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Stability in delay perturbed differential and difference equations

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1 Introduction

In this paper we summarize our earlier work concerning preserving stability under delay perturbation (see [**1**], [**8**]–[**10**]), and present some new stability theorems for certain classes of linear differential and difference equations. We will show that our results extend many known so-called 3/2-type or $\pi/2$-type stability theorems (see, e.g., [**14**]–[**16**], [**20**]–[**22**]). Our conditions are formulated with the help of the function

$$\Phi(\tau) = \int_0^\infty |u(t;\tau)|\, dt,$$

where $u(t;\tau)$ is the fundamental solution of the linear delay differential equation

$$\dot{x}(t) = -x(t-\tau), \qquad t \ge 0.$$

We also present some new exponential estimates for $u(t;\tau)$ and for $\Phi(\tau)$.

2 Fundamental solution of a linear delay differential equation

Let $\tau > 0$, and u be the solution of the initial value problem (IVP)

$$\dot{u}(t) = -u(t-\tau), \qquad t \ge 0, \tag{2.1}$$

$$u(t) = \begin{cases} 1, & t = 0, \\ 0, & t < 0, \end{cases} \tag{2.2}$$

i.e., u is the fundamental solution of the scalar delay differential equation

$$\dot{x}(t) = -x(t-\tau), \qquad t \ge 0. \tag{2.3}$$

If we want emphasize that the fundamental solution corresponds to delay τ, we use the notation $u(t;\tau)$.

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Let $\lambda = \alpha_0 + i\beta_0$ be the root of the characteristic equation

$$\lambda = -e^{-\lambda\tau} \tag{2.4}$$

of (2.3) with maximal real part. It is known (see, e.g., [**11**]) that $\alpha_0 < 0$ if and only if $\tau \in [0, \pi/2)$, and for any $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that $|u(t)| \le M_\varepsilon e^{(\alpha_0+\varepsilon)t}$ for $t \ge 0$. The following result gives the value of M_ε explicitly, and provides an exponential estimate of $|u(t)|$ with exponent $\alpha_0 t$, as well.

Theorem 2.1 *Let $\tau \in [0, \pi/2)$, $u(t) = u(t;\tau)$ be the fundamental solution of* (2.3)*, $\alpha_0 + i\beta_0$ be the root of* (2.4) *with maximal real part, and $\varepsilon > 0$ be such that $\alpha_0 + \varepsilon < 0$. Then the fundamental solution satisfies for $t \ge 0$*

$$|u(t)| \le \frac{1}{1-\gamma_\varepsilon} e^{(\alpha_0+\varepsilon)t}, \qquad \text{where} \quad \gamma_\varepsilon = e^{-\alpha_0\tau} \int_{-\tau}^{0} e^{-\varepsilon(s+\tau)} \cos\beta_0 s\, ds, \tag{2.5}$$

and

$$|u(t)| \le \frac{2t+\tau}{(1-\gamma)\tau} e^{\alpha_0 t}, \qquad \text{where} \quad \gamma = e^{-\alpha_0\tau} \int_{-\tau}^{-\tau/2} \cos\beta_0 s\, ds. \tag{2.6}$$

Proof Let $\tau \in [0, \pi/2)$ be fixed, and let $\alpha_0 + i\beta_0$ be the root of (2.4) with maximal real part. It is known (see, e.g., [**5**] or Theorem 2.3 below) that $\beta_0 \in [0, \pi/(2\tau))$, therefore

$$\cos\beta_0 s > 0, \qquad s \in [-\tau, 0]. \tag{2.7}$$

It follows from (2.4) that $\beta_0 = e^{-\alpha_0\tau} \sin\beta_0\tau$, therefore

$$e^{-\alpha_0\tau} \int_{-\tau}^{0} \cos\beta_0 s\, ds = 1. \tag{2.8}$$

This implies that $0 < \gamma_\varepsilon < 1$ and $0 < \gamma < 1$, where γ_ε and γ are defined by (2.5) and (2.6), respectively.

The function $y(t) = e^{\alpha_0 t} \cos\beta_0 t$ is a solution of (2.1), and so the variation-of-constants formula (see, e.g., [**11**]) yields

$$y(t) = u(t)y(0) - \int_{-\tau}^{0} u(t-s-\tau) e^{\alpha_0 s} \cos\beta_0 s\, ds.$$

Using (2.7) we get

$$|u(t)| \le e^{\alpha_0 t} + \int_{-\tau}^{0} |u(t-s-\tau)| e^{\alpha_0 s} \cos\beta_0 s\, ds, \qquad t \ge 0. \tag{2.9}$$

Multiplying this inequality by $e^{-(\alpha_0+\varepsilon)t}$, and using that $u(t) = 0$ for $t < 0$, we get that the function $w_\varepsilon(t) = e^{-(\alpha_0+\varepsilon)t}|u(t)|$ satisfies

$$w_\varepsilon(t) \le 1 + e^{-\alpha_0\tau} \int_{-\tau}^{0} w_\varepsilon(t-s-\tau) e^{-\varepsilon(s+\tau)} \cos\beta_0 s\, ds \le 1 + \gamma_\varepsilon \max_{0 \le s \le t} w_\varepsilon(s), \quad t \ge 0,$$

which proves (2.5).

