, \frac{1}{\tau})$.

Figure 3.4 shows the stability region $S(\tau)$. It is decomposed into three subregions, S_1, S_2 and S_3 by the curves $b = \frac{\pi}{2\tau}e^{-a\tau}$ and $b = \frac{1}{e\tau}e^{-a\tau}$. S_1 is the part of the stability region where estimates (2.8) and (2.11) can not be applied. In region S_3, Part (ii) (1) of Theorem 3.1 holds, i.e., $v(t) > 0$. In S_2 the leading root of (3.2) is complex, therefore $v(t)$ is oscillatory, but estimates (2.11) and (2.8) can be used. With the notation of Theorem 3.1 (ii) (2), $R(\tau) = S_2 \cup S_3$.

The proof of Theorem 3.1 will be based on a series of lemmas. First we need the characterization of the real roots of (3.2). The next lemma will prove Part (i) (1) of Theorem 3.1. Note that it follows from Corollary 2.2.1 in [13], as well.

Lemma 3.3 *Let $a, b \in \mathbb{R}$, $\tau > 0$.*

(i) If $0 < b\tau e^{a\tau} < \frac{1}{e}$, then (3.2) has exactly two real roots, λ_1, λ_2, which satisfy

$$\lambda_1 < -a + \frac{1}{\tau}\log(b\tau) < -a - \frac{1}{\tau} < \lambda_2 < -a.$$

(ii) If $b\tau e^{a\tau} = \frac{1}{e}$, then (3.2) has a unique real root, $\lambda_0 = -a - \frac{1}{\tau}$, which is a double root.

(iii) If $b\tau e^{a\tau} > \frac{1}{e}$, then (3.2) has no real root.

(iv) If $b < 0$, then (3.2) has a unique real root $\lambda_0 > -a$.

Moreover, in Case (i) and (iv) all real roots are simple, and all complex roots have smaller real part than the largest real root.

Proof Introducing the new variable $\mu = \lambda + a$, (3.2) is transformed into

$$\mu = -be^{a\tau}e^{-\mu\tau},$$

for which the above properties are well-known (see, e.g., [4] or [17]). □

Next we concentrate on the complex roots of (3.2), therefore we will assume that

$$b\tau e^{a\tau} > \frac{1}{e}. \tag{3.9}$$

Let $\lambda = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$, then (2.2) is equivalent to

$$\begin{aligned} \alpha + a &= -be^{-\alpha\tau}\cos\beta\tau, && (3.10)\\ \beta &= be^{-\alpha\tau}\sin\beta\tau. && (3.11) \end{aligned}$$

Since we investigate the case of the complex roots of (3.2), we can assume that $\beta \neq 0$. In this case simple algebraic manipulation of these equations yields the equivalent system

$$\alpha + a = -b\beta \cot \beta\tau, \tag{3.12}$$
$$\beta^2 = b^2 e^{-2\alpha\tau} - (\alpha + a)^2. \tag{3.13}$$

We define the function

$$h(t) = b^2 e^{2a\tau} e^{-2\tau t} - t^2. \tag{3.14}$$

Then, introducing the new variables $t = \alpha + a$ and $s = \beta$, system (3.12)–(3.13) is equivalent to

$$t = -bs \cot \tau s, \tag{3.15}$$
$$s^2 = h(t). \tag{3.16}$$

The graph of the function $-bs \cot \tau s$ can be seen in Figure 3.5.

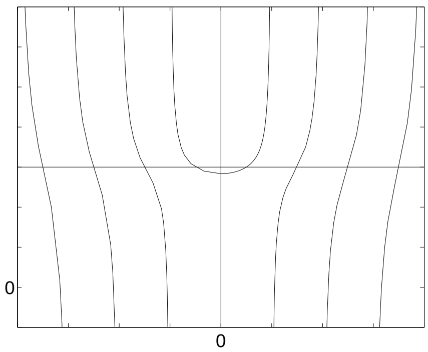

Figure 3.5: The graph of $-bs \cot \tau s$ for the case $b > 0$.

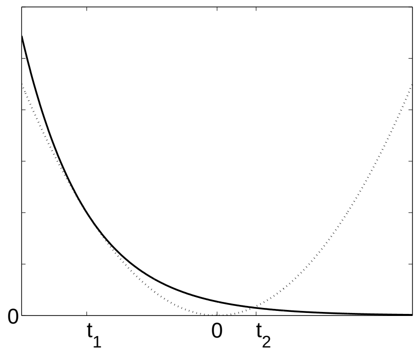

Figure 3.6: The graphs of the functions t^2 and $b^2 e^{2a\tau} e^{-2\tau t}$ in Case (ii) of Lemma 3.5.

The following lemmas describe the roots of h' and h.

