ON DIFFERENTIABILITY OF SOLUTIONS WITH RESPECT TO PARAMETERS IN A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study differentiability of solutions with respect to parameters in state-dependent delay equations. In particular, we give sufficient conditions for differentiability of solutions in the $W^{1,\infty}$ norm.

1. Introduction. We consider the state-dependent delay system

(1) \[ \dot{x}(t) = f\left(t, x(t), x(t - \tau(t, x(t, \sigma))), \theta \right), \quad t \in [0, T], \]

with initial condition

(2) \[ x(t) = \varphi(t), \quad t \in [-r, 0]. \]

Here $\theta \in \Theta$ and $\sigma \in \Sigma$ represent parameters in the function $f$ and in the delay function, $\tau$, where $\Theta$ and $\Sigma$ are normed linear spaces with norms $\| \cdot \|_{\Theta}$ and $\| \cdot \|_{\Sigma}$, respectively. The notation $x_t$ denotes the solution segment function, i.e., $x_t : [-r, 0] \rightarrow \mathbb{R}^n$, $x_t(s) \equiv x(t + s)$. (See Section 2 below for the detailed assumptions on the initial value problem (IVP) (1)-(2).)

In this paper we study differentiability of solutions of IVP (1)-(2) with respect to (wrt) the parameters $\varphi, \sigma$ and $\theta$. Differentiability wrt parameters in delay equations has been investigated, e.g., in [1], [5] and [6]. It has also been studied in state-dependent delay equations in [8], where sufficient conditions were given guaranteeing differentiability of the parameter map $\Gamma \rightarrow W^{1,p}$, $\gamma \mapsto x(\cdot; \gamma)$ (where $\gamma \in \Gamma$ is some parameter of the equation, and $1 \leq p < \infty$). In establishing this result a version of the Uniform Contraction Principle for quasi-Banach spaces was used. In many applications (e.g., in parameter identification problems, see, e.g., [2] and [3]) this sort of differentiability (i.e., differentiability in a $W^{1,p}$ norm) is too weak. In this paper we establish sufficient conditions implying “pointwise” differentiability of the parameter map, i.e., differentiability of $\Gamma \rightarrow \mathbb{R}^n$, $\gamma \mapsto x(t; \gamma)$, and the stronger property, differentiability of the map $\Gamma \rightarrow W^{1,\infty}$, $\gamma \mapsto x(\cdot; \gamma)$.

Our main results are contained in Section 3. In Section 2 we list our assumptions on IVP (1)-(2), introduce our notations, and give some necessary preliminary results.

2. Notations, assumptions and preliminaries. Throughout this paper a norm on $\mathbb{R}^n$ and the corresponding matrix norm on $\mathbb{R}^{n \times n}$ are denoted by $\| \cdot \|$ and $\| \cdot \|_1$, respectively.

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The notation \( f : (A \subset X) \to Y \) will be used to denote that the function maps the subset \( A \) of the normed linear space \( X \) to \( Y \). This notation emphasizes that the topology on \( A \) is defined by the norm of \( X \).

We denote the open ball around a point \( x_0 \) with radius \( R \) in a normed linear space \((X, |\cdot|_X)\) by \( B(x_0; R) \), i.e., \( B(x_0; R) = \{ x \in X : |x - x_0|_X < R \} \), and the corresponding closed ball by \( \overline{B}(x_0; R) \). Similarly, a neighborhood of a set \( M \subset X \) with radius \( R \) is denoted by \( G_M(M; R) \), i.e., \( G_M(M; R) = \{ x \in X : |x - y|_X < R \} \). The closure of this neighborhood is denoted by \( \overline{G}_M(M; R) \).

The space of continuous functions from \([-r, 0]\) to \( \mathbb{R}^n \) and the usual supremum norm on it are denoted by \( C \) and \( |\cdot|_C \), respectively. The space of absolutely continuous functions from \([-r, 0]\) to \( \mathbb{R}^n \) with essentially bounded derivatives is denoted by \( W^{1,\infty} \). The corresponding norm on \( W^{1,\infty} \) is \( |\cdot|_{W^{1,\infty}} \equiv \max\{|\psi|_C, \text{ess sup}\{\int_t^s |\dot{\psi}(s)| : s \in [-r, 0]\} \} \).

The partial derivatives of a function \( g(t, x_2, \ldots, x_n) \) w.r.t its second, third, etc. arguments are denoted by \( D_2g, D_3g, \) etc, and the derivative w.r.t \( t \) is denoted by \( \dot{g} \). Note that all derivatives we use in this paper are Fréchet-derivatives.

Next we consider a set of technical conditions, guaranteeing well-posedness and differentiability of solutions w.r.t parameters, for the state-dependent delay differential equation (1) with initial condition (2).

Let \( \Omega_1 \subset \mathbb{R}^n, \Omega_2 \subset \mathbb{R}^n, \Omega_3 \subset \Theta, \Omega_4 \subset C, \) and \( \Omega_5 \subset \Sigma \) be open subsets of the respective spaces. \( T > 0 \) is finite or \( T = \infty \), in which case \([0, T] \) denotes the interval \([0, \infty)\).

(A1) (i) \( f : [0, T] \times \Omega_1 \times \Omega_2 \times \Omega_3 \to \mathbb{R}^n \) is continuous,

(ii) \( f(t, v, w; \theta) \) is locally Lipschitz-continuous in \( v, w \) and \( \theta \) in the following sense: for every \( \alpha > 0 \), \( M_1 \subset \Omega_1, M_2 \subset \Omega_2, M_3 \subset \Omega_3 \), where \( M_1 \) and \( M_2 \) are compact subsets of \( \mathbb{R}^n \) and \( M_3 \) is a closed, bounded subset of \( \Theta \), there exists a constant \( L_1 = L_1(\alpha, M_1, M_2, M_3) \) such that

\[
|f(t, v, w; \theta) - f(t, \bar{v}, \bar{w}, \bar{\theta})| \leq L_1 \left( |v - \bar{v}| + |w - \bar{w}| + |\theta - \bar{\theta}|_\alpha \right),
\]

for \( t \in [0, \alpha] \), \( v, \bar{v} \in M_1, w, \bar{w} \in M_2, \) and \( \theta, \bar{\theta} \in M_3 \),

(iii) \( f : \left( [0, T] \times \Omega_1 \times \Omega_2 \times \Omega_3 \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \Theta \right) \to \mathbb{R}^n \) is continuously differentiable w.r.t its second, third and fourth arguments,

(A2) (i) \( \tau : [0, T] \times \Omega_4 \times \Omega_5 \to \mathbb{R}, \infty \) is continuous, and

\[
t - \tau(t, \psi, \sigma) \geq -r, \quad \text{for } t \in [0, T], \psi \in \Omega_4, \text{ and } \sigma \in \Omega_5,
\]

(ii) \( \tau(t, \psi, \sigma) \) is locally Lipschitz-continuous in \( \psi \) and \( \sigma \) in the following sense: for every \( \alpha > 0 \), \( M_4 \subset \Omega_4 \) and \( M_5 \subset \Omega_5 \), where \( M_4 \) is a compact subset of \( C \), and \( M_5 \) is a closed, bounded subset of \( \Sigma \), there exists a constant \( L_2 = L_2(\alpha, M_4, M_5) \) such that

\[
|\tau(t, \psi, \sigma) - \tau(t, \bar{\psi}, \bar{\sigma})| \leq L_2 \left( |\psi - \bar{\psi}|_C + |\sigma - \bar{\sigma}|_\Sigma \right)
\]

for \( t \in [0, \alpha], \psi, \bar{\psi} \in M_4, \) and \( \sigma, \bar{\sigma} \in M_5, \)
(iii) $\tau : \left[0, T \right] \times \Omega_4 \times \Omega_5 \subset [0, \alpha] \times C \times \Sigma \rightarrow \mathbb{R}$ is continuously differentiable wrt its second and third arguments.

