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Numerical Approximation of Neutral Differential Equations on Infinite Interval

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This paper is dedicated to Professor Allan C. Peterson on the occasion of his 60th birthday.

Abstract

In this paper we study numerical approximation of linear neutral differential equations on infinite interval using equations with piecewise constant arguments. As an application of our approximation results, we obtain stability theorems for some classes of linear delay and neutral difference equations.

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1 Introduction

There are several well-known results for both ordinary and delay differential equations which guarantee that the distance between a solution of the original and a related solution of the approximating one goes to zero uniformly on any compact time interval if the stepsize goes to zero. It is very rare the uniform approximation is proved on a halfline, for instance on $[0, \infty)$.

In this paper one of our main goals is to find the uniform numerical approximation on $[0, \infty)$ of the solutions of the neutral delay differential equation

$$\frac{d}{dt}\Big(x(t) - cx(t-\sigma)\Big) = \sum_{i=0}^{m} a_i x(t-\tau_i), \qquad t \geq 0, \tag{1.1}$$

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via the solution of the approximating equation

$$\frac{d}{dt}\Big(y_h(t) - cy_h(t-[\sigma]_h)\Big) = \sum_{i=0}^{m} a_i y_h([t]_h - [\tau_i]_h), \qquad t \geq 0. \tag{1.2}$$

Here $c \in [0,1)$, $\sigma > 0$, $a_i \in \mathbb{R}$, $\tau_i \in [0,\infty)$, $0 \leq i \leq m$, $h > 0$ is the stepsize,

$$[t]_h = \left[\frac{t}{h}\right] h,$$

where $[\cdot]$ denotes the greatest integer function.

Our approximating equation is a so-called equation with piecewise constant arguments (EPCA). This kind of equation was first introduced and studied by Cooke and Wiener in [2] and [3]. For further developments see [4] and [21]. EPCAs were used to generate numerical approximation schemes for linear delay and neutral differential equations with constant delays in [6] and [1], and later these schemes were extended to nonlinear delay and neutral differential equations with state-dependent delays in [9] and [13], respectively. Several variants of these schemes were used to generate numerical algorithms for parameter estimation problems in [12], [14] and [15].

In our main result (see Theorem 2.6 in Section 2) we prove that if the zero solution of Equation (1.1) is asymptotically stable, or equivalently, all the solutions of (1.1) tend to zero as $t \to \infty$, then the solutions of (1.2) uniformly approximate those of (1.1) on $[0,\infty)$.

From the proof of our main result it follows the existence of a constant depending on the parameters of Equation (1.1)

$$h_0 = h_0(c, \sigma, a_0, \ldots, a_m, \tau_0, \ldots, \tau_m) > 0 \tag{1.3}$$

such that for any $h \in (0, h_0)$ the difference between the related solutions of Equations (1.1) and (1.2), respectively, goes to zero exponentially as $t \to \infty$. From this it is clear that when the zero solution of (1.1) is exponentially stable and $h \in (0, h_0)$, then all solution of (1.2) tend to zero as $t \to \infty$.

Our second main goal in this paper is to give a method which allows us to reformulate well-known stability results from the theory of delay and neutral differential equations to discrete delay and neutral difference equations without repeating the proofs of the continuous case (see Theorem 3.1 in Section 3). We just remark that in several cases it is not self-evident how to convert some results from the continuous case to the discrete one. Our approach is based on the above mentioned main result and on the fact (see the begining of Section 2 of this paper) that Equation (1.2) is equivalent to a discrete difference equation.

Throughout this paper $r > 0$ is fixed, and C will be the Banach-space of continuous functions $[-r, 0] \to \mathbb{R}$ with the norm $\|\psi\| = \max\{|\psi(s)| : -r \leq s \leq 0\}$. The set of positive and nonnegative integers is denoted by $\mathbb{N}$ and $\mathbb{N}_0$, respectively, the set of integers is denoted by $\mathbb{Z}$. The forward difference of a sequence $u(n)$ is defined as $\Delta u(n) = u(n+1) - u(n)$.