Lemma 3.4 *Let h be defined by* (3.14), *and $b > 0$. Then*

(i) if $b\tau e^{a\tau} < \frac{1}{\sqrt{2e}}$, then h' has two real roots $u_1 < u_2 < 0$;

(ii) if $b\tau e^{a\tau} = \frac{1}{\sqrt{2e}}$, then h' has a unique real root $u_0 = -\frac{1}{2\tau}$;

(iii) if $b\tau e^{a\tau} > \frac{1}{\sqrt{2e}}$, then h' has no real root.

Proof We have $h'(t) = 2(-\tau b^2 e^{2a\tau} e^{-2\tau t} - t)$. Therefore the statement of this lemma follows easily from Lemma 3.3. $\square$

Lemma 3.5 *Let h be defined by (3.14), and $b > 0$. Then*

(i) if $b\tau e^{a\tau} > \frac{1}{e}$, then h has exactly one real root $t_0 > 0$, which is a simple root;

(ii) if $b\tau e^{a\tau} = \frac{1}{e}$, then h has exactly two real roots $t_1 = -\frac{1}{\tau}$ and $t_2 > 0$, where t_1 is a double root, t_2 is a single root;

(iii) if $b\tau e^{a\tau} < \frac{1}{e}$, then h has exactly three real roots t_1, t_2 and t_3 satisfying

$$t_1 < -\frac{1}{\tau} < t_2 < 0 < t_3.$$

All three roots are simple roots.

Proof Consider first Case (ii). Then simple substitution yields $h(t_1) = h'(t_1) = 0$ for $t_1 = -\frac{1}{\tau}$. Hence Lemma 3.4 yields $h(t) > 0$ for $t < t_1$ and $h(t) > 0$ for $t_1 < t < 0$. The existence of the root $t_2 > 0$ is trivial. Figure 3.6 contains the graphs of the functions $b^2 e^{2a\tau} e^{-2\tau t}$ and t^2 in this case.

Now consider Case (i), i.e., let a, b and τ be such that $b\tau e^{a\tau} > \frac{1}{e}$. Let $h_1(t) = b^2 e^{2a\tau} e^{-2\tau t}$. Now, decreasing b, we can find $\tilde{b} > 0$ such that $\tilde{b}\tau e^{a\tau} = \frac{1}{e}$, and define the corresponding function $\tilde{h}_1(t) = \tilde{b}^2 e^{2a\tau} e^{-2\tau t}$. Then $h(t) > \tilde{h}_1(t)$ for all t, so the graph of h has no intersection with that of t^2 for negative t, (see Figure 3.6), so the statement follows.

Case (iii) can be argued similarly. $\square$

Remark 3.6 *For the case when $b > 0$ by the help of the previous two lemmas we can easily draw the graph of $h(t)$, and therefore the graph of the curve $s^2 = h(t)$, as well. Five cases have to be distinguished: Case 1: $\frac{1}{\sqrt{2e}} < b\tau e^{a\tau}$, Case 2: $b\tau e^{a\tau} = \frac{1}{\sqrt{2e}}$, Case 3: $\frac{1}{e} < b\tau e^{a\tau} < \frac{1}{\sqrt{2e}}$, Case 4: $b\tau e^{a\tau} = \frac{1}{e}$, and Case 5: $0 < b\tau e^{a\tau} < \frac{1}{e}$. We can see the corresponding graphs of $h(t)$ and the curve $s^2 = h(t)$ in Figure 3.7.*

Consider again (3.10)-(3.11), where we replace $\alpha + a$ by t:

$$\begin{aligned} t &= -be^{a\tau} e^{-\tau t} \cos \tau s \\ s &= be^{a\tau} e^{-\tau t} \sin \tau s. \end{aligned}$$

Combining the two equations we get

$$t = -s \frac{\cos \tau s}{\sin \tau s},$$

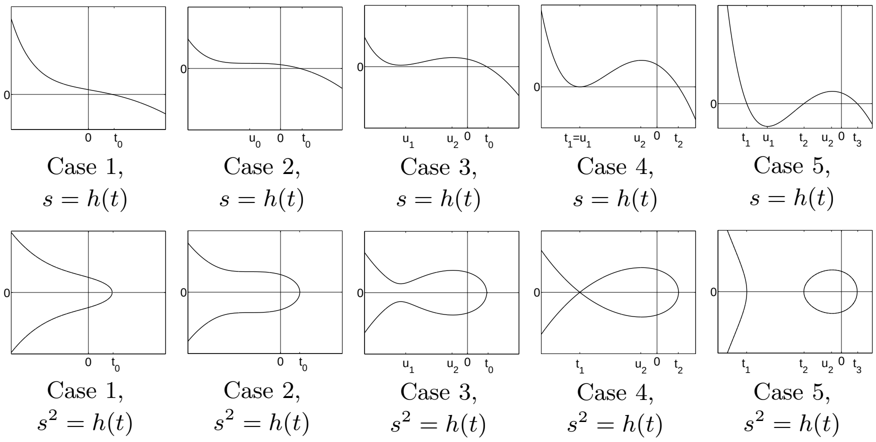

Figure 3.7: The graphs of $s = h(t)$ and $s^2 = h(t)$.