Note that (A1) (i), (ii) and (A2) (i), (ii) together with $\varphi \in W^{1,\infty}$ are standard assumptions in state-dependent delay equations guaranteeing the existence and uniqueness of the solution (see, e.g., [4] or [8]). If the parameter spaces $\Theta$ and $\Sigma$ are finite dimensional, then (A1) (ii) and (A2) (ii) follow from (A1) (iii) and (A2) (iii), respectively. We refer to [8] for further comments on the particular definition of local Lipschitz-continuity we use in (A1) (ii) and (A2) (ii).

We will use the following function to simplify the notation:

(3) $\Lambda : \left[0, T \right] \times \Omega_4 \times \Omega_5 \subset \mathbb{R} \times W^{1,\infty} \times \Sigma \rightarrow \mathbb{R}^n$, $\Lambda(t, \psi, \sigma) \equiv \psi(-\tau(t, \psi, \sigma))$.

With this notation we can rewrite (1) simply as:

$$\dot{x}(t) = f(t, x(t), \Lambda(t, x_1, \sigma, \theta)), \quad t \in [0, T].$$

It follows from the definition of $\Lambda$, (A2) (ii) and the Mean Value Theorem that

(4) $|\Lambda(t, \psi, \sigma) - \Lambda(t, \tilde{\psi}, \tilde{\sigma})|$

$$\leq |\tilde{\psi}(-\tau(t, \psi, \sigma)) - \tilde{\psi}(-\tau(t, \tilde{\psi}, \tilde{\sigma}))| + |\psi(-\tau(t, \psi, \sigma)) - \psi(-\tau(t, \tilde{\psi}, \tilde{\sigma}))|$

$$\leq L_t|\tilde{\psi}|_{W^{1,\infty}} |\psi - \tilde{\psi}|_C + |\sigma - \tilde{\sigma}|_C + |\psi - \tilde{\psi}|_C$

for $t \in [0, \alpha]$, $\psi, \tilde{\psi} \in M_4$, $\tilde{\psi} \in W^{1,\infty}$, and $\sigma, \tilde{\sigma} \in M_5$.

**Lemma 1.** Assume (A2), and let $\Lambda$ be defined by (3). Then $D_2\Lambda(t, \psi, \sigma)$ and $D_3\Lambda(t, \psi, \sigma)$ exist for $t \in [0, T]$, $\psi \in \Omega_4 \cap C^1$, $\sigma \in \Omega_5$, and

(5) $D_2\Lambda(t, \psi, \sigma)\psi = -\dot{\psi}(-\tau(t, \psi, \sigma))D_2\tau(t, \psi, \sigma)\psi + h(-\tau(t, \psi, \sigma))$, $h \in W^{1,\infty},$

(6) $D_3\Lambda(t, \psi, \sigma) = -\dot{\psi}(-\tau(t, \psi, \sigma))D_3\tau(t, \psi, \sigma)$.

Moreover, $D_2\Lambda(t, \cdot, \cdot)$ and $D_3\Lambda(t, \cdot, \cdot)$ are continuous on $(\Omega_4 \cap C^1) \times \Omega_5$ for $t \in [0, T]$.

Proof. Let $\psi \in \Omega_4 \cap C^1$, and introduce $\omega^0(\tilde{\sigma}; s) \equiv \psi(s) - \psi(\tilde{\sigma}) - \tilde{\psi}(s - \tilde{\sigma})$ for $\tilde{\sigma}, s \in [-\tau, 0]$, and $\omega^0(t, \psi, \sigma; \psi + h) \equiv \tau(t, \psi + h, \sigma) - \tau(t, \psi, \sigma) - D_2\sigma(t, \psi, \sigma)\psi + h \in W^{1,\infty}$, and $\sigma \in \Omega_5$. Let $t \in [0, T]$, $\psi + h \in \Omega_4$, and $\sigma \in \Omega_5$, and consider

$$\Lambda(t, \psi + h, \sigma) - \Lambda(t, \psi, \sigma)$$

$$= \psi(-\tau(t, \psi + h, \sigma)) - \psi(-\tau(t, \psi, \sigma)) + h(-\tau(t, \psi, \sigma))$$

$$= -\psi(-\tau(t, \psi, \sigma))(\tau(t, \psi + h, \sigma) - \tau(t, \psi, \sigma)) + h(-\tau(t, \psi, \sigma))$$

$$+ \omega^0(-\tau(t, \psi, \sigma) - \tau(t, \psi + h, \sigma)) + h(-\tau(t, \psi + h, \sigma)) - h(-\tau(t, \psi, \sigma))$$

$$= -\psi(-\tau(t, \psi, \sigma))D_2\tau(t, \psi, \sigma)h + h(-\tau(t, \psi, \sigma))$$

$$- \dot{\psi}(-\tau(t, \psi, \sigma))\omega^0(t, \psi, \sigma; \psi + h)$$

$$+ \omega^0(-\tau(t, \psi, \sigma) - \tau(t, \psi + h, \sigma)) + h(-\tau(t, \psi + h, \sigma)) - h(-\tau(t, \psi, \sigma))$$

Relation (5) follows from the last equation, using the continuity of $\tau$, the inequality

$$|h(-\tau(t, \psi + h, \sigma)) - h(-\tau(t, \psi, \sigma))| \leq |h|_{W^{1,\infty}}|\tau(t, \psi + h, \sigma) - \tau(t, \psi, \sigma)|$$
guaranteed by the Mean Value Theorem, $|\omega^h(\tilde{s}, s) - \tilde{s}| \leq h |\tilde{s} - s|$. Note that the last relation follows from $|\omega^h(t, \psi, \sigma; \psi + h) - h| = |h|$. Relation (6) is an immediate consequence of the Chain rule. The continuity of $D_2\lambda(t, \cdot, \cdot)$ and $D_3\lambda(t, \cdot, \cdot)$ follows readily from (5) and (6) and from the assumed continuity of $\tau, D_2\tau$ and $D_3\tau$. \]

We introduce the function

$$\omega^h(t, \tilde{\psi}, \tilde{\sigma}; \psi, \sigma) \equiv \Lambda(t, \psi, \sigma) - \Lambda(t, \tilde{\psi}, \tilde{\sigma}) - D_2\lambda(t, \tilde{\psi}, \tilde{\sigma})(\psi - \tilde{\psi}) - D_3\lambda(t, \tilde{\psi}, \tilde{\sigma})(\sigma - \tilde{\sigma})$$

for $t \in [0, T]$, $\tilde{\psi}, \psi \in \Omega_4$, $\tilde{\sigma}, \sigma \in \Omega_5$.

Let $\alpha > 0$, $M_4 \subset \Omega_4$ be a compact subset of $C$, $M_5 \subset \Omega_5$ be a closed and bounded subset of $\Sigma$. It is easy to prove, using the definition of $\omega^h$, (A2) (ii), (iii), (4), (5), and (6), that there exists a constant $K = K(\alpha, M_1, M_2)$ such that

$$\|D_2\lambda(t, \tilde{\psi}, \tilde{\sigma})\|_{C(\Theta^2, \mathbb{R}^n)} \leq K \quad \text{and} \quad \|D_3\lambda(t, \tilde{\psi}, \tilde{\sigma})\|_{C(\Sigma, \mathbb{R}^n)} \leq K,$$

and

$$\|\omega^h(t, \tilde{\psi}, \tilde{\sigma}, \psi, \sigma)\| \leq 2K(\|\psi - \tilde{\psi}\|_C + \|\sigma - \tilde{\sigma}\|_\infty)$$

for $t \in [0, \alpha]$, $\tilde{\psi}, \psi \in M_4$, $\tilde{\sigma}, \sigma \in M_5$.