Similarly, define $w(t) = e^{-\alpha_0 t}|u(t)|$. Then (2.9) yields for $t \ge 0$

$$\begin{aligned} w(t) \quad &\le \quad 1 + e^{-\alpha_0\tau} \int_{-\tau}^{-\tau/2} w(t-s-\tau) \cos\beta_0 s\, ds \\ &\qquad + e^{-\alpha_0\tau} \int_{-\tau/2}^{0} w(t-s-\tau) \cos\beta_0 s\, ds. \end{aligned} \tag{2.10}$$

Let M_n be defined by $M_n = \sup\{w(s)\colon n\tau/2 \le s \le (n+1)\tau/2\}$, $n = 0, 1, \ldots$. We show by induction that

$$M_n \le \frac{n+1}{1-\gamma}, \qquad n = 0, 1, \ldots . \tag{2.11}$$

We have for $t \in [n\tau/2, (n+1)\tau/2]$

$$(n-1)\frac{\tau}{2} \le t - s - \tau \le (n+1)\frac{\tau}{2}, \qquad \text{for} \quad s \in [-\tau, -\tau/2], \tag{2.12}$$

and

$$(n-2)\frac{\tau}{2} \le t - s - \tau \le n\frac{\tau}{2}, \qquad \text{for} \quad s \in [-\tau/2, 0]. \tag{2.13}$$

Therefore, using that $u(t) = 0$ for $t < 0$, (2.10) yields

$$w(t) \le 1 + \gamma M_0, \qquad t \in [0, \tau/2],$$

and so $M_0 \le 1/(1-\gamma)$. Suppose (2.11) is known for integers from 0 to $n-1$. The definitions of γ and M_n, relations (2.8), (2.10), (2.12) and (2.13), and the inductional hypothesis imply

$$\begin{aligned} w(t) &\le 1 + \gamma \max\{M_n, M_{n-1}\} + (1-\gamma)\max\{M_{n-1}, M_{n-2}\} \\ &\le 1 + \gamma \max\{M_n, M_{n-1}\} + n \qquad t \in [n\tau/2, (n+1)\tau/2]. \end{aligned}$$

If $M_n \le M_{n-1}$, then

$$w(t) \le n + 1 + \gamma\frac{n}{1-\gamma} < \frac{n+1}{1-\gamma}, \qquad t \in [n\tau/2, (n+1)\tau/2]$$

and so $M_n \le (n+1)/(1-\gamma)$. If $M_n > M_{n-1}$, then

$$w(t) \le n + 1 + \gamma M_n, \qquad t \in [n\tau/2, (n+1)\tau/2],$$

and hence $M_n \le n + 1 + \gamma M_n$, i.e., $M_n \le (n+1)/(1-\gamma)$. Therefore we proved (2.11) for all $n \ge 0$, but this yields (2.6), using the inequality $[2t/\tau] \le 2t/\tau$, where $[\cdot]$ is the greatest integer function. □

It follows from the above results that the trivial solution of (2.3) is asymptotically stable, if and only if $\int_0^\infty |u(t;\tau)|\,ds < \infty$. We introduce the function

$$\Phi(\tau) = \int_0^\infty |u(t;\tau)|\,dt. \tag{2.14}$$

Then $\Phi(\tau) = \infty$ for $\tau \ge \pi/2$. It is known (see, e.g., [**5**]) that $u(t;\tau) > 0$ for $t > 0$, if and only if $\tau \le 1/e$. For $\tau \le 1/e$ it follows easily from (2.1) that $\Phi(\tau) = \int_0^\infty u(t;\tau)\,dt = 1$. For $1/e < \tau < \pi/2$ numerical estimate of Φ yields Figure 1. Here we used a numerical approximation method introduced in [**6**] to obtain approximate values of u, and the simple trapezoidal method to estimate Φ.

As we will see in the next section, we can formulate stability theorems with the help of the function Φ, but in applying those results it is important to know an upper estimate of $\Phi(\tau)$. Theorem 2.1 has the following corollary in this direction.

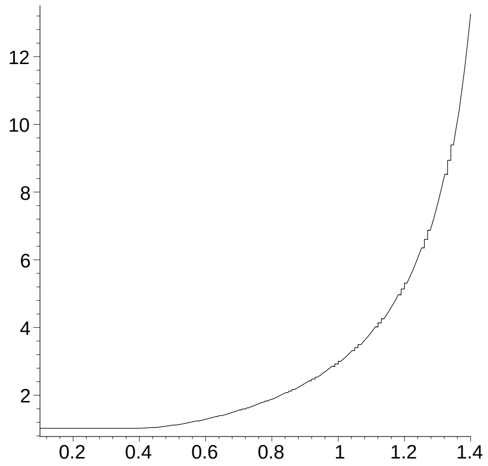

Figure 1 The graph of $\Phi(\tau)$

Corollary 2.2 *Using the notations of Theorem 2.1, we have*

$$\Phi(\tau) \le \frac{-1}{(1-\gamma_\varepsilon)(\alpha_0+\varepsilon)}, \qquad \tau \in [0,\pi/2), \tag{2.15}$$

and

$$\Phi(\tau) \le \frac{1}{1-\gamma}\left(\frac{2}{\alpha_0^2\tau} - \frac{1}{\alpha_0}\right), \qquad \tau \in [0,\pi/2). \tag{2.16}$$

Note that both estimates are worse than that given in [**5**].

Theorem 2.3 (Theorem 2.1, [5]) *For $\tau \in [0,\pi/2)$ the characteristic equation* (2.4) *has a root $\lambda_0 = \alpha_0 + i\beta_0$, such that $\alpha_0 < 0$, $\beta_0 \in [0,\pi/(2\tau))$, α_0 is the greatest real part of the roots of* (2.4), *and*

$$\Phi(\tau) \le \frac{\alpha_0^2+\beta_0^2}{\alpha_0^2}. \tag{2.17}$$

Inequality (2.17) is exact for $\tau \in [0,1/e]$, since then $\beta_0 = 0$. For a given $\tau \in (1/e,\pi/2)$ we can use Theorem 2.1 to estimate $\Phi(\tau)$ in the following way. Let u_n denote the restriction of u to the interval $[n\tau,(n+1)\tau]$. By integrating (2.1), it is easy to see that