and substituting this back to the second equation we get

$$s = be^{a\tau} \exp\left(\tau s \frac{\cos \tau s}{\sin \tau s}\right) \sin \tau s,$$

hence

$$\tau b e^{a\tau} = \frac{\tau s}{\sin \tau s} \exp\left(-\tau s \frac{\cos \tau s}{\sin \tau s}\right).$$

Introduce the function

$$g(u) := \frac{u}{\sin u} e^{-u \frac{\cos u}{\sin u}}. \tag{3.17}$$

With this notation we have that if t and s solve (3.12)-(3.13), then

$$\tau b e^{a\tau} = g(\tau s). \tag{3.18}$$

Some properties of function g are given in the next lemma.

Lemma 3.7 *Let g be defined by* (3.17). *Then g is strictly monotone increasing on the interval* $(0, \pi)$, *and*

$$\lim_{u \to 0+} g(u) = \frac{1}{e}, \qquad g\left(\frac{\pi}{2}\right) = \frac{\pi}{2}, \qquad \lim_{u \to \pi-} g(u) = \infty.$$

Proof The above limits are obvious. Since

$$g'(u) = e^{-\frac{u \cos u}{\sin u}} \frac{1 - 2u\frac{\cos u}{\sin u} + \frac{u^2}{\sin^2 u}}{\sin u} = e^{-\frac{u \cos u}{\sin u}} \frac{\left(\frac{u}{\sin u} - \cos u\right)^2 + 1 - \cos^2 u}{\sin u},$$

it follows $g'(u) > 0$ on $(0, \pi)$, and the monotonicity of g follows. $\square$

In general, it is easy to show that

$$\lim_{u\to 0+} g(k\pi) = 0, \quad \lim_{u\to 2k\pi-} g(u) = -\infty, \quad \text{and} \quad \lim_{u\to(2k+1)\pi-} g(u) = \infty.$$

See the graph of $g(u)$ in Figure 3.8.

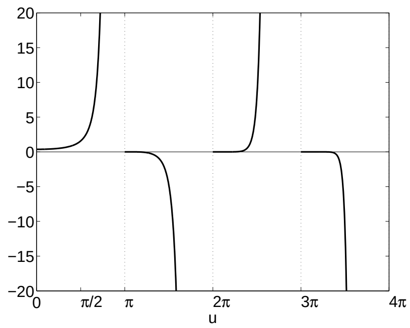

Figure 3.8: The graph of $g(u)$.

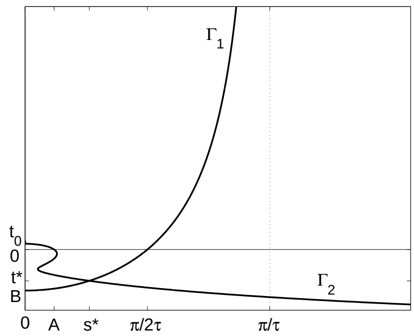

Figure 3.9: The graphs of Γ_1 and Γ_2 in Case (iii) of Remark 3.6.

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1 (i) (2)

Let denote the graph of the function $-bs\cos\alpha s$, $s \in (0, \frac{\pi}{2\tau})$ by Γ_1, and the part of the curve of (3.16) belonging to the half-plane $s \geq 0$ by Γ_2. We first show that under assumption (3.5), Γ_1 and Γ_2 always has at least one intersection (s^*, t^*).

Clearly,

$$A := \sqrt{h(0)} = be^{a\tau} < \frac{\pi}{2\tau}, \qquad B := \lim_{s\to 0} -bs\cot\tau s = -\frac{b}{\tau}.$$

Let t_0 denote the real root of h. Lemma 3.5 yields that $t_0 > 0$. Curve Γ_1 starts at $(0, B)$, intersects the s-axis at $s = \frac{\pi}{2\tau}$, and it is monotone increasing. Curve Γ_2 starts at the point $(0, t_0)$, intersects the s-axis at the point $s = A$.

Assumption (3.5) yields that the graph of curve Γ_2 has type (i), (ii) or (iii) of Remark 3.6. We plotted Γ_1 and Γ_2 in Figure 3.9 for Case (iii) of Remark 3.6. In Cases (i) and (ii) Γ_2 is a monotone decreasing curve. In these cases the intersection of Γ_1 and Γ_2 is unique, and clearly, all other intersections of Γ_2 and other branches of the curve (3.15) have smaller t-coordinate.

In Case (iii) of Remark 3.6, it is easy to see that Γ_1 and Γ_2 has at least one intersection (s^*, t^*). It follows from Lemma 3.7, (3.18) and (3.5) that in this case also the intersection is unique. Since the part of curve Γ_2 between the points $(0, t_0)$ and (s^*, t^*) has no intersection with other branches of the curve (3.15), for all other intersections $(\tilde{s}, \tilde{t})$ it follows $\tilde{t} < t^*$. This completes the proof, since $\alpha_0 = t^* - a$ and $\beta_0 = s^*$ is a leading characteristic root.