2 Approximation of linear neutral equations

Consider the linear neutral differential equation

$$\frac{d}{dt}\Big(x(t) - cx(t-\sigma)\Big) = \sum_{i=0}^{m} a_i x(t-\tau_i), \qquad t \geq 0, \tag{2.1}$$

together with the initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{2.2}$$

(H1) $|c| < 1$, $a_i \in \mathbb{R}$, $\tau_i \in [0, \infty)$, $(i = 0, \ldots, m)$, $r \equiv \max(\sigma, \tau_0, \ldots, \tau_m)$,

(H2) $\varphi \in C$,

(H3) There exists $K \geq 1$ and $\alpha > 0$ such that for any initial function φ the corresponding solution of initial value problem (IVP) (2.1)-(2.2) satisfies $|x(t)| \leq Ke^{-\alpha t}\|\varphi\|$, $t \geq 0$, i.e., the trivial solution of (2.1) is exponentially stable.

We may (and do) assume that α in (H3) is selected so small that

(H4) $|c|e^{\alpha\sigma} < 1$

holds.

Fix $0 < h < \sigma$, and to IVP (2.1)-(2.2) we associate the EPCA

$$\frac{d}{dt}\Big(y_h(t) - cy_h(t-[\sigma]_h)\Big) = \sum_{i=0}^{m} a_i y_h([t]_h - [\tau_i]_h), \qquad t \geq 0, \tag{2.3}$$

and the initial condition

$$y_h(t) = \varphi_h(t), \tag{2.4}$$

where φ_h is the linear interpolate of φ using the mesh points $-r, -jh, -(j-1)h, \ldots, 0$, $j \equiv [r/h]$. Then $\|\varphi - \varphi_h\| \to 0$ as $h \to 0+$.

It is easy to show by the method of steps (see also [6]) that the solution of IVP (2.3)-(2.4) is a continuous function, which is linear between the mesh points nh.

Integrating (2.3) from nh to t and taking the limit $t \to (n+1)h-$ we get for $n \in \mathbb{N}_0$

$$y_h((n+1)h) - cy_h((n+1)h - [\sigma]_h) - y_h(nh) + cy_h(nh - [\sigma]_h) = h\sum_{i=0}^{m} a_i y_h(nh - [\tau_i]_h).$$

Therefore the sequence $u(n) = y_h(nh)$ satisfies the neutral difference equation

$$\Delta\Big(u(n) - cu(n - [\sigma/h])\Big) = h\sum_{i=0}^{m} a_i u(n - [\tau_i/h]), \qquad n \in \mathbb{N}_0. \tag{2.5}$$

This equation together with the initial condition

$$u(n) = \varphi(nh), \qquad n \in \mathbb{Z}, \quad nh \geq -r \tag{2.6}$$

can be solved recursively. This sequence determines the solution of IVP (2.3)-(2.4) uniquely. Therefore IVPs (2.3)-(2.4) and (2.5)-(2.6) are equivalent in many sense, but, as we shall see later, instead of studying the discrete equation it is more convenient to study the continuous equation (2.3) and use the tools of the differential equations in out proofs.

We recall the following result from [6], which yields that the solutions of IVP (2.3)-(2.4) approximate the solution of (2.1)-(2.2) uniformly on compact time intervals as $h \to 0+$.

Theorem 2.1 *Assume (H1)–(H2). Then for any* $T > 0$

$$\lim_{h\to 0+} \max_{-r\leq t\leq T} |x(t) - y_h(t)| = 0.$$

We show that under the additional assumption (H3) this result can be extended for the interval $[-r, \infty)$.

Introduce the function

$$\eta_h(t) \equiv x(t) - y_h(t), \qquad t \geq -r. \tag{2.7}$$

Clearly,

$$\frac{d}{dt}\Big(\eta_h(t) - c\eta_h(t-\sigma) - g_h(t)\Big) = \sum_{i=0}^{m} a_i\eta_h(t-\tau_i) + f_h(t), \qquad t \geq 0, \tag{2.8}$$

where

$$g_h(t) \equiv c\Big(y_h(t-\sigma) - y_h(t - [\sigma]_h)\Big) \tag{2.9}$$

and

$$f_h(t) = \sum_{i=0}^{m} a_i\Big(y_h(t-\tau_i) - y_h([t]_h - [\tau_i]_h)\Big). \tag{2.10}$$

Let v be the fundamental solution of (2.1), i.e., the solution of the IVP

$$\frac{d}{dt}\Big(v(t) - cv(t-\sigma)\Big) = \sum_{i=0}^{m} a_i v(t-\tau_i), \qquad t \geq 0, \tag{2.11}$$

$$v(t) = \begin{cases} 1, & t = 0, \\ 0, & t \in [-r, 0). \end{cases} \tag{2.12}$$

It is known (see, e.g., [11]) that (H3) implies that there exists $K_0 \geq 0$ such that

$$|v(t)| \leq K_0 e^{-\alpha t}, \qquad t \geq 0. \tag{2.13}$$

Remark 2.2 It is easy to check by the method of steps that v is continuously differentiable on the intervals $(k\sigma, (k+1)\sigma)$, $(k = 0, 1, \ldots,)$, and it has jumps $v(k\sigma+) - v(k\sigma-) = c^k$ $(k = 0, 1, \ldots,)$.