Similarly to $\omega^h$, we define

$$\omega^h(t, \tilde{x}, \tilde{y}, \tilde{\theta}; x, y, \theta) \equiv f(t, x, y, \theta) - f(t, \tilde{x}, \tilde{y}, \tilde{\theta}) - D_2 f(t, \tilde{x}, \tilde{y}, \tilde{\theta})(x - \tilde{x}) - D_3 f(t, \tilde{x}, \tilde{y}, \tilde{\theta})(y - \tilde{y})$$

for $t \in [0, T]$, $\tilde{x}, x \in \Omega_1$, $\tilde{y}, y \in \Omega_2$, and $\tilde{\theta}, \theta \in \Omega_3$. Assumption (A1) (iii) implies that

$$\frac{|\omega^h(t, \tilde{x}, \tilde{y}, \tilde{\theta}; x, y, \theta)|}{|x - \tilde{x}| + |y - \tilde{y}| + |\theta - \tilde{\theta}|_\Theta} \to 0, \quad \text{as } |x - \tilde{x}| + |y - \tilde{y}| + |\theta - \tilde{\theta}|_\Theta \to 0.$$ 

Let $\alpha > 0$ be fixed, $M_i \subset \Omega_i$ ($i = 1, 2, 3$) be such that $M_1$ and $M_2$ be compact subsets of $\mathbb{R}^n$ and $M_3$ be a closed and bounded subset of $\Theta$, and let $L_1 = L_1(\alpha, M_1, M_2, M_3)$ be the constant from (A1) (ii). Then assumptions (A1) (ii) and (iii) yield that

$$\|D_2 f(t, \tilde{x}, \tilde{y}, \tilde{\theta})\| \leq L_1, \quad \|D_3 f(t, \tilde{x}, \tilde{y}, \tilde{\theta})\| \leq L_1, \quad \|D_4 f(t, \tilde{x}, \tilde{y}, \tilde{\theta})\|_{C(\Theta, \mathbb{R}^n)} \leq L_1$$

and

$$|\omega^h(t, \tilde{x}, \tilde{y}, \tilde{\theta}; x, y, \theta)| \leq 2L_1(\|x - \tilde{x}\| + |y - \tilde{y}| + |\theta - \tilde{\theta}|_\Theta)$$

for $t \in [0, \alpha]$, $x, \tilde{x} \in M_1$, $y, \tilde{y} \in M_2$, and $\theta, \tilde{\theta} \in M_3$.

We define the parameter space $\Gamma = W^{1, \infty} \times \Sigma \times \Theta$, and use the notation $\gamma = (\varphi, \sigma, \theta)$ (or $\gamma = (\gamma^\varphi, \gamma^\sigma, \gamma^\theta)$) for the components of $\gamma \in \Gamma$, and $|\gamma|_\Gamma \equiv |\varphi|_{W^{1, \infty}} + |\sigma|_\Sigma + |\theta|_\Theta$ for the norm on
The solution of IVP (1)-(2) corresponding to a parameter \( \gamma \) and its segment function at \( t \) are denoted by \( x(t; \gamma) \) and \( x(; \gamma)_t \), respectively.

Introduce

\[
\Pi \equiv \left\{ \gamma = (\varphi, \sigma, \theta) \in \Omega_1 \times \Omega_2 \times \Omega_3 : \quad \varphi \in W^{1, \infty}, \quad \varphi(0) \in \Omega_1, \quad \Lambda(0, \varphi, \sigma) \in \Omega_2 \right\}
\]

and

\[
\mathcal{M} \equiv \left\{ \gamma = (\varphi, \sigma, \theta) \in \Pi : \quad \varphi \in C^1, \quad \varphi(0-) = f(0, \varphi(0), \Lambda(0, \varphi, \sigma), \theta) \right\}.
\]

**Theorem 1.** Assume (A1) (i), (ii), (A2) (i), (ii), and let \( \bar{\gamma} \in \Pi \). Then there exist \( \delta > 0 \) and \( 0 < \alpha \leq T \) such that

(i) \( \mathcal{G}_T(\bar{\gamma}; \delta) \subset \Pi \),

(ii) IVP (1)-(2) has a unique solution, \( x(t; \gamma) \), on \([0, \alpha]\) for all \( \gamma \in \mathcal{G}_T(\bar{\gamma}; \delta) \).

(iii) there exist \( M_1 \subset \Omega_1, M_2 \subset \Omega_2 \) and \( M_3 \subset \Omega_3 \) compact subsets of \( \mathbb{R}^n \) and \( C \), respectively, such that

\[
\text{for } t \in [0, \alpha], \quad \gamma \in \mathcal{G}_T(\bar{\gamma}; \delta), \quad x(; \gamma)_t \in W^{1, \infty} \text{ for } t \in [0, \alpha], \quad \gamma \in \mathcal{G}_T(\bar{\gamma}; \delta), \quad \text{and there exists } L = L(\alpha, \delta), \text{ such that}
\]

\[
|x(; \gamma)_t - x(; \bar{\gamma})_t|_{W^{1, \infty}} \leq L(\gamma - \bar{\gamma})_t \quad \text{for } t \in [0, \alpha], \quad \gamma \in \mathcal{G}_T(\bar{\gamma}; \delta).
\]

(v) the function \( x(; \gamma) : [-\tau, \alpha] \to \mathbb{R}^n \) is continuously differentiable for \( \gamma \in \mathcal{M} \cap \mathcal{G}_T(\bar{\gamma}; \delta) \).

**Proof.** Part (i) and (v) are obvious (see also [7]). For the proof of (ii) we refer to [8], [9] or [4]. Part (iii) and (iv) will be essential in our proofs in the next section, therefore we prove them here. Let \( \delta^i > 0 \) and \( \alpha > 0 \) be such that they satisfy (i) and (ii). We will show that \( 0 < \delta \leq \delta^i \) can be selected so that (iii) and (iv) are also satisfied.

Let \( \bar{\gamma} = (\bar{\varphi}, \bar{\sigma}, \bar{\theta}) \in \Pi \), and define \( M_1^* \equiv \{ x(t; \gamma) : t \in [0, \alpha] \} \), \( M_2^* \equiv \{ \Lambda(t, x(; \gamma)_t, \bar{\sigma}) : t \in [0, \alpha] \} \), and \( M_3^* \equiv \{ x(; \gamma)_t : t \in [0, \alpha] \} \). From part (ii) of the theorem that \( M_i^* \subset \Omega_i \) \( (i = 1, 2, 4) \). Moreover, \( M_1^* \) and \( M_2^* \) are compact subsets of \( \mathbb{R}^n \) since \( t \to x(t; \gamma) \) is continuous on \([0, \alpha] \). Therefore there exist \( \varepsilon > 0 \) \( (i = 1, 2, 4) \) such that \( M_1 \equiv \overline{\mathcal{G}_R}^+(M_1^*; \varepsilon) \subset \Omega_1 \), \( M_2 \equiv \overline{\mathcal{G}_R}^+(M_2^*; \varepsilon) \subset \Omega_2 \), and \( \mathcal{G}_C(M_3^*; \varepsilon) \subset \Omega_3 \) since \( \Omega_4 \) \((i = 1, 2, 4) \) are open sets in \( \mathbb{R}^n \) and \( C \), respectively. Let \( M_4 \equiv \overline{\mathcal{G}}_{W^{1, \infty}}(M_4^*; \varepsilon) \). Clearly, \( M_1 \) and \( M_2 \) are compact subsets of \( \mathbb{R}^n \). We have \( M_4 \subset \Omega_4 \), and it is compact in \( C \) by Arzela-Ascoli’s Theorem, since it is a bounded subset of \( W^{1, \infty} \).