Conversly, if (3.5) does not hold, then $A \geq \frac{\pi}{2\tau}$, so Γ_1 and Γ_2 intersect each other at a point with first coordinate greater or equal than $\frac{\pi}{2\tau}$ (see Figure 3.9). □

Proof of Theorem 3.1 (ii) Consider first part (1). Proposition 2.1 (ii) yields that $v(t) > 0$ for $t \geq 0$, if and only if (3.2) has a real root, which, by (3.4), is equivalent to that

$$b \leq \frac{1}{e\tau} e^{-a\tau}.$$

Proposition 2.1 (i) implies that $v(t) \to 0$ as $t \to \infty$, if and only of the trivial solution of (3.1) is asymptotically stable. Therefore Lemma 3.2 yields $v(t) \to 0$, if (3.7) is satisfied. See Figure 3.4, where S_3 denotes the set of parameters satisfying condition (3.7).

Relation (3.8) is the restatement of Proposition 2.1 (iii) for Equation (3.1).

Statement (ii) (2) is a simple application of Lemma 3.2 and Proposition 2.1 (i). □

4 Stability of linear systems

Consider the delay system

$$\dot{x}_k(t) = -\sum_{\ell=0}^{m} \sum_{j=1}^{n} a_{kj}^{(\ell)} x_j(t - \tau_{kj}^{(\ell)}), \qquad k = 1, \ldots, n, \quad t \geq 0, \tag{4.1}$$

where we assume

(A) $a_{kj}^{(\ell)} \in \mathbb{R}$, $(k, j = 1, \ldots, n,\ \ell = 0, \ldots, m)$, $\sum_{\ell=0}^{m} a_{kk}^{(\ell)} \neq 0$, $(k = 1, \ldots, n)$, $\tau_{kj}^{(0)} = 0$, $\tau_{kj}^{(\ell)} \geq 0$, $\quad (k, j = 1, \ldots, n,\ \ell = 1, \ldots, m)$.

For given delays $0 \leq \sigma_k^{(\ell)} \leq \tau_{kk}^{(\ell)}$ $k = 1, \ldots, n$ and $\ell = 1, \ldots, m$ and $\sigma_k^{(0)} = 0$ $(k = 1, \ldots, n)$ we associate the scalar equations

$$\dot{y}_k(t) = -\sum_{\ell=0}^{m} a_{kk}^{(\ell)} y_k(t - \sigma_k^{(\ell)}), \qquad t \geq 0, \qquad k = 1, \ldots, n, \tag{4.2}$$

and let v_k denote the fundamental solution of (4.2). Let γ_k be such that

$$\int_0^\infty |v_k(t)|\, dt \leq \frac{1}{\sum_{\ell=0}^{m} a_{kk}^{(\ell)}} \gamma_k. \tag{4.3}$$

If the trivial solution of (4.2) is asymptotically stable, such estimate can be given using, e.g., (2.8), (2.11) or (2.14), or in case of a positive fundamental solution, $\gamma_k = 1$ satisfies (4.3) with equality.

Theorem 4.1 *Suppose (A), and let σ_k $(k = 1, \dots, n)$ be such that the trivial solution of* (4.2) *is asymptotically stable, γ_k $(k = 1, \dots, n)$ be defined by* (4.3), *and let H be the associated $n \times n$ matrix with components*

$$h_{kj} = \begin{cases} \frac{1}{\gamma_k} \sum_{\ell=0}^{m} a_{kk}^{(\ell)} - (\sum_{\ell=0}^{m} |a_{kk}^{(\ell)}|) \sum_{\ell=1}^{m} |a_{kk}^{(\ell)}| (\tau_{kk}^{(\ell)} - \sigma_k^{(\ell)}), & k = j, \\ -(\sum_{\ell=0}^{m} |a_{kj}^{(\ell)}|) \Big(\sum_{\ell=1}^{m} |a_{kk}^{(\ell)}| (\tau_{kk}^{(\ell)} - \sigma_k^{(\ell)}) + 1 \Big), & k \neq j. \end{cases}$$

Suppose matrix H is a nonsingular M-matrix. Then the trivial solution of (4.1) *is asymptotically stable for any selection of the delays $\tau_{kj}^{(\ell)} \geq 0$, $k \neq j$ $(k, j = 1, \dots, n,\ \ell = 1, \dots, m)$.*