Relation (2.13) implies the next result.

Lemma 2.3 *Assume (H1)–(H4). Then there exists* $\tilde{K}_0 \geq 0$ *such that*

$$|\dot{v}(t)| \leq \tilde{K}_0 e^{-\alpha t}, \qquad t > 0, \quad t \neq k\sigma, (k \in \mathbb{N}).$$

Proof We have for $t > 0$, $t \neq k\sigma$

$$\dot{v}(t) = c\dot{v}(t-\sigma) + \sum_{i=0}^{m} a_i v(t-\tau_i),$$

therefore

$$|\dot{v}(t)| \leq |c||\dot{v}(t-\sigma)| + Ae^{-\alpha t}, \qquad t > 0, \quad t \neq k\sigma, \tag{2.14}$$

where

$$A = K_0 \sum_{i=0}^{m} |a_i| e^{\alpha \tau_i}.$$

Since $\dot{v}(t-\sigma) = 0$ for $t \in (0, \sigma)$, we have

$$|\dot{v}(t)| \leq Ae^{-\alpha t}, \qquad t \in (0, \sigma).$$

Hence (2.14) yields

$$|\dot{v}(t)| \leq (|c|e^{\alpha\sigma} + 1)Ae^{-\alpha t}, \qquad t \in (\sigma, 2\sigma),$$

and, in general,

$$|\dot{v}(t)| \le (|c|^k e^{k\alpha\sigma} + \cdots + |c|e^{\alpha\sigma} + 1)Ae^{-\alpha t}, \qquad t \in (k\sigma, (k+1)\sigma),$$

so the statement of the lemma follows with $\tilde{K}_0 = A/(1 - |c|e^{\alpha\sigma})$. □

We introduce the following notations:

$$\omega(u) = \max\{|x(s_2) - x(s_1)|: \ -r \le s_1 < s_2 \le 2r, \ s_2 - s_1 \le u\}, \tag{2.15}$$

and

$$z_h(t) = \max_{-r \le u \le t} e^{\alpha u}|\eta_h(u)|, \qquad t \ge -r. \tag{2.16}$$

With this notations we can estimate the difference of function values of y_h, which will be essential later.

Lemma 2.4 *Suppose (H1)–(H4), and let η_h and z_h be defined by* (2.7) *and* (2.16)*, respectively. Then for any $0 < h < \sigma$ and $0 \le t_1 < t_2$ such that $t_2 - t_1 \le r$ any solution y_h of IVP* (2.3)-(2.4) *satisfies*

$$\begin{aligned} |y_h(t_2) - y_h(t_1)| \ &\le \ |c|^{\frac{t_1}{\sigma}} \Big(e^{-\alpha(t_2 - t_1 - \sigma)} z_h(t_2 - t_1) + \|\varphi - \varphi_h\| + \omega(t_2 - t_1) \Big) \\ &\quad + \frac{\sum_{i=0}^m |a_i|}{1 - |c|e^{\alpha\sigma}} (t_2 - t_1) e^{-\alpha(t_1 - r - h)} (z_h(t_2) + K\|\varphi\|). \end{aligned}$$

Proof Let $l \equiv [t_1/[\sigma]_h]$. Then $t_1 - (l+1)[\sigma]_h < 0 \le t_1 - l[\sigma]_h$. Therefore

$$\begin{aligned} y_h(t_2) - y_h(t_1) \ &= \ c\Big(y_h(t_2 - [\sigma]_h) - y_h(t_1 - [\sigma]_h)\Big) + \sum_{i=0}^m a_i \int_{t_1}^{t_2} y_h([s]_h - [\tau_i]_h)\, ds \\ &= \ c^{l+1}\Big(y_h(t_2 - (l+1)[\sigma]_h) - y_h(t_1 - (l+1)[\sigma]_h)\Big) \\ &\quad + \sum_{k=0}^{l} c^k \sum_{i=0}^m a_i \int_{t_1 - k[\sigma]_h}^{t_2 - k[\sigma]_h} y_h([s]_h - [\tau_i]_h)\, ds. \end{aligned}$$