Let \( \delta^2 \equiv \min\{\delta^1, \varepsilon^1, \varepsilon^2 / (L_2\|\bar{\varphi}\|_{W^{1, \infty}} + 1), \varepsilon^4 \} \). Let \( \gamma = (\varphi, \sigma, \theta) \in \mathcal{G}_T(\bar{\gamma}; \delta^2) \). We have from (4) and the definition of \( \| \_ \|_T \) that \( |x(0) - x(0)| < \varepsilon^1 \), \( |\Lambda(0, \varphi, \sigma) - \Lambda(0, \bar{\varphi}, \bar{\sigma})| \leq L_2\|\bar{\varphi}\|_{W^{1, \infty}} |\varphi - \bar{\varphi}| + |\sigma - \bar{\sigma}| + |\varphi - \bar{\varphi}| < \varepsilon^3 \), and \( |\varphi - \bar{\varphi}| < \varepsilon^4 \). Therefore there exists \( 0 < \alpha^* \leq \alpha \) such that

\[
|x(t; \gamma) - x(t; \bar{\gamma})| < \varepsilon^1, \quad |\Lambda(t, x(; \gamma)_t, \sigma) - \Lambda(t, x(; \bar{\gamma})_t, \bar{\sigma})| < \varepsilon^3,
\]

and

\[
|x(; \gamma)_t - x(; \bar{\gamma})_t| < \varepsilon^4
\]
for \( t \in [0, \alpha^\gamma] \).

Let \( L_1 = L_1(\alpha, M_1, M_2, M_3) \) and \( L_2 = L_2(\alpha, M_4, M_5) \) be the constants from (A1) (ii) and (A2) (ii), respectively. We have for \( t \in [0, \alpha^\gamma] \):

\[
\begin{align*}
|x(t; \gamma) - x(t; \bar{\gamma})| &
\leq |\varphi(0) - \bar{\varphi}(0)| + \int_0^t \left| f(s, x(s; \gamma), \Lambda(s, x(s; \gamma), , \sigma), \theta) - f(s, x(s; \bar{\gamma}), \Lambda(s, x(s; \bar{\gamma}), , \bar{\sigma}), \bar{\theta}) \right| \, ds \\
&\leq |\gamma - \bar{\gamma}| r + L_1 \int_0^t \left( |x(s; \gamma) - x(s; \bar{\gamma})| + |\Lambda(s, x(s; \gamma), , \sigma) - \Lambda(s, x(s; \bar{\gamma}), , \bar{\sigma})| + |\theta - \bar{\theta}| r \right) \, ds.
\end{align*}
\]

Let \( N \equiv \max\{\max\{|x(t; \gamma)| : t \in [-r, \alpha]\}, \text{ess sup}\{|x(t; \bar{\gamma})| : t \in [-r, \alpha]\}\}. \) Then (4) yields

\[
|x(t; \gamma) - x(t; \bar{\gamma})| \leq |\gamma - \bar{\gamma}| r + L_1 \int_0^t \left( |x(s; \gamma) - x(s; \bar{\gamma})| + L_2 N |x(s; \gamma) - x(s; \bar{\gamma})| c + |\sigma - \bar{\sigma}| r + |\theta - \bar{\theta}| r \right) \, ds.
\]

Introduce \( \eta(t; \gamma) \equiv \text{sup}\{|x(s; \gamma) - x(s; \bar{\gamma})| : s \in [-r, t]\}. \) With this notation we get

\[
|x(t; \gamma) - x(t; \bar{\gamma})| \leq (1 + L_1 + L_2 N) |\gamma - \bar{\gamma}| r + L_1 (2 + L_2 N) \int_0^t |\eta(s; \gamma, \bar{\gamma})| \, ds,
\]

for \( t \in [0, \alpha^\gamma]. \) The monotonicity of the right-hand side in \( t \) and \( \eta(t; \gamma, \bar{\gamma}) \leq |\gamma - \bar{\gamma}| r \) for \( t \in [-r, 0] \) yield

\[
\eta(t; \gamma, \bar{\gamma}) \leq (1 + L_1 + L_2 N) |\gamma - \bar{\gamma}| r + L_1 (2 + L_2 N) \int_0^t \eta(s; \gamma, \bar{\gamma}) \, ds, \quad t \in [0, \alpha^\gamma].
\]

Applying the Gronwall-Bellmann inequality we get

(16) \[ |x(t; \gamma) - x(t; \bar{\gamma})| \leq \eta(t; \gamma, \bar{\gamma}) \leq L^* |\gamma - \bar{\gamma}| r, \quad t \in [-r, \alpha^\gamma], \]

where \( L^* \equiv (1 + L_1 + L_1 N) e^{L^* (1 + L_2 N)^{\gamma}} \). Let \( \delta \equiv \min\{\delta^2, \varepsilon^2 / L^*, \varepsilon^2 / (L_2 N (L^* + 1) + L^*) \}. \) Then it is easy to show, using (16), that \( \alpha^\gamma = \alpha \) can be used in (14) and (15) for \( \gamma \in \mathcal{G}; \delta \).

This proves (12) as well.

It follows from (1), (16), (A1) (ii) and (A2) (ii) that

(17) \[
\begin{align*}
|\dot{x}(t; \gamma) - \dot{x}(t; \bar{\gamma})| &
\leq |f(t, x(t; \gamma), \Lambda(t, x(s; \gamma), , \sigma), \theta) - f(t, x(t; \bar{\gamma}), \Lambda(t, x(s; \bar{\gamma}), , \bar{\sigma}), \bar{\theta})| \\
&\leq L_1 \left( |\dot{x}(t; \gamma) - \dot{x}(t; \bar{\gamma})| + L_2 N |x(s; \gamma) - x(s; \bar{\gamma})| c + |\sigma - \bar{\sigma}| r + |\theta - \bar{\theta}| r \right) \\
&\leq L^* |\gamma - \bar{\gamma}| r, \quad t \in [0, \alpha^\gamma],
\end{align*}
\]

where \( L^* \equiv L_1 (2 + L_2 N) L^* + L_1 (L_2 N + 1). \) Therefore (13) follows from (16), (17) and from

\[
|\dot{x}(t) - \dot{x}(t)| \leq |\gamma - \bar{\gamma}| r \quad \text{for almost every } t \in [-r, 0] \quad \text{with } \max\{L^*, L^{**}\}. \]

3. Differentiability wrt parameters. In this section we study differentiability of solutions of IVP (1)-(2) wrt the initial function, $\varphi$, the parameter $\sigma$ of the delay function $\tau$, and the parameter $\theta$ of the function $f$.