Proof Let x_k and y_k denote the solutions of (4.1) and (4.2), respecively, associated to the same initial functions

$$x_k(t) = y_k(t) = \varphi_k(t), \qquad k = 1, \dots, n, \quad t \in [-r, 0],$$

where $r = \max\{\tau_{kj}^{(\ell)} :\ k, j = 1, \dots, n,\ \ell = 1, \dots, m\}$. We rewrite (4.1) as

$$\begin{aligned} \dot{x}_k(t) &= -\sum_{\ell=0}^{m} a_{kk}^{(\ell)} x_k(t - \sigma_k^{(\ell)}) - \sum_{\ell=1}^{m} a_{kk}^{(\ell)} (x_k(t - \tau_{kk}^{(\ell)}) - x_k(t - \sigma_k^{(\ell)})) \\ &\quad - \sum_{\ell=0}^{m} \sum_{\substack{j=1, \\ j \neq k}}^{n} a_{kj}^{(\ell)} x_j(t - \tau_{kj}^{(\ell)}), \qquad k = 1, \dots, n, \quad t \geq 0, \end{aligned}$$

and therefore

$$\begin{aligned} x_k(t) &= y_k(t) - \sum_{\ell=1}^{m} a_{kk}^{(\ell)} \int_0^t v_k(t-s) \Big(x_k(s - \tau_{kk}^{(\ell)}) - x_k(s - \sigma_k^{(\ell)}) \Big)\, ds \\ &\quad - \sum_{\ell=0}^{m} \sum_{\substack{j=1, \\ j \neq k}}^{n} a_{kj}^{(\ell)} \int_0^t v_k(t-s) x_j(s - \tau_{kj}^{(\ell)})\, ds \end{aligned}$$

for $k = 1, \dots, n$, $\quad t \geq 0$. Suppose $t \geq r$, then we get

$$\begin{aligned} |x_k(t)| &\leq |y_k(t)| + \sum_{\ell=1}^{m} |a_{kk}^{(\ell)}| \int_0^{\tau_{kk}^{(\ell)}} |v_k(t-s)| \Big| x_k(s - \tau_{kk}^{(\ell)}) - x_k(s - \sigma_k^{(\ell)}) \Big|\, ds \\ &\quad + \sum_{\ell=1}^{m} |a_{kk}^{(\ell)}| \int_{\tau_{kk}^{(\ell)}}^{t} |v_k(t-s)| \Big| \int_{s-\sigma_k^{(\ell)}}^{s-\tau_{kk}^{(\ell)}} \dot{x}_k(u)\, du \Big|\, ds \\ &\quad + \sum_{\ell=0}^{m} \sum_{\substack{j=1, \\ j \neq k}}^{n} |a_{kj}^{(\ell)}| \int_0^t |v_k(t-s)| |x_j(s - \tau_{kj}^{(\ell)})|\, ds \end{aligned}$$

$$\begin{aligned}
&\leq \ |y_k(t)| + \sum_{\ell=1}^{m} |a_{kk}^{(\ell)}| \int_0^{\tau_{kk}^{(\ell)}} |v_k(t-s)| \Big| x_k(s-\tau_{kk}^{(\ell)}) - x_k(s-\sigma_k^{(\ell)}) \Big| \, ds \\
&\quad + \sum_{\ell=1}^{m} |a_{kk}^{(\ell)}| \int_{\tau_{kk}^{(\ell)}}^{t} |v_k(t-s)| \Big| \int_{s-\sigma_k^{(\ell)}}^{s-\tau_{kk}^{(\ell)}} \sum_{p=0}^{m} \sum_{j=1}^{n} a_{kj}^{(p)} x_j(u-\tau_{kj}^{(p)}) \, du \Big| \, ds \\
&\quad + \sum_{\ell=0}^{m} \sum_{\substack{j=1,\\ j\neq k}}^{n} |a_{kj}^{(\ell)}| \int_0^t |v_k(t-s)| |x_j(s-\tau_{kj}^{(\ell)})| \, ds, \qquad t \geq r. \qquad (4.4)
\end{aligned}$$

Introduce the functions

$$z_k(t) = \max\{|x_k(s)| \colon \ s \in [-r,t]\}, \qquad t \geq 0, \quad k = 1, \ldots, n,$$

and the constants

$$M_k = \max\{|y_k(s)| \colon \ s \in [-r,\infty)\} \quad \text{and} \quad B_k = \sum_{\ell=0}^{m} a_{kk}^{(\ell)}, \qquad k = 1, \ldots, n.$$

Then

$$\begin{aligned}
|x_k(t)| \quad &\leq \quad M_k + \sum_{\ell=1}^{m} |a_{kk}^{(\ell)}| 2 z_k(r) \int_0^{\tau_{kk}^{(\ell)}} |v_k(t-s)| \, ds \\
&\quad + \sum_{\ell=1}^{m} |a_{kk}^{(\ell)}| (\tau_{kk}^{(\ell)} - \sigma_k^{(\ell)}) \Big(\sum_{p=0}^{m} \sum_{j=1}^{n} |a_{kj}^{(p)}| z_j(t) \Big) \int_{\tau_{kk}^{(\ell)}}^{t} |v_k(t-s)| \, ds \\
&\quad + \Big(\sum_{\ell=0}^{m} \sum_{\substack{j=1,\\ j\neq k}}^{n} |a_{kj}^{(\ell)}| z_j(t) \Big) \int_0^t |v_k(t-s)| \, ds, \qquad t \geq r,
\end{aligned}$$