Thus, using the definition of η_h,

$$\begin{aligned} |y_h(t_2) - y_h(t_1)| \ &\le \ |c|^{l+1}\Big|\eta_h(t_2 - (l+1)[\sigma]_h) - \eta_h(t_1 - (l+1)[\sigma]_h)\Big| \\ &\quad + |c|^{l+1}\Big|x(t_2 - (l+1)[\sigma]_h) - x(t_1 - (l+1)[\sigma]_h)\Big| \\ &\quad + \sum_{k=0}^{l} |c|^k \sum_{i=0}^m |a_i| \int_{t_1 - k[\sigma]_h}^{t_2 - k[\sigma]_h} |\eta_h([s]_h - [\tau_i]_h)|\, ds \\ &\quad + \sum_{k=0}^{l} |c|^k \sum_{i=0}^m |a_i| \int_{t_1 - k[\sigma]_h}^{t_2 - k[\sigma]_h} |x([s]_h - [\tau_i]_h)|\, ds. \end{aligned}$$

Hence, relations $l+1 \geq t_1/\sigma$, $t_2-t_1-\sigma \leq t_2-(l+1)[\sigma]_h \leq t_2-t_1$, $t_1-(l+1)[\sigma]_h \leq 0$, $|x(t)| \leq Ke^{-\alpha t}\|\varphi\|$ for $t \geq -r$, assumption (H4), and the definition of z_h imply

$$\begin{aligned}
&|y_h(t_2) - y_h(t_1)| \\
&\quad \leq |c|^{\frac{t_1}{\sigma}} \Big(e^{-\alpha(t_2-(l+1)[\sigma]_h)} z_h(t_2-(l+1)[\sigma]_h) + \|\varphi-\varphi_h\| + \omega(t_2-t_1) \Big) \\
&\qquad + \sum_{k=0}^{l} |c|^k \sum_{i=0}^{m} |a_i| \int_{t_1-k[\sigma]_h}^{t_2-k[\sigma]_h} e^{\alpha([s]_h-[\tau_i]_h)} |\eta_h([s]_h-[\tau_i]_h)| e^{-\alpha([s]_h-[\tau_i]_h)}\, ds \\
&\qquad + \sum_{k=0}^{l} |c|^k \sum_{i=0}^{m} |a_i| K e^{-\alpha(t_1-k\sigma-h-r)} \|\varphi\| (t_2-t_1) \\
&\quad \leq |c|^{\frac{t_1}{\sigma}} \Big(e^{-\alpha(t_2-t_1-\sigma)} z_h(t_2-t_1) + \|\varphi-\varphi_h\| + \omega(t_2-t_1) \Big) \\
&\qquad + z_h(t_2) e^{\alpha r} \sum_{i=0}^{m} |a_i| \sum_{k=0}^{l} |c|^k \int_{t_1-k[\sigma]_h}^{t_2-k[\sigma]_h} e^{-\alpha(s-h)}\, ds \\
&\qquad + \frac{K\|\varphi\| \sum_{i=0}^{m} |a_i|}{1-|c|e^{\alpha\sigma}} e^{-\alpha(t_1-h-r)} (t_2-t_1).
\end{aligned}$$

This implies the statement of the lemma using the estimates

$$\begin{aligned}
\sum_{k=0}^{l} |c|^k \int_{t_1-k[\sigma]_h}^{t_2-k[\sigma]_h} e^{-\alpha s}\, ds &= \sum_{k=0}^{l} |c|^k \frac{e^{-\alpha(t_1-k[\sigma]_h)} - e^{-\alpha(t_2-k[\sigma]_h)}}{\alpha} \\
&\leq \frac{e^{-\alpha t_1} - e^{-\alpha t_2}}{\alpha} \sum_{k=0}^{l} |c|^k e^{k\alpha\sigma} \\
&\leq \frac{e^{-\alpha t_1}(t_2-t_1)}{1-|c|e^{\alpha\sigma}}.
\end{aligned}$$

□

Lemma 2.4 has the following immediate consequences.