$$\begin{aligned} u_0(t) &= 1, \qquad t \in [0,\tau], \\ u_n(t) &= u_{n-1}(n\tau) - \int_{n\tau}^{t} u_{n-1}(s-\tau)\,ds, \qquad t \in [n\tau,(n+1)\tau], \quad n \ge 1, \end{aligned}$$

and therefore u_n is an nth order polynomial, which can easily be generated, e.g., using a computer algebra system like Maple V. Since u_n is a polynomial, Maple V can symbolically integrate $\int_{n\tau}^{(n+1)\tau} |u_n(s)|\,ds$. Therefore if we write $\Phi(\tau) = \int_0^{M\tau} |u(t)|\,dt + \int_{M\tau}^{\infty} |u(t)|\,dt$, then we can compute the exact value of the first integral, and, using Theorem 2.1, we have an upper estimate

$$E_M(\tau) = \frac{1}{(1-\gamma)\tau}\int_{M\tau}^{\infty} (2t+\tau)e^{\alpha_0 t}\,dt$$

of the second one. Denoting the first integral by $I_M(\tau)$, we have $\Phi(\tau) \le I_M(\tau) + E_M(\tau)$. Unfortunately, as numerical experiments show, this computation of u_n is not stable, i.e., for large n the computed formula for u_n contains significant round-off errors. In Table 1 the numerical result of our computer experiment can be seen where we selected M by a certain algorithm so that M be reasonably small, and computed $I_M(\tau)$ over $[0, M\tau]$ (by computing the integral exactly over subintervals where the function u_n has constant sign by the symbolic integration of Maple V, and adding up those values). Note that $\tau = 0.2$ and 0.3 is computed only to test the method.

Table 1

τ	0.2	0.3	0.4	0.5	0.6	0.7	0.8
M	22	13	7	8	9	11	15
$I_M(\tau)$	0.997	0.998	1.001	1.083	1.260	1.511	1.846
$E_M(\tau)$	0.156	0.044	0.019	0.040	0.084	0.112	0.082
$I_M(\tau) + E_M(\tau)$	1.153	1.042	1.02	1.123	1.344	1.623	1.928
τ	0.9	1.0	1.1	1.2	1.3	1.4	1.5
M	17	20	26	26	26	25	24
$I_M(\tau)$	2.289	2.895	3.803	5.390	8.027	18.795	18.907
$E_M(\tau)$	0.191	0.402	0.591	4.254	29.77	243.5	3275
$I_M(\tau) + E_M(\tau)$	2.48	3.297	4.394	9.644	37.80	262.3	3294

Open problem This numerical estimate of Φ certainly requires a lot of computations. It is still an interesting open problem to give a (computable) formula for an upper estimate of $\Phi(\tau)$ better than (2.17). Find estimates for $\int_0^\infty |u(t;\tau)|\,dt$, where u is the fundamental solution of the multiple delay equation

$$\dot{x}(t) = -\sum_{i=1}^{m} a_i x(t - \tau_i).$$

The next theorem shows that Φ is a continuous function.

Theorem 2.4 *The function Φ is continuous on $[0, \pi/2)$.*

Proof Fix $\tau_0 \in [0, \pi/2)$, and let $\tau \ne \tau_0$. The characteristic root with greatest real part of (2.3) corresponding to τ_0 and τ is denoted by $\alpha_0 + i\beta_0$ and $\alpha + i\beta$, respectively. It is easy to see that $\alpha \to \alpha_0$ and $\beta \to \beta_0$ as $\tau \to \tau_0$ (see also [**6**]). It is known (see, e.g., [**11**]) that $u(t;\tau) \to u(t;\tau_0)$ as $\tau \to \tau_0$ for every fixed $t > 0$. Let $\varepsilon > 0$ be such that $\alpha_0 + 2\varepsilon < 0$, and let τ be such that the corresponding α satisfies $\alpha \le \alpha_0 + \varepsilon$. Let $\gamma_{\alpha,\varepsilon}$ and $\gamma_{\alpha_0,\varepsilon}$ be the constants defined by (2.5) corresponding to $\varepsilon > 0$ and to τ, α and τ_0, α_0, respectively. Then Theorem 2.1 yields that

$$|u(t;\tau) - u(t;\tau_0)| \le \frac{1}{1-\gamma_{\alpha,\varepsilon}} e^{(\alpha+\varepsilon)t} + \frac{1}{1-\gamma_{\alpha_0,\varepsilon}} e^{(\alpha_0+\varepsilon)t}, \qquad t \ge 0.$$

Since $\gamma_{\alpha,\varepsilon} \to \gamma_{\alpha_0,\varepsilon}$ as $\tau \to \tau_0$, there exists $M > 0$ such that $|u(t;\tau) - u(t;\tau_0)| \le Me^{(\alpha_0+2\varepsilon)t}$, for $t \ge 0$. Then Lebesgue's Dominated Convergence Theorem yields

$$|\Phi(\tau) - \Phi(\tau_0)| \le \int_0^\infty |u(t;\tau) - u(t;\tau_0)|\,dt \to 0, \qquad \text{as } \tau \to \tau_0.$$

□

Open problem Prove that τ is a monotone increasing function (as Figure 1 indicates).

3 Stability of linear delay differential equations

The function Φ introduced in the previous section plays an important role in the stability theory of delay differential equations. We just recall two examples from the literature. In [**5**] global attractivity results was proved for equations of the form

$$\dot{x}(t) = -ax(t-\tau) + f(t, x(t-\eta(t)))$$

with the help of estimate (2.17) of Φ. In [**9**] the following theorem was proved for the asymptotic stability of

$$\dot{x}(t) = -\sum_{i=1}^{m} a_i x(t-\tau_i-\eta_i(t)), \qquad t \geq 0, \tag{3.1}$$

comparing its stability to the "unperturbed" equation

$$\dot{y}(t) = -\sum_{i=1}^{m} a_i y(t-\tau_i), \qquad t \geq 0. \tag{3.2}$$

Here $\eta_i\colon [0,\infty) \to [0,\infty)$ are piecewise continuous bounded functions.