and so

$$\begin{aligned}
|x_k(t)| \quad &\leq \quad M_k + z_k(r) + \frac{\gamma_k}{B_k} \sum_{\ell=1}^{m} |a_{kk}^{(\ell)}| 2 z_k(r) \\
&\quad + \sum_{j=1}^{n} \Big(\frac{\gamma_k}{B_k} \sum_{\ell=1}^{m} |a_{kk}^{(\ell)}| (\tau_{kk}^{(\ell)} - \sigma_k^{(\ell)}) \sum_{p=0}^{m} |a_{kj}^{(p)}| \Big) z_j(t) \\
&\quad + \sum_{\substack{j=1,\\ j\neq k}}^{n} \Big(\frac{\gamma_k}{B_k} \sum_{\ell=0}^{m} |a_{kj}^{(\ell)}| \Big) z_j(t), \ \ t \geq 0.
\end{aligned}$$

Since the right-hand-side is monotone in t, it implies

$$z_k(t) \quad \leq \quad M_k + z_k(r) + \frac{\gamma_k}{B_k} \sum_{\ell=1}^{m} |a_{kk}^{(\ell)}| 2 z_k(r)$$

$$+ \sum_{j=1}^{n}\Big(\frac{\gamma_k}{B_k}\sum_{\ell=1}^{m}|a_{kk}^{(\ell)}|(\tau_{kk}^{(\ell)}-\sigma_k^{(\ell)})\sum_{p=0}^{m}|a_{kj}^{(p)}|\Big)z_j(t)$$
$$+ \sum_{\substack{j=1,\\ j\neq k}}^{n}\Big(\frac{\gamma_k}{B_k}\sum_{\ell=0}^{m}|a_{kj}^{(\ell)}|\Big)z_j(t), \quad t\geq 0.$$

Hence the definition of H yields

$$H\mathbf{z}(t)\leq \mathbf{w}, \qquad t\geq 0,$$

where $\mathbf{z}(t)=(z_1(t),\ldots,z_n(t))^T$, $\mathbf{w}=(w_1,\ldots,w_n)^T$ and

$$w_k=\frac{(M_k+z_k(r))B_k}{\gamma_k}+\sum_{\ell=1}^{m}|a_{kk}^{(\ell)}|2z_k(r), \qquad k=1,\ldots,n,$$

and $\leq$ denotes componentwise inequality for vectors. Therefore

$$\mathbf{z}(t)\leq H^{-1}\mathbf{w}, \qquad t\geq 0,$$

since H is a nonsingular M-matrix, and so we get that the functions x_k are bounded on $[-r,\infty)$ for $k=1,\ldots,n$.

Next we show that $\lim_{t\to\infty}|x_k(t)|=0$ for $k=1,\ldots,n$. Denote

$$m_k=\overline{\lim_{t\to\infty}}\,|x_k(t)|, \qquad k=1,\ldots,n.$$

Using the relation

$$\overline{\lim_{t\to\infty}}\int_T^t |v_k(t-s)||\alpha(s)|\,ds\leq\int_0^\infty |v_k(t)|\,dt\,\overline{\lim_{t\to\infty}}\,|\alpha(t)|, \qquad T\geq 0,$$

(see, e.g., Lemma 2.3 in [7]), $\lim_{t\to\infty}|v_k(t)|=0$ and $\lim_{t\to\infty}|y_k(t)|=0$, inequality (4.4) yields

$$\begin{aligned}
m_k \;\leq\; & \sum_{\ell=1}^{m}|a_{kk}^{(\ell)}|\int_0^\infty |v_k(t)|\,dt\,\overline{\lim_{s\to\infty}}\Big|\int_{s-\sigma_k^{(\ell)}}^{s-\tau_{kk}^{(\ell)}}\sum_{p=0}^{m}\sum_{j=1}^{n}a_{kj}^{(p)}x_j(u-\tau_{kj}^{(p)})\,du\Big| \\
& + \int_0^\infty |v_k(t)|\,dt\sum_{\ell=0}^{m}\sum_{\substack{j=1,\\ j\neq k}}^{n}|a_{kj}^{(\ell)}|m_j \\
\leq\; & \sum_{j=1}^{n}\Bigg(\frac{\gamma_k}{B_k}\sum_{\ell=1}^{m}|a_{kk}^{(\ell)}|(\tau_{kk}^{(\ell)}-\sigma_k^{(\ell)})\sum_{p=0}^{m}|a_{kj}^{(p)}|\Bigg)m_j+\sum_{\substack{j=1,\\ j\neq k}}^{n}\Bigg(\frac{\gamma_k}{B_k}\sum_{\ell=0}^{m}|a_{kj}^{(\ell)}|\Bigg)m_j.
\end{aligned}$$