Lemma 2.5 *Suppose (H1)–(H4), $0 < h < \sigma$. Then there exist nonnegative constants A_1, A_2, B_1, B_2 such that*

$$|g_h(t)| \leq A_1 |c|^{\frac{t}{\sigma}} (z_h(h) + \|\varphi-\varphi_h\| + \omega(h)) + A_2 e^{-\alpha(t-h)} (z_h(t) + \|\varphi\|) h, \qquad t \geq \sigma,$$

and

$$|f_h(t)| \leq B_1 |c|^{\frac{t}{\sigma}} (z_h(h) + \|\varphi-\varphi_h\| + \omega(2h)) + B_2 e^{-\alpha(t-2h)} (z_h(t) + \|\varphi\|) h, \qquad t \geq r.$$

Now we are ready to prove our main theorem.

Theorem 2.6 *Assume (H1)–(H4), and let x be the solution of* (2.1)-(2.2)*, and for any $0 < h < \sigma$ let y_h be the solution of the corresponding IVP* (2.3)-(2.4)*. Then*

$$\lim_{h\to 0+} \max_{-r\le t<\infty} |x(t) - y_h(t)| = 0.$$

Proof Fix $T \ge r$. Let $X_{h,T}$ be the solution of (2.1) for $t \ge T$, corresponding to the initial function

$$X_{h,T}(t) = x(t) - y_h(t), \qquad t \in [T-r, T].$$

Then the variation of constant formula (see, e.g., [11] for the general case, and see [8] for this special case of one delay in the neutral term) yields

$$\begin{aligned} \eta_h(t) &= X_{h,T}(t) - v(t-T)g_h(T) + \sum_{k=0}^{\left[\frac{t-T}{\sigma}\right]} c^k g_h(t-k\sigma) \\ &\quad + \int_T^t \dot v(t-s)g_h(s)\,ds + \int_T^t v(t-s)f_h(s)\,ds. \end{aligned} \tag{2.17}$$

Since $X_{h,T}$ is a solution of the homogeneous equation (2.1), there exist $K \ge 1$ such that

$$|X_{h,T}(t)| \le Ke^{-\alpha(t-T)} \max_{T-r\le u\le T} |x(u) - y_h(u)|, \qquad t \ge T.$$

Therefore

$$|X_{h,T}(t)| \le Ke^{-\alpha(t-T)} \max_{T-r\le u\le T} |e^{\alpha u}\eta_h(u)e^{-\alpha u}| \le Ke^{-\alpha(t-T)} z_h(T)e^{-\alpha(T-r)}, \qquad t \ge T. \tag{2.18}$$

Let $s = \left[\frac{t-T}{\sigma}\right]$. Then

$$\begin{aligned} \sum_{k=0}^{s} |c|^k |g_h(t-k\sigma)| &\le \sum_{k=0}^{s} |c|^k \Big(A_1 |c|^{\frac{t-k\sigma}{\sigma}} (z_h(h) + \|\varphi - \varphi_h\| + \omega(h)) \\ &\quad + A_2 e^{-\alpha(t-k\sigma-h)} (z_h(t-k\sigma) + \|\varphi\|)h \Big) \\ &\le A_1(z_h(h) + \|\varphi - \varphi_h\| + \omega(h))|c|^{\frac{t}{\sigma}}(s+1) \\ &\quad + A_2 e^{-\alpha(t-h)}(z_h(t) + \|\varphi\|)h \sum_{k=0}^{m} |c|^k e^{k\alpha\sigma}, \\ &\le A_1(z_h(h) + \|\varphi - \varphi_h\| + \omega(h))|c|^{\frac{t}{\sigma}} \frac{t-T+\sigma}{\sigma} \\ &\quad + \frac{A_2}{1 - |c|e^{\alpha\sigma}} e^{-\alpha(t-h)}(z_h(t) + \|\varphi\|)h. \end{aligned} \tag{2.19}$$

Therefore it follows from (2.13), (2.17), (2.18), (2.19), and Lemma 2.3 for $t \geq T$

$$\begin{aligned}
|\eta_h(t)| \leq\ & Ke^{-\alpha(t-r)} z_h(T) + K_0 e^{-\alpha(t-T)} |g_h(T)| \\
&+ A_1(z_h(h) + \|\varphi - \varphi_h\| + \omega(h)) |c|^{\frac{t}{\sigma}} \frac{t - T + \sigma}{\sigma} \\
&+ \frac{A_2}{1 - |c| e^{\alpha\sigma}} e^{-\alpha(t-h)} (z_h(t) + \|\varphi\|) h \\
&+ \int_T^t \tilde{K}_0 e^{-\alpha(t-s)} A_1 |c|^{\frac{s}{\sigma}} (z_h(h) + \|\varphi - \varphi_h\| + \omega(h))\, ds \\
&+ \int_T^t \tilde{K}_0 e^{-\alpha(t-s)} A_2 e^{-\alpha(s-h)} (z_h(s) + \|\varphi\|) h\, ds \\
&+ \int_T^t K_0 e^{-\alpha(t-s)} B_1 |c|^{\frac{s}{\sigma}} (z_h(h) + \|\varphi - \varphi_h\| + \omega(2h))\, ds \\
&+ \int_T^t K_0 e^{-\alpha(t-s)} B_2 e^{-\alpha(s-2h)} (z_h(s) + \|\varphi\|) h\, ds.
\end{aligned}$$