Theorem 3.1 (Theorem 3.1, [9]) *Suppose that the trivial solution of* (3.2) *is asymptotically stable, and*

$$\sum_{i=1}^{m} |a_i| \overline{\lim_{t\to\infty}} |\eta_i(t)| < \frac{1}{\left(\sum_{i=1}^{m} |a_i|\right) \int_0^\infty |v(t)|\, ds}, \tag{3.3}$$

where v is the fundamental solution of (3.2). *Then the trivial solution of* (3.1) *is asymptotically stable, as well.*

In the application of this theorem we need either the exact value of $\int_0^\infty |v(t)|\, ds$, which is known if $v(t) > 0$ (see [**9**]), or an upper estimate of it, which is known so far only for the single delay case (see Theorem 2.3).

Let $a_i > 0$ $(i = 1, \ldots, m)$, and consider the linear delay equation

$$\dot{x}(t) = -\sum_{i=1}^{m} a_i x(t-\sigma_i(t)), \qquad t \geq 0. \tag{3.4}$$

We can consider Equation (3.4) as the delay perturbation of

$$\dot{y}(t) = -\left(\sum_{i=1}^{m} a_i\right) y(t-\tau) \tag{3.5}$$

with the perturbations $\eta_i(t) = \sigma_i(t) - \tau$, where $\tau \geq 0$. Let v denote the fundamental solution of (3.5), then $\dot{v}(t) = -(\sum_{i=1}^{m} a_i) v(t-\tau)$. Therefore an application of Theorem 3.1 yields that if $0 \leq \tau \sum_{i=1}^{m} a_i < \pi/2$, and

$$\sum_{i=1}^{m} a_i \overline{\lim_{t\to\infty}} |\sigma_i(t) - \tau| < \frac{1}{\left(\sum_{i=1}^{m} a_i\right) \int_0^\infty |v(t)|\, dt}, \tag{3.6}$$

then the trivial solution of (3.4) is asymptotically stable. Introducing $u(t) = v(t/\sum_{i=1}^m a_i)$ we get

$$\dot{u}(t) = \frac{1}{\sum_{i=1}^m a_i}\dot{v}\left(\frac{t}{\sum_{i=1}^m a_i}\right) = -v\left(\frac{t}{\sum_{i=1}^m a_i} - \tau\right) = -u\left(t - \tau\sum_{i=1}^m a_i\right).$$

On the other hand,

$$\Phi\left(\tau\sum_{i=1}^m a_i\right) = \int_0^\infty |u(t)|\,dt = \int_0^\infty \left|v\left(\frac{t}{\sum_{i=1}^m a_i}\right)\right| dt = \left(\sum_{i=1}^m a_i\right)\int_0^\infty |v(t)|\,dt.$$

Therefore, using the relation

$$\overline{\lim_{t\to\infty}} |f(t)| = \max\left\{\overline{\lim_{t\to\infty}} f(t), -\underline{\lim}_{t\to\infty} f(t)\right\}, \tag{3.7}$$

we get immediately the following result.

Theorem 3.2 *Suppose $a_i > 0$, $\sigma_i : [0,\infty) \to [0,\infty)$ is piecewise continuous $(i = 1,\dots,m)$, and there exists $\tau \in [0, \pi/(2a))$ such that*

$$\tau a - \frac{1}{\Phi(\tau a)} < \sum_{i=1}^m a_i \underline{\lim}_{t\to\infty} \sigma_i(t) \le \sum_{i=1}^m a_i \overline{\lim_{t\to\infty}} \sigma_i(t) < \tau a + \frac{1}{\Phi(\tau a)}, \tag{3.8}$$

where $a \equiv \sum_{i=1}^m a_i$. Then the trivial solution of (3.4) *is asymptotically stable.*

Note that the first inequality of (3.8) is automatically satisfied if $0 \le \tau a \le 1/e$, since then $\Phi(\tau a) = 1$. See Figure 2 for the numerically generated graph of the functions $\tau + 1/\Phi(\tau)$ and $\tau - 1/\Phi(\tau)$.

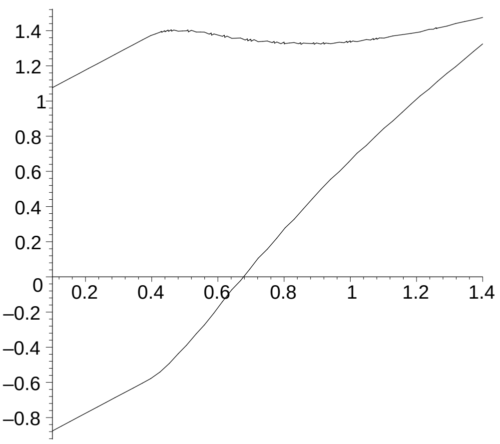

Figure 2 The graphs of $\tau + 1/\Phi(\tau)$ and $\tau - 1/\Phi(\tau)$

Suppose there exists $\tau \in [0, \pi/2)$ such that $\tau + 1/\Phi(\tau) > \pi/2$. Then, applying Theorem 3.2 for $m = 1$ and $a = 1$, we could find a constant delay $\sigma(t) = \sigma \ge \pi/2$, such that the trivial solution of $\dot{x}(t) = -x(t - \sigma)$ was asymptotically stable, which is impossible for such σ. Therefore we have the following corollary of the theorem.