Let $\bar{\gamma} = (\bar{\varphi}, \bar{\sigma}, \bar{\theta}) \in \mathcal{M}$, and $x(\cdot; \bar{\gamma})$ be the corresponding solution of IVP (1)-(2) on $[0, \alpha]$. Fix $h = (h^x, h^\sigma, h^\theta) \in \Gamma$ and consider the variational equation

\begin{align}
(18) \quad z(t; \bar{\gamma}, h) &= D_2 f(t, x(t; \bar{\gamma}), \Lambda(t, x(t; \bar{\gamma}), \bar{\sigma}), \bar{\theta}) z(t; \bar{\gamma}, h) \\
&\quad + D_2 f(t, x(t; \bar{\gamma}), \Lambda(t, x(t; \bar{\gamma}), \bar{\sigma}), \bar{\theta}) \left(D_3 \Lambda(t, x(t; \bar{\gamma}), \bar{\sigma}) z(t; \bar{\gamma}, h) + D_4 f(t, x(t; \bar{\gamma}), \Lambda(t, x(t; \bar{\gamma}), \bar{\sigma}), \bar{\theta}) h^\sigma, \right) \\
&\quad + D_3 \Lambda(t, x(t; \bar{\gamma}), \bar{\sigma}) h^\sigma, \right) + D_4 f(t, x(t; \bar{\gamma}), \Lambda(t, x(t; \bar{\gamma}), \bar{\sigma}), \bar{\theta}) h^\theta, \right), \\
&\quad t \in [0, \alpha],
\end{align}

\begin{align}
(19) \quad z(t; \bar{\gamma}, h) = h^\theta(t), \quad t \in [-r, 0],
\end{align}

This is a linear state-independent delay equation for $z(\cdot; \bar{\gamma}, h)$, and the right-hand side of (18) depends continuously on $t$ and $z(\cdot; \bar{\gamma}, h)_t$ since $x(\cdot; \bar{\gamma})_t \in C^1$ by Theorem 1 (v). Therefore this IVP has a unique solution, $z(\cdot; \bar{\gamma}, h)$, which depends linearly on $h$.

First we study differentiability of the function $x(t; \gamma) = x(t; (\varphi, \sigma, \theta))$ wrt $\varphi$ and $\theta$ only. We denote this differentiation by $D_{(\varphi, \theta)}x$. Let

\begin{align}
(20) \quad G^{\sigma, \theta}(\delta, \bar{\theta}) \equiv \{ (\varphi, \bar{\sigma}, \bar{\theta}) \in W^{1,\infty} \times \Theta : (\varphi, \bar{\sigma}, \bar{\theta}) \in \mathcal{G}(\bar{\gamma}; \delta) \}.
\end{align}

**Theorem 2.** Assume (A1), (A2), and let $\bar{\gamma} \in \mathcal{M}$ be fixed. Let $\delta > 0$ and $\alpha > 0$ be defined by Theorem 1, and $x(t; \gamma)$ be the solution of IVP (1)-(2) on $[0, \alpha]$ for $\gamma \in \mathcal{G}(\bar{\gamma}; \delta)$, and $G^{\sigma, \theta}(\delta, \bar{\theta})$ be defined by (20). Then the function $x(t; (\cdot, \bar{\sigma}, \bar{\theta})): G^{\sigma, \theta}(\delta, \bar{\theta}) \to \mathbb{R}^n$ is differentiable at $(\bar{\varphi}, \bar{\theta})$ for $t \in [0, \alpha]$, and

\begin{align}
D_{(\sigma, \theta)}x(t; (\cdot, \bar{\sigma}, \bar{\theta}))(h^\sigma, h^\theta) = z(t; \bar{\gamma}, (h^\sigma, 0, h^\theta)),
\end{align}

where $z$ is the solution of IVP (18)-(19), and $(h^\sigma, h^\theta) \in W^{1,\infty} \times \Theta$.

**Proof.** Let $\bar{\gamma} \in \mathcal{M}$, $\delta > 0$, $\alpha$, and $G^{\sigma, \theta}(\delta, \bar{\theta})$ be as in the assumption of the theorem. We can and do assume that $\delta$ is such that $M_3 \equiv \mathcal{G}_3(\bar{\sigma}; \bar{\theta}) \subset \Omega_3$ and $M_5 \equiv \mathcal{G}_5(\bar{\sigma}; \bar{\theta}) \subset \Omega_5$. Let $h = (h^\sigma, h^\theta) \in \Gamma$ such that $|h|_{TV} < \delta$. (Here, for our future purposes, we do not assume yet that $h^\theta = 0$.) Note that $z(t; \bar{\gamma}, h)$ is well-defined since, by our assumptions, $x(\cdot; \bar{\gamma})_t \in C^1$. Integrating (1) and (18), and using the definition of $\omega^J$ and $\omega^\Lambda$ we get

\begin{align}
x(t; \bar{\gamma} + h) - x(t; \bar{\gamma}) &= \int_0^t \left(f(s, x(s; \bar{\gamma} + h), \Lambda(s, x(s; \bar{\gamma} + h), \bar{\sigma} + h^\sigma, \bar{\theta} + h^\theta) \\
&\quad - f(s, x(s; \bar{\gamma}), \Lambda(s, x(s; \bar{\gamma}), \bar{\sigma}), \bar{\theta}) - D_2 f(s, x(s; \bar{\gamma}), \Lambda(s, x(s; \bar{\gamma}), \bar{\sigma}), \bar{\theta}) z(s; \bar{\gamma}, h) \\
&\quad - D_3 f(s, x(s; \bar{\gamma}), \Lambda(s, x(s; \bar{\gamma}), \bar{\sigma}), \bar{\theta}) \left(D_3 \Lambda(s, x(s; \bar{\gamma}), \bar{\sigma}) z(s; \bar{\gamma}, h) \\
&\quad + D_4 f(s, x(s; \bar{\gamma}), \Lambda(s, x(s; \bar{\gamma}), \bar{\sigma}), \bar{\theta}) h^\sigma, \right) ds + D_3 \Lambda(s, x(s; \bar{\gamma}), \bar{\sigma} + h^\sigma, \bar{\theta} + h^\theta) h^\theta\right) ds
\end{align}

\begin{align}
&\quad = \int_0^t \left(\omega^J(s, x(s; \bar{\gamma}), \Lambda(s, x(s; \bar{\gamma}), \bar{\sigma}), \bar{\theta}; x(s; \bar{\gamma} + h), \Lambda(s, x(s; \bar{\gamma} + h), \bar{\sigma} + h^\sigma, \bar{\theta} + h^\theta) + h^\theta\right)
\end{align}
+ D_2 f(s, x(s; \bar{\gamma}), \Lambda(s, x(s; \bar{\gamma}), \bar{\sigma}), \bar{\delta}) \left( x(s; \bar{\gamma} + h) - x(s; \bar{\gamma}) - z(s; \bar{\gamma}, h) \right)
+ D_3 f(s, x(s; \bar{\gamma}), \Lambda(s, x(s; \bar{\gamma}), \bar{\sigma}), \bar{\delta}) \left( \omega^\Lambda(s, x(s; \bar{\gamma}), \bar{\sigma}, x(s; \bar{\gamma} + h)_s, \bar{\sigma} + h^n \right)
+ D_2 \Lambda(s, x(s; \bar{\gamma}), \bar{\sigma}) (x(s; \bar{\gamma} + h)_s - x(s; \bar{\gamma}) - z(s; \bar{\gamma}, h)_s) \right) ds. 

Let \( M_i \ (i = 1, 2, 4) \) be defined by Theorem 1. Let \( L_1 = L_1(\alpha, M_1, M_2, M_3) \) and \( L_2 = L_2(\alpha, M_4, M_5) \) be the constants from (A1) (ii) and (A2) (ii), respectively, and \( K = K(\alpha, M_4, M_5) \) be the constant from (7)-(8). Then (10) yields

\[
| x(t; \bar{\gamma} + h) - x(t; \bar{\gamma}) - z(t; \bar{\gamma}, h) | 
\leq \int_0^t \left( G^f(s; \bar{\gamma}, h) + L_1 \left| x(s; \bar{\gamma} + h) - x(s; \bar{\gamma}) - z(s; \bar{\gamma}, h) \right| + L_1 G^\Lambda(s; \bar{\gamma}, h) + L_1 K | x(s; \bar{\gamma} + h)_s - x(s; \bar{\gamma}) - z(s; \bar{\gamma}, h)_s | \right) ds, \quad t \in [0, \alpha].
\]

where \( G^f(s; \bar{\gamma}, h) \equiv | f(s, x(s; \bar{\gamma}), \Lambda(s, x(s; \bar{\gamma}), \bar{\sigma}), \bar{\delta}, x(s; \bar{\gamma} + h), \Lambda(s, x(s; \bar{\gamma} + h)_s, \bar{\sigma} + h^n), \bar{\delta} + h^n) | \) and \( G^\Lambda(s; \bar{\gamma}, h) \equiv | \omega^\Lambda(s, x(s; \bar{\gamma}), \bar{\sigma}, x(s; \bar{\gamma} + h)_s, \bar{\sigma} + h^n) | \). Introduce \( \eta(t; \bar{\gamma}, h) \equiv \sup_{s \geq s_0} \sup_{r \leq t} | x(s; \bar{\gamma} + h) - x(s; \bar{\gamma}) - z(s; \bar{\gamma}, h) | \). Inequality (21) implies

\[
| x(t; \bar{\gamma} + h) - x(t; \bar{\gamma}) - z(t; \bar{\gamma}, h) | 
\leq \int_0^t \left( G^f(s; \bar{\gamma}, h) + L_1 G^\Lambda(s; \bar{\gamma}, h) \right) ds + L_1 (1 + K) \int_0^t \eta(s; \bar{\gamma}, h) ds.
\]