We define the constants

$$\begin{aligned}
C_h &= z_h(h) + \|\varphi - \varphi_h\| + \omega(2h), \\
\beta &= -\alpha - \frac{\log |c|}{\sigma}.
\end{aligned}$$

Assumption (H4) yields $\beta > 0$. Multiplying both sides of the last inequality by $e^{\alpha t}$ gives

$$\begin{aligned}
e^{\alpha t} |\eta_h(t)| \leq\ & Ke^{\alpha r} z_h(T) + K_0 e^{\alpha T} |g_h(T)| + \frac{A_1 C_h}{\sigma} e^{-\beta t} (t - T + \sigma) \\
&+ \frac{A_2}{1 - |c| e^{\alpha\sigma}} e^{\alpha h} (z_h(t) + \|\varphi\|) h \\
&+ \int_T^t \tilde{K}_0 (A_1 C_h e^{-\beta s} + A_2 e^{\alpha h} (z_h(s) + \|\varphi\|) h)\, ds \\
&+ \int_T^t K_0 (B_1 C_h e^{-\beta s} + B_2 e^{\alpha 2h} (z_h(s) + \|\varphi\|) h)\, ds.
\end{aligned}$$

We introduce

$$\begin{aligned}
M_T &= \max\{e^{-\beta t}(t - T + \sigma) \colon t \geq T\}, \\
A_{h,T} &= Ke^{\alpha r} z_h(T) + K_0 e^{\alpha T} |g_h(T)| + \frac{A_1 C_h M_T}{\sigma} \\
&\quad + \frac{A_2}{1 - |c| e^{\alpha\sigma}} e^{\alpha h} \|\varphi\| h + \frac{\tilde{K}_0 A_1 C_h}{\beta} + \frac{K_0 B_1 C_h}{\beta}, \\
D_1 &= \frac{A_2}{1 - |c| e^{\alpha\sigma}} e^{\alpha\sigma}, \\
D_2 &= (\tilde{K}_0 A_2 + K_0 B_2) e^{\alpha 2\sigma}.
\end{aligned}$$

Then

$$e^{\alpha t}|\eta_h(t)| \le A_{h,T} + D_1 z_h(t)h + \int_T^t D_2(z_h(s) + \|\varphi\|)h\,ds, \qquad t \ge T,$$

which implies, using the monotonicity of the right-hand-side in t and $z_h(t) \le A_{h,T}$ for $t \le T$, that

$$z_h(t) \le A_{h,T} + D_1 z_h(t)h + \int_T^t D_2(z_h(s) + \|\varphi\|)h\,ds, \qquad t \ge T,$$

and hence, for $h < 1/D_1$, we get

$$z(t) \le \frac{A_{h,T}}{1-hD_1} + \frac{D_2 h}{1-hD_1}(t-T)\|\varphi\| + \frac{D_2 h}{1-hD_1}\int_T^t z_h(s)\,ds, \qquad t \ge T.$$

Gronwall's inequality implies

$$z(t) \le \left(\frac{A_{h,T}}{1-hD_1} + \frac{D_2 h}{1-hD_1}(t-T)\|\varphi\|\right) e^{\frac{D_2 h}{1-hD_1}(t-T)},$$

so

$$|x(t) - y_h(t)| \le e^{-\alpha t} z_h(t) \le \left(\frac{A_{h,T}}{1-hD_1} + \frac{D_2 h}{1-hD_1}(t-T)\|\varphi\|\right) e^{-\left(\alpha - \frac{D_2 h}{1-hD_1}\right)t - \frac{D_2 T h}{1-hD_1}}.$$