Corollary 3.3 *The function* Φ *satisfies*

1. $\dfrac{1}{\frac{\pi}{2}-\tau} \le \Phi(\tau), \quad \tau \in [0, \pi/2),$
2. $\lim_{\tau\to\frac{\pi}{2}-} \Phi(\tau) = +\infty.$

We get a special case of Theorem 3.2 in the following way. Define

$$\tau_0 = \inf\{t\colon\ t - 1/\Phi(t) \ge 0\}. \tag{3.9}$$

Part 2 of Corollary 3.3 and $1/e < 1/\Phi(1/e)$ yields that such τ_0 exists, and since Φ is continuous, $\tau_0 = 1/\Phi(\tau_0)$. The numerical study of Figure 2 indicates that equation $\tau = 1/\Phi(\tau)$ has exactly one solution, and $\tau_0 \approx 0.65$.

Corollary 3.4 *Suppose* $a_i > 0$*,* $\sigma_i : [0,\infty) \to [0,\infty)$ *is piecewise continuous* ($i = 1,\dots,m$)*, and let* τ_0 *be defined by* (3.9)*. Assume*

$$\overline{\lim_{t\to\infty}}\, \sigma_i(t) < \frac{2\tau_0}{\sum_{i=1}^m a_i} \qquad \text{for } i = 1,\dots,m.$$

Then the trivial solution of (3.4) *is asymptotically stable.*

Proof Let $a = \sum_{i=1}^m a_i$, and fix $\tau > 0$ such that $2\tau < 2\tau_0/a$ and $\overline{\lim}_{t\to\infty}\sigma_i(t) < 2\tau$ for $i = 1,\dots,m$. For this τ we have

$$\sum_{i=1}^m a_i \overline{\lim_{t\to\infty}}\, \sigma_i(t) < 2a\tau < a\tau + \frac{1}{\Phi(a\tau)},$$

since $a\tau < \tau_0$. On the other hand $a\tau - 1/\Phi(a\tau) < 0$, therefore Theorem 3.2 proves the corollary. $\square$

Consider the delay equation

$$\dot{x}(t) = -x(t - \sigma(t)), \qquad t \ge 0. \tag{3.10}$$

Myshkis showed in [**17**], that if $\sup\{\sigma(t)\colon\ t \ge 0\} < 3/2$, then the trivial solution of (3.10) is asymptotically stable, and he gave an example, where $\sup\{\sigma(t)\colon\ t \ge 0\} \in (3/2, \pi/2)$ and the corresponding equation has unstable trivial solution. Note that in his example $\underline{\lim}_{t\to\infty}\sigma(t) = 0$. Many other papers generalized this 3/2-type result (see, e.g., [**14**], [**20**]–[**22**]). Ladas et al. showed [**15**] that if $\lim_{t\to\infty}\sigma(t) \in [0, \pi/2)$, then the trivial solution of (3.10) is asymptotically stable.

Our Theorem 3.2 generalizes both results. Ladas' condition is included in (3.8) using $\tau = \lim_{t\to\infty}\sigma(t)$. Myshkis' condition can be weaker than (3.8) in the case $0 < \tau - 1/\Phi(\tau)$. On the other hand, we formulate our condition in terms of $\overline{\lim}_{t\to\infty}\sigma(t)$ and $\underline{\lim}_{t\to\infty}\sigma(t)$ instead of $\sup_{t\ge 0}\sigma(t)$ and $\inf_{t\ge 0}\sigma(t)$. Moreover, if $\lim_{t\to\infty}\sigma(t)$ does not exist, and $\overline{\lim}_{t\to\infty}\sigma(t) \in (3/2, \pi/2)$, then Theorem 3.2 and Corollary 3.3 imply that if $\overline{\lim}_{t\to\infty}\sigma(t)$ is "not too small", then the trivial solution of (3.10) is asymptotically stable.

Corollary 3.5 *For any* $c \in (3/2, \pi/2)$ *there exists* $b < c$*, such that the trivial solution of* (3.10) *is asymptotically stable, if*

$$b < \underline{\lim_{t\to\infty}}\, \sigma(t) \le \overline{\lim_{t\to\infty}}\, \sigma(t) < c.$$

Now we give another application of Theorem 3.2. Consider the time-dependent scalar delay equation

$$\dot{x}(t) = -a(t)x(t-\sigma(t)), \qquad t \geq 0, \tag{3.11}$$

where $a: [0,\infty) \to [0,\infty)$ is continuous such that $\int_0^\infty a(t)\,dt = \infty$. The next theorem extends the result of Yoneyama [**19**], where it was proved that

$$0 < \inf_{t\geq 0} \int_t^{t+q} a(s)\,ds \leq \sup_{t\geq 0} \int_t^{t+q} a(s)\,ds < \frac{3}{2},$$

where $q = \sup_{t\geq 0} \sigma(t)$, implies the asymptotic stability of the trivial solution of (3.11). Ladas et al. [**15**] proved, that if $\sigma(t) = \sigma$ is constant, and

$$\lim_{t\to\infty} \int_{t-\sigma}^t a(s)\,ds \in [0, \pi/2),$$

then the trivial solution of (3.11) is asymptotically stable. We have the following result.