Using that \( \eta(0; \bar{\gamma}, h) = 0 \), and the right-hand side of (22) is monotone in \( t \), we get from (22)

\[
\eta(t; \bar{\gamma}, h) \leq \int_0^t \left( G^f(s; \bar{\gamma}, h) + L_1 G^\Lambda(s; \bar{\gamma}, h) \right) ds + L_1 (1 + K) \int_0^t \eta(s; \bar{\gamma}, h) ds,
\]

which, by the Gronwall-Bellman inequality, implies

\[
\eta(t; \bar{\gamma}, h) \leq \int_0^t \left( G^f(s; \bar{\gamma}, h) + L_1 G^\Lambda(s; \bar{\gamma}, h) \right) ds e^{L_1 (1 + K) \alpha}, \quad t \in [0, \alpha].
\]

Applying (23) we get

\[
| x(t; \bar{\gamma} + h) - x(t; \bar{\gamma}) - z(t; \bar{\gamma}, h) | / | h | \rho 
\leq \eta(t; \bar{\gamma}, h) / | h | \rho 
\leq \int_0^t \left( G^f(s; \bar{\gamma}, h) / | h | \rho + L_1 G^\Lambda(s; \bar{\gamma}, h) / | h | \rho \right) ds e^{L_1 (1 + K) \alpha}, \quad t \in [-r, \alpha].
\]

Here we used that \( x(t; \bar{\gamma} + h) - x(t; \bar{\gamma}) - z(t; \bar{\gamma}, h) = 0 \) for \( t \in [-r, 0] \). We will show that \( \int_0^t G^f(s; \bar{\gamma}, h) / | h | \rho ds \to 0 \) and \( \int_0^t G^\Lambda(s; \bar{\gamma}, h) / | h | \rho ds \to 0 \) as \( | h | \rho \to 0 \).

Using (4) and (13), we get that there exists \( K^* = K^*(\alpha, M_4, M_5) \) such that

\[
| \Lambda(s, x(s; \bar{\gamma} + h)_s, \bar{\sigma} + h^n) - \Lambda(s, x(s; \bar{\gamma})_s, \bar{\sigma}) | \leq K^* | h | \rho, \quad | h | \rho < \delta, \quad s \in [0, \alpha].
\]
Using the obvious relation
\[
\frac{G^f(s; \gamma, h)}{|h|^r} = \frac{\omega''(s, x(s; \gamma), \lambda(s, x(s; \gamma), \sigma), \tilde{\sigma}; x(s; \gamma + h), \lambda(s, x(s; \gamma + h), \tilde{\sigma} + h^\sigma), \tilde{\sigma} + h^\sigma)}{|x(s; \gamma + h) - x(s; \gamma)|^r} = \frac{\omega''(s, x(s; \gamma), \lambda(s, x(s; \gamma), \sigma), \tilde{\sigma}; x(s; \gamma + h), \lambda(s, x(s; \gamma + h), \tilde{\sigma} + h^\sigma), \tilde{\sigma} + h^\sigma)}{|x(s; \gamma + h) - x(s; \gamma)|^r} \leq \frac{1}{2L1 (L + K^* + 1)} \frac{\omega''(s, x(s; \gamma), \lambda(s, x(s; \gamma), \sigma), \tilde{\sigma}; x(s; \gamma + h), \lambda(s, x(s; \gamma + h), \tilde{\sigma} + h^\sigma), \tilde{\sigma} + h^\sigma)}{|x(s; \gamma + h) - x(s; \gamma)|^r}.
\]

(11), (12), (13), (24) and (25) yield \( G^f(s; \gamma, h)/|h|^r \leq 2L1 (L + K^* + 1) \). On the other hand, (9) and (25) imply \( G^f(s; \gamma, h)/|h|^r \to 0 \) as \( |h|^r \to 0 \) for \( s \in [0, \alpha] \). Therefore \( \int_0^\alpha G^f(s; \gamma, h)/|h|^r \, ds \to 0 \) as \( |h|^r \to 0 \) by the Lebesgue’s Dominated Convergence Theorem.

Similarly, inequalities (8) and (13) imply \( G^\lambda(s; \gamma, h)/|h|^r \leq 2K (L + 1) \). To show that \( G^\lambda(s; \gamma, h)/|h|^r \to 0 \) we now assume that \( h^\sigma = 0 \). Lemma 1 implies \( G^\lambda(s; \gamma, h)/|h|^r \to 0 \) as \( |h|^r \to 0 \) for \( s \in [0, \alpha] \), since, by (13), \( |x(s; \gamma + h) - x(s; \gamma)|^r \to 0 \) as \( |h|^r \to 0 \). Therefore \( \int_0^\alpha G^\lambda(s; \gamma, h)/|h|^r \, ds \to 0 \) as \( |h|^r \to 0 \).

We conclude that \( |x(t; \gamma + h) - x(t; \gamma)|/|h|^r \to 0 \) as \( |h|^r \to 0 \), which proves the theorem.

The proof of the previous theorem implies immediately:

**Corollary 1.** Assume the conditions of Theorem 2. Then the function \( G^{\phi, \theta}(s; \gamma, \delta) \to \mathbb{C} \), \( (\varphi, \theta) \to x(\varphi; \gamma, \delta) \), is differentiable at \( (\frac{\varphi}{\theta}, \theta) \) for \( t \in [0, \alpha] \), and its derivative is given by \( D_t(x(\varphi; \gamma, \delta))(\phi^\sigma, h^\sigma) = \frac{\partial}{\partial \phi^\sigma} x(\varphi; \gamma, \delta), (\phi^\sigma, h^\sigma) \in \mathbb{W}^{1, \infty} \times \Theta \).

Next we study differentiability wrt \( \sigma \) as well. We will need the following definition.

**Definition 1.** Let \( X \) and \( Y \) be normal linear spaces, \( M \subset X \), and \( x_0 \in M \) be an accumulation point of \( M \). We say that \( f : \overline{M} \subset X \to Y \) is differentiable at the point \( x_0 \) with respect to the set \( M \) if there exists \( L \in \mathcal{L}(X, Y) \) such that
\[
\lim_{x \to x_0 \in M} \frac{|f(x) - f(x_0) - L(x - x_0)|}{|x - x_0|} = 0.
\]

We have the following result.