Therefore

$$|x(t) - y_h(t)| \le F_h(t), \qquad t \ge 0, \quad 0 < h < h_1,$$

where $0 < h_1 < \min(\sigma, 1/D_1)$, and

$$\begin{aligned} F_h(t) &= (M_1(h) + M_2\|\varphi\|ht)e^{-(\alpha - M_2 h)t}, \\ M_1(h) &= \frac{A_{h,T}}{1-hD_1} - \frac{D_2 h}{1-hD_1}T\|\varphi\|, \\ M_2 &= \frac{D_2}{1-h_1 D_1}. \end{aligned}$$

Let $0 < h < h_0 < \min(h_1, \alpha/M_2)$. The function F_h has the maximum at

$$t^* = \frac{1}{\alpha - M_2 h} - \frac{M_1(h)}{M_2\|\varphi\|h}.$$

Therefore if $t^* > 0$, then

$$F_h(t) \le F_h(t^*) = \frac{M_2\|\varphi\|h}{\alpha - M_2 h} e^{-(\alpha - M_2 h)t^*}.$$

On the other hand, if $t^* \leq 0$, then $F_h(t) \leq F_h(0)$ for $t \geq 0$. Therefore

$$F_h(t) \leq \max\Big(M_1(h), \frac{M_2\|\varphi\|h}{\alpha - M_2 h}\Big), \qquad t \geq 0.$$

This concludes the proof, since

$$\lim_{h \to 0+} M_1(h) = 0,$$

using the definition of $A_{h,T}$ and Theorem 2.1. □

The following estimate is the immediate consequence of the proof of this theorem.

Corollary 2.7 *Assume (H1)–(H4), and let x be the solution of* (2.1)-(2.2)*, and for any $0 < h < \sigma$ let y_h be the solution of the corresponding IVP* (2.3)-(2.4)*. Then there exist a function $M_1 : (0, \sigma) \to [0, \infty)$ satisfying $\lim_{h \to 0+} M_1(h) = 0$, and constants $M_2 \geq 0$ and $0 < h_0 < \sigma$ such that*

$$|x(t) - y_h(t)| \leq (M_1(h) + M_2\|\varphi\|ht)e^{-(\alpha - M_2 h)t}, \qquad t \geq 0, \quad 0 < h < h_0.$$

3 Stability of difference equations

Stability of several classes of difference equations has been studied extensively in the recent literature. Without completeness, we refer to [1], [5], [10], [16], [17], [19] and [20]. In [7] we obtained stability results for difference delay equations rewriting them in an equivalent form as an EPCA and applying our earlier stability results for differential delay equations.

Here we use a similar approach, but, as an application of the previous section, we use Corollary 2.7 together with known stability conditions for delay and neutral differential equations. First consider the abstract version of this approach.

Consider the neutral differential equation

$$\frac{d}{dt}\Big(x(t) - cx(t - \sigma)\Big) = -\sum_{i=0}^{m} a_i x(t - \tau_i), \qquad t \geq 0. \tag{3.1}$$

We say that the parameter set $\mathcal{S}$ is a stability region to (3.1), if for any parameters $(c, \sigma, a_0, \ldots, a_m, \tau_0, \ldots, \tau_m) \in \mathcal{S}$ all the solutions of the corresponding Equation (3.1) tend to zero as $t \to \infty$.

For any $h > 0$ we define the set

$$\mathcal{S}_h = \Big\{(c, s, \alpha_0, \ldots, \alpha_m, l_0, \ldots, l_m) \in (-1,1) \times \mathbb{N}_0 \times \mathbb{R}^{m+1} \times \mathbb{N}_0^{m+1} \colon\ s = \left[\frac{\sigma}{h}\right], \\ \alpha_i = a_i h,\ l_i = \left[\frac{\tau_i}{h}\right],\ i = 0, \ldots, m,\ \text{where } (c, \sigma, a_0, \ldots, a_m, \tau_0, \ldots, \tau_m) \in \mathcal{S}\Big\}. \tag{3.2}$$

Theorem 3.1 *For any $(c, \sigma, a_0, \ldots, a_m, \tau_0, \ldots, \tau_m) \in \mathcal{S}$ there exists a constant $h_0 = h_0(c, \sigma, a_0, \ldots, a_m, \tau_0, \ldots, \tau_m) > 0$ such that any solution of the difference equation*

$$\Delta\Big(u(n) - cu(n-s)\Big) = \sum_{i=0}^{m} \alpha_i u(n - l_i), \qquad n \in \mathbb{N}_0 \tag{3.3}$$

tends to zero as $n \to \infty$, if $(c, s, \alpha_0, \ldots, \alpha_m, l_0, \ldots, l_m) \in \mathcal{S}_h$ and $0 < h < h_0$.