Theorem 3.6 *Suppose $a\colon [0,\infty) \to [0,\infty)$ is continuous, the function $A(t) = \int_0^t a(s)\,ds$ is strictly monotone increasing, $\int_0^\infty a(t)\,dt = \infty$, and $\sigma\colon [0,\infty) \to [0,\infty)$ is piecewise continuous and bounded, and assume there exists $\tau \in [0,\pi/2)$ such that*

$$\tau - \frac{1}{\Phi(\tau)} < \varliminf_{t\to\infty} \int_{t-\sigma(t)}^t a(s)\,ds \leq \varlimsup_{t\to\infty} \int_{t-\sigma(t)}^t a(s)\,ds < \tau + \frac{1}{\Phi(\tau)}. \tag{3.12}$$

Then the trivial solution of (3.11) *is asymptotically stable.*

Proof The inverse of A exists, $\lim_{t\to\infty} A^{-1}(t) = \infty$, and A^{-1} is continuous and differentiable. Define the function

$$\eta(t) = \int_{A^{-1}(t)-\sigma(A^{-1}(t))}^{A^{-1}(t)} a(s)\,ds.$$

Then $\eta\colon [0,\infty) \to [0,\infty)$ is piecewise continuous, and

$$\eta(t) = \int_0^{A^{-1}(t)} a(s)\,ds - \int_0^{A^{-1}(t)-\sigma(A^{-1}(t))} a(s)\,ds = t - A\Big(A^{-1}(t) - \sigma(A^{-1}(t))\Big),$$

and hence

$$A^{-1}(t-\eta(t)) = A^{-1}(t) - \sigma(A^{-1}(t)). \tag{3.13}$$

Let $y(t) = x(A^{-1}(t))$. Then

$$\dot{y}(t) = \frac{d}{dt}(A^{-1}(t))\dot{x}(A^{-1}(t)) = -x\Big(A^{-1}(t) - \sigma(A^{-1}(t))\Big),$$

therefore, using (3.13), y satisfies

$$\dot{y}(t) = -y(t-\eta(t)). \tag{3.14}$$

We have $\lim_{t\to\infty} y(t) = 0$, if and only if $\lim_{t\to\infty} x(t) = 0$, since $\lim_{t\to\infty} A^{-1}(t) = \infty$. Hence Theorem 3.2 implies the statement of this theorem, using

$$\varliminf_{t\to\infty} \eta(t) = \varliminf_{t\to\infty} \int_{t-\sigma(t)}^t a(s)\,ds \qquad \text{and} \qquad \varlimsup_{t\to\infty} \eta(t) = \varlimsup_{t\to\infty} \int_{t-\sigma(t)}^t a(s)\,ds.$$

□

4 Stability of linear delay difference equations

We denote the set of nonnegative integers by $\mathbb{N}_0$, and define the forward difference operator by $\Delta x(n) \equiv x(n+1) - x(n)$. Consider the linear delay difference equation

$$\Delta x(n) = -\sum_{i=1}^{m} a_i x(n - k_i(n)), \qquad n \in \mathbb{N}_0, \tag{4.1}$$

where $a_i > 0$ and $k_i\colon \mathbb{N}_0 \to \mathbb{N}_0$, $(i = 1, \dots, m)$, and there exists $r > 0$ such that $k_i(n) \le r$ for $n \in \mathbb{N}_0$ and $i = 1, \dots, m$. Equation (4.1) has a unique solution, assuming that

$$x(n) = \varphi(n), \tag{4.2}$$

for some $\varphi\colon [-r, 0] \to \mathbb{R}$.

In [**1**] it was proved that if $k_i(n) = k_i$ are constants for $i = 1, \dots, m$ and $\sum_{i=1}^{m} a_i k_i < 1$, then the trivial solution of (4.1) is asymptotically stable. In [**8**] it was shown that either one of the two conditions

1. there exists $T > 0$ such that $k_i(n) \le 1/(4\sum_{j=0}^{m} a_j)$ for $n > T$ and $i = 0, 1, \dots, m$;
2. There exists $T > 0$ and $0 \le \alpha \le 1$ such that $\alpha/(4\sum_{j=0}^{m} a_j) \in \mathbb{N}_0$, $k_i(n) \ge \alpha/(4\sum_{j=0}^{m} a_j)$ for $n > T$ and all $i = 0, 1, \dots, m$, and $\displaystyle\sum_{i=0}^{m} a_i \varlimsup_{n\to\infty} k_i(n) < 1 + \frac{\alpha}{4}$

implies the asymptotic stability of the trivial solution of (4.1). The idea of the proof was to compare the stability of (4.1) to that of the equation $\Delta y(n) = -(\sum_{i=1}^{m} a_i) y(n - l)$, and use the discrete version of Theorem 2.3 (see [**8**] for details).

In this paper we compare the stability of the discrete equation (4.1) to that of a differential equation. We associate the linear delay differential equation

$$\dot y(t) = -\sum_{i=1}^{m} a_i y\Big([t] - k_i([t])\Big), \qquad t \ge 0, \tag{4.3}$$

and the initial condition

$$y(t) = \varphi(t), \qquad t \in [-r, 0], \tag{4.4}$$

to (4.1)-(4.2), where $[\cdot]$ is the greatest integer function. Equation (4.3) is a so-called equation with piecewise constant argument (EPCA). EPCAs were first introduced and studied by Cooke and Wiener in [**2**] and [**3**]. For further developments see [**4**] and [**18**]. EPCAs were also used in [**1**], [**6**], [**8**] and [**12**] to get numerical approximations for different classes of differential equations.

Integrating both sides of (4.3) from n to $t \in [n, (n+1))$, we get

$$y(t) - y(n) = -\sum_{i=1}^{m} a_i y\Big(n - k_i(n)\Big)(t - n).$$

Therefore IVP (4.3)-(4.4) has a unique solution, which is piecewise linear between nonnegative integers, and

$$y(n+1) - y(n) = -\sum_{i=1}^{m} a_i y\Big(n - k_i(n)\Big), \qquad n \in \mathbb{N}_0. \tag{4.5}$$

We can observe that the solutions of (4.1) and (4.3) are related by $y(n) = x(n)$. Therefore the trivial solution of (4.1) is asymptotically stable, if and only if, so is the trivial solution of (4.3).