**Theorem 3.** Assume (A1), (A2), and let \( \gamma \in M \) be an accumulation point of \( M \). Let \( \delta > 0 \) and \( \alpha > 0 \) be defined by Theorem 1, and \( x(t; \gamma) \) be the solution of IVP (1)-(2) on \([0, \alpha]\) for \( \gamma \in G_T(\gamma; \delta) \). Then the function \( x(t; \gamma) : \{G_T(\gamma; \delta) \cap M\} \to \mathbb{R}^n \) is differentiable at \( \gamma \) wrt \( G_T(\gamma; \delta) \cap M \) for \( t \in [0, \alpha] \), and its derivative is \( D_t x(t; \gamma)^h = z(t; \gamma, h) \), where \( z \) is the solution of IVP (18)-(19), \( h \in \Gamma \) is such that \( \gamma + h \in M \).

**Proof.** We proceed as in the proof of Theorem 2. The only step needs a different argument here is the last one, to show that \( G^\lambda(s; \gamma, h)/|h|^r \to 0 \) as \( |h|^r \to 0 \). We have \( G^\lambda(s; \gamma, h) = |\lambda(s, x(s; \gamma + h), \tilde{\sigma} + h^\sigma) - \lambda(s, x(s; \gamma), \tilde{\sigma}) - D_t \lambda(s, x(s; \gamma), \tilde{\sigma})(x(s; \gamma + h) - x(s; \gamma), -x(s; \gamma), -D_t \lambda(s, x(s; \gamma), \tilde{\sigma})h^\sigma)|/|h|^r \).

Let \( h \) be such that \( \gamma + h \in M \). Then, using that \( \Lambda(t, \cdot, \cdot) \) is continuously differentiable on
\[ \Omega_1 \cap C^1 \times \Omega_2, \text{and } x(\cdot; \bar{\gamma} + h)_s \in C^1 \text{ for } s \in [0, \alpha], \text{ we get} \]

\[ G^h(s; \bar{\gamma}, h) \]
\[ \leq \sup_{0 < c < 1} \left\| D_2 \Lambda(s, (1 - \nu)x(\cdot; \bar{\gamma})_s + \nu x(\cdot; \bar{\gamma} + h)_s, \bar{\sigma} + \nu h^\delta) \right\|_{\mathcal{L}(W^{1,\infty}, \mathbb{R}^n)} - D_2 \Lambda(s, x(\cdot; \bar{\gamma})_s, \bar{\sigma}) \left\|_{\mathcal{L}(W^{1,\infty}, \mathbb{R}^n)} \right\|
\]
\[ + \sup_{0 < c < 1} \left\| D_2 \Lambda(s, (1 - \nu)x(\cdot; \bar{\gamma})_s + \nu x(\cdot; \bar{\gamma} + h)_s, \bar{\sigma} + \nu h^\delta) \right\|_{\mathcal{L}(W^{1,\infty}, \mathbb{R}^n)} - D_2 \Lambda(s, x(\cdot; \bar{\gamma})_s, \bar{\sigma}) \left\|_{\mathcal{L}(W^{1,\infty}, \mathbb{R}^n)} \right\|
\]

Therefore the continuity of \( D_2 \Lambda(s, \cdot; \cdot) \) and \( D_3 \Lambda(s, \cdot; \cdot) \) (see Lemma 1), and (13) imply \( G^h(s; \bar{\gamma}, h)/|h|_r \to 0 \) as \( |h|_r \to 0 \). \( \square \)

Next we show that, under the assumptions of the previous theorem, \( x(\cdot; \gamma)_t \) is differentiable wrt \( \gamma \) (in the sense of Definition 1) if we use \( W^{1,\infty} \) as the space-state of the solutions.

**Theorem 4.** Assume (A1), (A2), and let \( \bar{\gamma} \in \mathcal{M} \) be an accumulation point of \( \mathcal{M} \). Let \( \delta > 0 \) and \( \alpha > 0 \) be defined by Theorem 1, and \( z(\cdot; \gamma) \) be the solution of IVP (1)-(2) on \( [0, \alpha] \) for \( \gamma \in G_T(\bar{\gamma}; \bar{\theta}) \). Then the function \( \left( G_T(\bar{\gamma}; \bar{\delta}) \cap \mathcal{M} \right) \to W^{1,\infty}, \gamma \mapsto x(\cdot; \gamma)_t \) is differentiable at \( \bar{\gamma} \) wrt \( G_T(\bar{\gamma}; \bar{\delta}) \cap \mathcal{M} \) for \( t \in [0, \alpha] \), and \( D_x x(\cdot; \bar{\gamma})_t h = z(\cdot; \bar{\gamma}, h)_t \), where \( z \) is the solution of IVP (18)-(19), and \( h \in \Gamma \) is such that \( \bar{\gamma} + h \in \mathcal{M} \).

**Proof.** We use all the notations introduced in the proof of Theorem 2. It follows from the proofs of Theorems 2 and 3 that \( |x(\cdot; \bar{\gamma} + h)_t - x(\cdot; \bar{\gamma})_t|/|h|_r \to 0 \) as \( \bar{\gamma} + h \in \mathcal{M} \) and \( |h|_r \to 0 \). Similarly to (22) we get

\[ |\dot{x}(t; \bar{\gamma} + h)_t - \dot{x}(t; \bar{\gamma})_t - \dot{z}(t; \bar{\gamma}, h)| \]
\[ \leq G^h(t; \bar{\gamma}, h) + L_1 G^h(t; \bar{\gamma}, h) + L_1(1 + K)|\eta(t; \bar{\gamma}, h)|, \quad t \in [0, \alpha]. \]

Clearly, \( \dot{x}(t; \bar{\gamma} + h)_t - \dot{x}(t; \bar{\gamma})_t - \dot{z}(t; \bar{\gamma}, h) = 0 \) for \( t \in [-r, 0] \). Therefore, in view of (23), it suffices to show that \( G^h(t; \bar{\gamma}, h)/|h|_r \to 0 \) and \( G^h(t; \bar{\gamma}, h)/|h|_r \to 0 \) as \( \bar{\gamma} + h \in \mathcal{M} \) and \( |h|_r \to 0 \) uniformly in \( t \in [0, \alpha] \). Consider a sequence \( h^k = (h^{k,\gamma}, h^{k,\delta}, h^{k,\theta}) \in \Gamma \) such that \( \bar{\gamma} + h^k \in \mathcal{M} \) for \( k \in \mathbb{N} \) and \( |h^k|_r \to 0 \) as \( k \to \infty \). We have

\[ G^h(t; \bar{\gamma}, h^k) \]
\[ \leq \sup_{0 < c < 1} \left\| D_3 f(t, (1 - \nu)x(t; \bar{\gamma}) + \nu x(t; \bar{\gamma} + h^k), \left(1 - \nu\right)\Lambda(t, x(\cdot; \bar{\gamma})_t, \bar{\sigma}) + \nu \Lambda(t, x(\cdot; \bar{\gamma} + h^k)_t, \bar{\sigma} + h^k, \bar{\delta}), \bar{\theta} + \nu h^k, \bar{\delta} \right\|_{\mathcal{L}(W^{1,\infty}, \mathbb{R}^n)} - D_3 f(t, x(t; \bar{\gamma})_t, \Lambda(t, x(\cdot; \bar{\gamma})_t, \bar{\sigma}), \bar{\theta}) \left\|_{\mathcal{L}(W^{1,\infty}, \mathbb{R}^n)} \right\|
\]
\[ + \sup_{0 < c < 1} \left\| D_3 f(t, (1 - \nu)x(t; \bar{\gamma}) + \nu x(t; \bar{\gamma} + h^k), \left(1 - \nu\right)\Lambda(t, x(\cdot; \bar{\gamma})_t, \bar{\sigma}) + \nu \Lambda(t, x(\cdot; \bar{\gamma} + h^k)_t, \bar{\sigma} + h^k, \bar{\delta}), \bar{\theta} + \nu h^k, \bar{\delta} \right\|_{\mathcal{L}(W^{1,\infty}, \mathbb{R}^n)} - D_3 f(t, x(t; \bar{\gamma})_t, \Lambda(t, x(\cdot; \bar{\gamma})_t, \bar{\sigma}), \bar{\theta}) \left\|_{\mathcal{L}(W^{1,\infty}, \mathbb{R}^n)} \right\|
\]
\[ - \Lambda(t, x(\cdot; \bar{\gamma} + h^k)_t, \bar{\sigma} + h^k, \bar{\delta}) - \Lambda(t, x(\cdot; \bar{\gamma})_t, \bar{\sigma}) \right\|_{\mathcal{L}(W^{1,\infty}, \mathbb{R}^n)} \]
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\[ + \sup_{0 < t < 1} \left\| \frac{D_t f(t, (1 - \nu)x(t; \bar{\gamma}) + \nu x(t; \bar{\gamma} + h^k)}{\nu} \right\| C_{\Theta} \cdot |h|^6 \]