Proof Let $(c, \sigma, a_0, \ldots, a_m, \tau_0, \ldots, \tau_m) \in \mathcal{S}$, and h_0 be the constant defined by Corollary 2.7. We associate EPCA

$$\frac{d}{dt}\Big(y_h(t) - cy_h(t - [\sigma]_h)\Big) = \sum_{i=0}^{m} a_i y_h([t]_h - [\tau_i]_h), \qquad t \geq 0 \tag{3.4}$$

to (3.1). Fix any $0 < h < h_0$, and let $\alpha_i = a_i h$, $s = \left[\frac{\sigma}{h}\right]$, and $l_i = \left[\frac{\tau_i}{h}\right]$. Then

$$(c, s, \alpha_0, \ldots, \alpha_m, l_1, \ldots, l_m) \in \mathcal{S}_h.$$

Integrating both sides of (3.4) from nh to $(n+1)h$ we get

$$y_h((n+1)h) - cy_h((n+1-s)h) - y_h(nh) + cy_h((n-s)h) = -\sum_{i=0}^{m} \alpha_i y_h((n-l_i)h), \ \ n \in \mathbb{N}.$$

Therefore $u(n) = y_h(nh)$. On the other hand, Corollary 2.7 yields

$$\lim_{t\to\infty} |y_h(t)| \leq \lim_{t\to\infty} (|x(t)| + |x(t) - y_h(t)|) = 0,$$

hence the trivial solution of (3.4) is asymptotically stable, which proves the theorem. □

Next we give an application of this result for delay equations.

Theorem 3.2 *Suppose $a_i \geq 0$, $l_i \in \mathbb{N}_0$ $(i = 0, \ldots, m)$, and*

$$0 < \sum_{i=0}^{m} a_i l_i < \frac{\pi}{2}. \tag{3.5}$$

Then there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ the trivial solution of

$$\Delta u(n) = -\sum_{i=0}^{m} \frac{a_i}{k} u(n - l_i k), \qquad n \geq 0 \tag{3.6}$$

is asymptotically stable.

Proof Consider the delay equation

$$\dot{x}(t) = -\sum_{i=0}^{m} a_i x(t - l_i), \qquad t \geq 0. \tag{3.7}$$

Our assumption (3.5) and a result of Krisztin (see [18]) imply that the trivial solution of (3.7) is asymptotically stable. Then Theorem 3.1 implies the statement of this theorem using discretization parameters of the form $h = 1/k$. □

In the next two theorems we apply conditions which imply asymptotic stability of the trivial solution of neutral differential equations independently of the delay in the neutral term. In this case we can get conditions for some associated neutral difference equations which are also independent from the delay in the neutral term.

Theorem 3.3 *Suppose $c \in \mathbb{R}$, $s \in \mathbb{N}$, $a_i \in \mathbb{R}$, $l_i \in \mathbb{N}_0$ $(i = 0, \ldots, m)$, and*

$$|c| + \sum_{i=0}^{m} |a_i| l_i < 1, \qquad \text{and} \qquad \sum_{i=0}^{m} a_i > 0. \tag{3.8}$$

Then there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ the trivial solution of

$$\Delta\Big(u(n) - cu(n - s)\Big) = -\sum_{i=0}^{m} \frac{a_i}{k} u(n - l_i k), \qquad n \geq 0 \tag{3.9}$$

is asymptotically stable.

Proof Consider the scalar neutral equation

$$\frac{d}{dt}\Big(x(t) - cx(t - s/k)\Big) = -\sum_{i=0}^{m} a_i x(t - l_i), \qquad t \geq 0. \tag{3.10}$$

Then a result from [5] yields that the trivial solution of (3.10) is asymptotically stable. Then the proof follows from Theorem 3.1 with $h = 1/k$. □

Another stability condition (see [22]) gives the next result.

Theorem 3.4 *Suppose* $c \in [0, 1/2)$, $s \in \mathbb{N}$, $a \geq 0$, $l \in \mathbb{N}_0$, *and*

$$2c(2-c) + al < \frac{3}{2}. \tag{3.11}$$

Then there exists $k_0 \in \mathbb{N}$ *such that for every* $k \geq k_0$ *the trivial solution of*

$$\Delta\Big(u(n) - cu(n-s)\Big) = -\frac{a}{k}u(n-lk), \qquad n \geq 0 \tag{3.12}$$

is asymptotically stable.

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