Rewrite (4.3) as

$$\dot{y}(t) = -\sum_{i=1}^{m} a_i y\Big(t - \sigma_i(t)\Big), \qquad t \geq 0, \tag{4.6}$$

where

$$\sigma_i(t) \equiv k_i([t]) + t - [t]. \tag{4.7}$$

Theorem 3.2 yields that the trivial solution of (4.6) (i.e., that of (4.3)) is asymptotically stable, if for some $\tau \in [0, \pi/(2a))$ it follows

$$\tau a - \frac{1}{\Phi(\tau a)} < \sum_{i=1}^{m} a_i \varliminf_{t\to\infty} \sigma_i(t) \leq \sum_{i=1}^{m} a_i \varlimsup_{t\to\infty} \sigma_i(t) < \tau a + \frac{1}{\Phi(\tau a)}, \tag{4.8}$$

where $a \equiv \sum_{i=1}^{m} a_i$. Since

$$\varliminf_{n\to\infty} k_i(n) \leq \varliminf_{t\to\infty} \sigma_i(t) \qquad \text{and} \qquad \varlimsup_{t\to\infty} \sigma_i(t) \leq \varlimsup_{n\to\infty} k_i(n) + 1,$$

we get the following result.

Theorem 4.1 *Suppose* $a_i > 0$ *(*$i = 1, \ldots, m$*),* $a \equiv \sum_{i=1}^{m} a_i$*, and for some* $\tau \in [0, \pi/(2a))$

$$\tau a - \frac{1}{\Phi(\tau a)} < \sum_{i=1}^{m} a_i \varliminf_{n\to\infty} k_i(n) \leq \sum_{i=1}^{m} a_i \varlimsup_{n\to\infty} k_i(n) < (\tau - 1)a + \frac{1}{\Phi(\tau a)} \tag{4.9}$$

holds. Then the trivial solution of (4.1) *is asymptotically stable.*

Note that the right-hand-side of (4.9) can not be replaced by $\tau a + 1/\Phi(\tau a)$, since that would imply, using Corollary 3.3, that if $m = 1$ and $k_i(n) = k$ constant, then the trivial solution of (4.1) was asymptotically stable, if and only if $ak < \pi/2$. This contradicts to the known condition (see, e.g., [**13**]) that the trivial solution of $\Delta x(n) = -ax(n-k)$ is asymptotically stable if and only if

$$0 < ak < 2k \cos \frac{k\pi}{2k+1}.$$

Applying Theorem 4.1 with $\tau = 1/(ea)$ the theorem has the following corollary.

Corollary 4.2 *Suppose* $0 < a_i$ *(*$i = 1, \ldots, m$*), and*

$$\sum_{i=1}^{m} a_i \varlimsup_{n\to\infty} k_i(n) < 1 + \frac{1}{e} - \sum_{i=1}^{m} a_i. \tag{4.10}$$

Then the trivial solution of (4.1) *is asymptotically stable.*

Similarly to Corollary 3.4 we get the next result.

Corollary 4.3 *Let* τ_0 *be defined by* (3.9). *Assume* $a_i > 0$ $(i = 1, \dots, m)$, $\sum_{i=1}^m a_i < 2\tau_0$, *and*

$$\overline{\lim_{n\to\infty}} k_i(n) < \frac{2\tau_0}{\sum_{i=1}^m a_i} - 1 \qquad \text{for } i = 1, \dots, m.$$

Then the trivial solution of (4.1) *is asymptotically stable.*

Note that Corollaries 4.2 and 4.3 improve the results of [**8**].

The method of Theorem 3.6 can be applied for discrete equations, as well. Consider the time-dependent scalar linear delay difference equation

$$\Delta x(n) = -a(n)x(n - k(n)), \qquad n \in \mathbb{N}_0, \tag{4.11}$$

where $a\colon \mathbb{N}_0 \to [0, \infty)$, $k\colon \mathbb{N}_0 \to \mathbb{N}_0$. Ladas et al. [**16**] proved that if

$$k(n) = k, \qquad \sum_{n=0}^{\infty} a(n) = \infty \qquad \text{and} \qquad \overline{\lim_{n\to\infty}} \sum_{i=n-k}^{n} a(i) < 1,$$

then the trivial solution of (4.11) is asymptotically stable. Győri and Pituk [**10**] showed that

$$k(n) = k \qquad \text{and} \qquad \overline{\lim_{n\to\infty}} \sum_{i=n-k}^{n-1} a(i) < 1$$

imply the asymptotic stability or (4.11). In some cases the following theorem extends these results.

Theorem 4.4 *Assume* $\sum_{n=0}^{\infty} a(n) = \infty$, *and there exists* $\tau \in [0, \pi/2)$ *such that*

$$\tau - \frac{1}{\Phi(\tau)} < \underline{\lim_{n\to\infty}} \sum_{i=n-k(n)}^{n-1} a(i) \le \overline{\lim_{n\to\infty}} \sum_{i=n-k(n)}^{n} a(i) < \tau + \frac{1}{\Phi(\tau)}. \tag{4.12}$$