Rearranging the inequalities and using the definition of H we get

$$H\mathbf{m}\leq\mathbf{0},$$

where $\mathbf{m} = (m_1, \ldots, m_n)^T$. H is a nonsingular M-matrix, therefore $\mathbf{m} \le \mathbf{0}$, and the proof is completed, since, on the other hand, $\mathbf{m} \ge \mathbf{0}$. $\square$

Note that if $m = 1$ and we take $\sigma_k = \tau_{kk}^{(1)}$ in the previous theorem, we get back Theorem 1.2. The advantage of this formulation is that if $\sigma_k^{(\ell)}$ can be selected so that the fundamental solution of (4.2) is positive, than the exact value of its absolute integral can be used in the theorem.

Example 4.2 Consider the two-dimensional delay system

$$\begin{aligned} \dot{x}_1(t) &= -0.1x_1(t) - 0.3x_1(t-0.2) - 0.5x_1(t-0.4) \\ &\quad - 0.5x_2(t-\tau_{12}^{(1)}) + 0.2x_2(t-\tau_{12}^{(2)}) \\ \dot{x}_2(t) &= 0.2x_1(t-\tau_{21}^{(1)}) + 0.3x_1(t-\tau_{21}^{(2)}) \\ &\quad - 0.2x_2(t) - 0.4x_2(t-0.2) - 0.1x_2(t-0.5), \end{aligned} \tag{4.5}$$

where $\tau_{12}^{(1)}, \tau_{12}^{(2)}, \tau_{21}^{(1)}, \tau_{21}^{(2)} \ge 0$. We select $\sigma_1^{(1)} = 0.2$, $\sigma_1^{(2)} = 0.4$, $\sigma_2^{(1)} = 0.2$, $\sigma_2^{(2)} = 0.5$. Then the associated equations (4.2) have positive fundamental solution by Proposition 2.1 (v), therefore $\gamma_1 = 1$ and $\gamma_2 = 1$ can be used in Theorem 4.1. We get the matrix

$$H = \begin{pmatrix} 0.7830 & -0.7910 \\ -0.5600 & 0.6160 \end{pmatrix},$$

which is an M-matrix. Therefore Theorem 4.1 yields that the trivial solution of (4.5) is asymptotically stable independently of the selection of $\tau_{12}^{(1)}, \tau_{12}^{(2)}, \tau_{21}^{(1)}, \tau_{21}^{(2)} \ge 0$.

Consider again the special case of (4.1):

$$\dot{x}_k(t) = -\sum_{j=1}^{n} b_{kj} x_j(t-\tau_{kj}), \qquad k = 1, \ldots, n, \quad t \ge 0. \tag{4.6}$$

Corollary 4.3 *Suppose*

$$0 < b_{kk}\tau_{kk} < 1 + \frac{1}{e}, \qquad k = 1, \ldots, n, \tag{4.7}$$

and matrix $\tilde{H}$ *with components*

$$\tilde{h}_{kj} = \begin{cases} \dfrac{1 - (b_{kk}\tau_{kk} - \frac{1}{e})_+}{1 + (b_{kk}\tau_{kk} - \frac{1}{e})_+} b_{kk}, & k = j, \\ -|b_{kj}|, & k \ne j. \end{cases} \tag{4.8}$$

is a nonsingular M-matrix. Then the trivial solution of (4.6) *is asymptotically stable for any selection of the delays* $\tau_{kj} \ge 0$, $k \ne j$ $(k, j = 1, \ldots, n)$.

Proof The result follows immediately from Theorem 4.1 by taking

$$\sigma_k = \begin{cases} \frac{1}{eb_{kk}}, & \frac{1}{e} < b_{kk}\tau_{kk} < 1 + \frac{1}{e}, \\ \tau_{kk}, & 0 \le b_{kk}\tau_{kk} \le \frac{1}{e}, \end{cases}$$

since in this case the fundamental solution of (4.2) is positive, and so $\gamma_k = 1$ can be used in (4.3). □

We comment that Corollary 4.3 follows from Theorems 1.2 and 2.4, as well. We also note that if condition (1.10) is satisfied, then (4.8) in Corollary 4.3 reduces to (1.11).

Example 4.4 Consider the two-dimensional delay system

$$\begin{aligned} \dot{x}_1(t) &= -0.6x_1(t-1) - 0.2x_2(t-\tau_{12}) \\ \dot{x}_2(t) &= 0.8x_1(t-\tau_{21}) - 0.5x_2(t-0.8), \end{aligned} \tag{4.9}$$

where $\tau_{12}, \tau_{21} \ge 0$. Condition (4.7) in Corollary 4.3 is satisfied, and the matrix

$$\tilde{H} = \begin{pmatrix} 0.3739 & -0.2000 \\ -0.8000 & 0.4689 \end{pmatrix}$$

defined by (4.8) is an M-matrix. Therefore the trivial solution of (4.9) is asymptotically stable independently of the selection of $\tau_{12} \ge 0$ and $\tau_{21} \ge 0$. One can check easily that the matrices defined by (1.3) and (1.16) are not M-matrices, so the results of [3], [10], [15] and Theorem 1.3 (see also in [19]) do not apply for this example.