\[ - D_t f(t, x(t; \bar{\gamma}), \Lambda(t, x(t; \bar{\gamma} + h^k), \bar{\sigma} + h^k, \bar{\sigma}) + \nu h^k, \bar{\sigma}) \right\| C_{\Theta} \cdot |h|^6 \]

Let \( M_2^k \equiv \{ h, h^k, \bar{\sigma} : k \in \mathbb{N}, \nu \in [0, 1] \} \), and \( A \equiv [0, \alpha] \times M_1 \times M_2 \times M_3 \). The set \( A \) is a compact subset of \( \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \Theta \), since \( M_1 \) and \( M_2 \) are compact subsets of \( \mathbb{R}^n \), and it is easy to see that \( M_3 \) is a compact subset of \( \Theta \). By (A1) (iii) \( D_2 f, D_3 f \) and \( D_4 f \) are continuous, therefore uniformly continuous on \( A \). Therefore (28), together with (13) and (24), yields \( G^2(t; \bar{\gamma}, h^k)/|h|^6 \right| \rightarrow 0 \) as \( k \rightarrow \infty \) uniformly in \( t \in [0, \alpha] \).

Similarly, define \( M_3^k \equiv \{ \bar{\sigma} + h^k, \bar{\sigma} : k \in \mathbb{N}, \nu \in [0, 1] \} \), and \( B \equiv [0, \alpha] \times M_4 \times M_3 \). Then \( B \) is a compact subset of \( \mathbb{R} \times \mathbb{R} \times \Sigma \), therefore (13) and (25) imply that \( G^3(t; \bar{\gamma}, h^k)/|h|^6 \right| \rightarrow 0 \) as \( k \rightarrow \infty \) uniformly in \( t \in [0, \alpha] \). This concludes the proof of the theorem.

The next two examples show cases when the differentiability property of the solution wrt some parameter guaranteed by Theorem 4 equals to the usual Frechét-differentiability of the solution wrt the parameter.

**Example 1.** Suppose \( f \) satisfies (A1) and has the form

\[ f(t, x, y, \theta) = f^1(t, x, y) + f^2(t, x, y, \theta), \]

where \( f^2(0, x, y, \theta) = 0 \) for all \( x \in \Omega, y \in \Omega \) and \( \theta \in \Theta \). Then if \( \bar{\gamma} = (\bar{\gamma}, \bar{\sigma}, \bar{\theta}) \in \Pi \) satisfies \( \bar{\gamma} \in C^1 \) and \( \bar{\gamma}(0) = f^1(0, \bar{\gamma}(0), \Lambda(0, \bar{\gamma}, \bar{\sigma})) \), then the solution of IVP (1)-(2), \( x(\cdot; \bar{\theta}) \), is differentiable wrt \( \theta \) on \( \Omega_3 \) for \( t \in [0, \alpha] \) in the usual Frechét-sense as a function \( \left( \Omega_3 \subset \Theta \right) \rightarrow W^{1, \infty}, \theta \mapsto x(\cdot; \bar{\theta}) \).

**Example 2.** Suppose the function \( \tau \) satisfies (A2) and \( \tau(t, \psi, \sigma) = \tau^1(t, \psi) + \tau^2(t, \psi, \sigma) \), where \( \tau^2(0, \psi, \sigma) = 0 \) for all \( \psi \in \Omega \) and \( \sigma \in \Sigma \). Then if \( \bar{\gamma} = (\bar{\gamma}, \bar{\sigma}, \bar{\theta}) \in \Pi \) satisfies \( \bar{\gamma} \in C^1 \) and \( \bar{\gamma}(0) = f(0, \bar{\gamma}(0), \bar{\gamma}(0), \bar{\theta}) \), then the solution, \( x(\cdot; \sigma) \), is differentiable wrt \( \sigma \) on \( \Omega_3 \) for \( t \in [0, \alpha] \) (in Frechét-sense) as a function \( \left( \Omega_3 \subset \Sigma \right) \rightarrow W^{1, \infty}, \sigma \mapsto x(\cdot; \sigma) \).

Finally, we consider the state-independent version of IVP (1)-(2), i.e., we assume that \( \tau(t, \psi, \sigma) \) is independent of \( \psi \). Let \( \bar{\psi} \in C^1 \). First we note that (5) yields in this case that \( D_3 \lambda(t, \bar{\psi}, \bar{\sigma}) h = h(\tau(t, \bar{\psi}, \bar{\sigma})) \), therefore a simple calculation and (6) imply

\[ |\omega^\lambda(t, \bar{\psi}, \bar{\sigma}; \psi, \sigma)| = |\bar{\psi}(\tau(t, \bar{\psi}, \sigma)) - \bar{\psi}(\tau(t, \bar{\psi}, \bar{\sigma}))| \]

\[ + |\psi(\tau(\psi, \psi)) - \psi(\tau(\bar{\psi}, \bar{\sigma}))| \]

\[ \leq |\bar{\psi}(\tau(t, \psi, \sigma) - \tau(t, \bar{\psi}, \bar{\sigma}))| + |\psi(\psi(\bar{\psi}, \bar{\sigma})) - \psi(\tau(t, \bar{\psi}, \bar{\sigma}))| \]

Therefore (A2) (iii), the Chain-rule and the Mean Value Theorem yield

\[ \frac{|\omega^\lambda(t, \bar{\psi}, \bar{\sigma}; \psi, \sigma)|}{|\psi - \bar{\psi}|_{W^{1, \infty}} + |\sigma - \bar{\sigma}|_{\Sigma}} \rightarrow 0 \quad \text{as} \quad |\psi - \bar{\psi}|_{W^{1, \infty}} + |\sigma - \bar{\sigma}|_{\Sigma} \rightarrow 0. \]
Consequently, $G^3(t;\bar{\tau},h)/|h|_{\Gamma} \to 0$ as $|h|_{\Gamma} \to 0$. Using this relation, it follows easily from the proof of Theorem 4:

**Corollary 2.** Assume (A1), (A2), and let $\bar{\gamma} \in \mathcal{M}$ be fixed. Assume moreover that $\tau(t,\psi,\sigma)$ is independent of $\psi$. Let $\delta > 0$ and $\alpha > 0$ be defined by Theorem 1, and $x(t;\gamma)$ be the solution of IVP (11)-(12) on $[0,\alpha]$ for $\gamma \in G_{\tau}(\bar{\gamma},\delta)$. Then the function $\bar{G}_{\tau}(\bar{\gamma};\delta) \subset \Gamma \to W^{1,\infty}, \gamma \mapsto x(t;\gamma)$ is differentiable at $\bar{\gamma}$ for $t \in [0,\alpha]$, and $D_{t}x(t;\gamma)h = z(t;\bar{\gamma},h)$, where $z$ is the solution of IVP (18)-(19), and $h \in \Gamma$.

REFERENCES