Then the trivial solution of (4.11) *is asymptotically stable.*

Proof Let $b\colon [0, \infty) \to [0, \infty)$ be the continuous function satisfying $b(n) = 0$ and $b(n + 1/2) = 2a(n)$, and which is piecewise linear between these values. Then $\int_n^{n+1} b(s)\,ds = a(n)$, and the function

$$B\colon [0, \infty) \to [0, \infty), \qquad B(t) = \int_0^t b(s)\,ds$$

is a bijective, strictly monotone increasing function. Associate the delay differential equation

$$\dot{y}(t) = -b(t)y([t] - k([t])) \tag{4.13}$$

to (4.11). Integrating (4.13) from n to $t \in (n, n + 1)$ and taking the limit as $t \to (n + 1)-$ we get

$$y(n + 1) - y(n) = -\left(\int_n^{n+1} b(s)\,ds\right) y(n - k(n)),$$

i.e., $y(n) = x(n)$ for $n \in \mathbb{N}_0$. The function $z(t) = y(B^{-1}(t))$ satisfies

$$\dot{z}(t) = -y\Big([B^{-1}(t)] - k([B^{-1}(t)])\Big). \tag{4.14}$$

Let

$$\eta(t) = \int_{[B^{-1}(t)]-k([B^{-1}(t)])}^{B^{-1}(t)} b(s)\,ds,$$

then η satisfies $[B^{-1}(t)] - k([B^{-1}(t)]) = B^{-1}(t - \eta(t))$, therefore (4.14) yields

$$\dot{z}(t) = -z(t - \eta(t)). \tag{4.15}$$

We have

$$\varliminf_{t\to\infty} \eta(t) \geq \varliminf_{t\to\infty} \int_{[B^{-1}(t)]-k([B^{-1}(t)])}^{[B^{-1}(t)]} b(s)\,ds = \varliminf_{n\to\infty} \sum_{i=n-k(n)}^{n-1} a(i)$$

and

$$\varlimsup_{t\to\infty} \eta(t) \leq \varlimsup_{t\to\infty} \int_{[B^{-1}(t)]-k([B^{-1}(t)])}^{[B^{-1}(t)]+1} b(s)\,ds = \varlimsup_{n\to\infty} \sum_{i=n-k(n)}^{n} a(i),$$

therefore the theorem follows from Theorem 3.2. □

The theorem has the following corollary.

Corollary 4.5 *Assume* $\sum_{n=0}^{\infty} a(n) = \infty$, *and*

$$\varlimsup_{n\to\infty} \sum_{i=n-k(n)}^{n} a(i) < 1 + \frac{1}{e}.$$

Then the trivial solution of (4.11) *is asymptotically stable.*

References

[1] K. L. Cooke and I. Győri, *Numerical approximation of the solutions of delay differential equations on an infinite interval using piecewise constant arguments*, **28:1–3** (1994) 81–92.

[2] K. L. Cooke and J. Wiener, *Retarded differential equations with piecewise constant delays*, J. Math. Anal. Appl., **99** (1984) 265–297.

[3] K. L. Cooke and J. Wiener, *Stability regions for linear equations with piecewise continuous delays*, Computers Math. Applic., **12A** (1986) 695–701.

[4] K. L. Cooke and J. Wiener, *A survey of differential equations with piecewise continuous arguments*, in Delay Differential Equations and Dynamical Systems, eds. S. Busenberg and M. Martelli (Lecture Notes in Math. **1475**, Springer-Verlag, Berlin, 1991, 1–15.

[5] I. Győri, *Global attractivity in a perturbed linear delay differential equation*, Applicable Analysis, **34** (1989) 167–181.

[6] I. Győri, *Approximation of the solution of delay differential equations by using piecewise constant arguments*, International J. of Math. Math. Sciences, **14** (1991) 111–126.

[7] I. Győri, F. Hartung and J. Turi, *Numerical approximations for a class of differential equations with time- and state-dependent delays*, Applied Math. Letters, **8:6** (1995) 19–24.

[8] I. Győri, F. Hartung and J. Turi, *On the effect of delay perturbations on the stability of delay difference equations*, Proceedings of the First International Conference on Difference Equations, San Antonio, Texas, May 1994, eds. S. N. Elaydi, J. R. Graef, G. Ladas and A. C. Peterson, Gordon and Breach, 1995, 237–253.

[9] I. Győri, F. Hartung and J. Turi, *Preservation of Stability in Delay Equations under Delay Perturbations*, J. Math. Anal. Appl., **220** (1998), 290–312.

[10] I. Győri and M. Pituk, *Asymptotic stability in a linear delay difference equation*, in Advances in Difference Equations, Proceedings of the Second International Conference on Difference Equations, eds. S. Elaydi, I. Győri and G. Ladas, Gordon and Breach Science Publishers, Amsterdam, 1997.

[11] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Spingler-Verlag, New York, 1993.

[12] F. Hartung, T. L. Herdman, and J. Turi, *On existence, uniqueness and numerical approximation for neutral equations with state-dependent delays*, Appl. Numer. Math. **24** (1997) 393–409.

[13] V. L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, 1993.

[14] T. Krisztin, *On stability properties for one-dimensional functional differential equations*, Funkcional Ekvacioj **34** (1991) 241–256.

[15] G. Ladas, Y. G. Sficas, and I. P. Stavroulakis, *Asymptotic behavior of solutions of retarded differential equations*, Proc. Amer. Math. Soc., **88** (1983) 247–253.

[16] G. Ladas, C. Qian, P. N. Vlahos, and J. Yan, *Stability of solutions of linear nonautonomous difference equations*, Appl. Anal., **41** (1991) 183–191.

[17] A. D. Myshkis, *Linear differential equations with retarded arguments* [Russian], Nauka, Moscow, 1972.

[18] J. Wiener, *Generalized solutions of functional differential equations*, World Scientific, Singapore, 1993.

[19] T. Yoneyama, *On the 3/2 stability theorem for one-dimensional delay-differential equations*, J. Math. Anal. Appl. **125** (1987) 161–173.

[20] T. Yoneyama and J. Sugie, *On the stability region of scalar delay-differential equation*, J. Math. Anal. Appl. **134** (1988) 408–425.

[21] T. Yoneyama and J. Sugie, *On the stability region of differential equation with two delays*, Funkcialaj Ekvacioj **31** (1988) 233–240.

[22] J. A. Yorke,*Asymptotic stability for one dimensional differential-delay equations*, J. Differential Equations **7** (1988) 189–202.