Finally, we compare the conditions of Theorems 1.1, 1.2, 1.3 and Corollary 4.3 applied for (4.6). In these conditions it is required, that the matrix (1.7), (1.17) and (1.19), respectively, be a nonsingular M-matrix. The larger is the coefficient of b_{kk} in the respective matrix, the larger is the class of matrices satisfying this condition, i.e., the weaker is the condition of the theorem. We fix $b = b_{kk} = 1$ in (1.8), and we consider the inverse of this coefficient as a parameter of the delay $\tau = \tau_{kk}$ of Equation (1.8), since it is related to the estimate (1.14). In Figure 4.10 we have plotted the value of estimate (2.8) (crosses) and estimate (2.11) (circles) for several values of the delay τ, and the coefficients $\omega_1(\tau) = \frac{1+\frac{1}{9}\tau(3+2\tau)}{1-\frac{1}{9}\tau(3+2\tau)}$ (solid line) and $\omega_2(\tau) = \frac{1+(\tau-\frac{1}{e})_+}{1-(\tau-\frac{1}{e})_+}$ (dotted line) in (1.16) and (4.8), respectively, as a function of τ. The graph of $\omega_1(\tau)$ and $\omega_2(\tau)$ intersect at $\tau_0 = 0.7289341695$. Since the function $\omega_1(\tau) \to \infty$ as $\tau \to \frac{3}{2}-$, at some value τ_1 close to $\frac{3}{2}$ the graph of ω_1 will intersect the graph corresponding to estimate (2.8).

We can see that for $\tau \in (0, \frac{1}{e}]$ the graph of ω_2 is above the graph of ω_1 and the estimates (2.8) and (2.11), so in this region of the parameters Theorem 1.3 has more restrictive condition than that of Theorem 1.1 or Corollary 4.3.

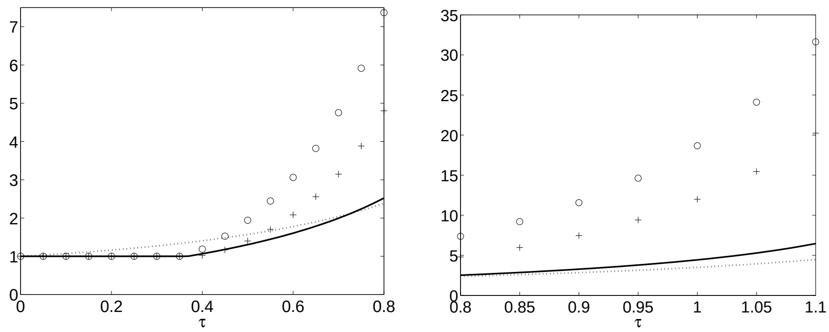

Figure 4.10: Comparison of Theorems 1.1, 1.2, 1.3 and Corollary 4.3.

For $\tau \in (\frac{1}{e}, \tau_0]$ Corollary 4.3 has the best condition. The graph of estimate (2.8) crosses that of ω_1 at about 0.55, so beyond that Theorem 1.2 combined with estimates (2.8) gives the worst condition up to τ_1.

For $\tau \in (\tau_0, 1 + \frac{1}{e})$ Theorem 1.3 has the best condition. Corollary 4.3 is not applicable beyond $1 + \frac{1}{e}$.

For $\tau \in [1 + \frac{1}{e}, \tau_1)$ Theorem 1.3 gives better condition, but for $\tau \in [\tau_1, \frac{3}{2})$ our Theorem 1.2 combined with (2.8) has a weaker condition.

For $\tau \in [\frac{3}{2}, \frac{\pi}{2})$ Theorem 1.3 is no longer applicable. In this region only Theorem 1.2 or Theorem 4.1 combined with estimates (2.8) or (2.11) can be applied.

We have summarized these observations in Table 1.

parameter region	Theorem 1.2 with (2.8)	Corollary 4.3	Theorem 1.3 (see [19])
$(0, 1/e]$	exact	exact	applicable
$(1/e, \tau_0]$	applicable	best	applicable
$(\tau_0, 1 + 1/e)$	applicable	applicable	best
$[1 + 1/e, \tau_1)$	applicable	not applicable	best
$[\tau_1, 3/2)$	best	not applicable	applicable
$[3/2, \pi/2)$	applicable	not applicable	not applicable

Table 1: Comparison of Theorems 1.2, 1.3 and Corollary 4.3 for different values of τ.

It is still an open and interesting problem to improve estimate (2.8), or find explicit estimates of the absolute integral of the fundamental solution, and obtain weaker stability conditions for Equation (1.1) or (4.1).

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