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ON DIFFERENTIABILITY OF SOLUTIONS WITH RESPECT TO PARAMETERS IN A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study differentiability of solutions with respect to parameters in state-dependent delay equations. In particular, we give sufficient conditions for differentiability of solutions in the $W^{1,\infty}$ norm.

1. Introduction.

We consider the state-dependent delay system

$$\dot{x}(t) = f\Big(t, x(t), x(t-\tau(t, x_t, \sigma)), \theta\Big), \qquad t \in [0, T], \tag{1}$$

with initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{2}$$

Here $\theta \in \Theta$ and $\sigma \in \Sigma$ represent parameters in the function f and in the delay function, τ, where Θ and Σ are normed linear spaces with norms $|\cdot|_\Theta$ and $|\cdot|_\Sigma$, respectively. The notation x_t denotes the solution segment function, i.e., $x_t : [-r, 0] \to \mathbb{R}^n$, $x_t(s) \equiv x(t+s)$. (See Section 2 below for the detailed assumptions on the initial value problem (IVP) (1)-(2).)

In this paper we study differentiability of solutions of IVP (1)-(2) with respect to (wrt) the parameters φ, σ and θ. Differentiability wrt parameters in delay equations has been investigated, e.g., in [1], [5] and [6]. It has also been studied in state-dependent delay equations in [8], where sufficient conditions were given guaranteeing differentiability of the parameter map $\Gamma \to W^{1,p}$, $\gamma \mapsto x(\cdot;\gamma)_t$ (where $\gamma \in \Gamma$ is some parameter of the equation, and $1 \le p < \infty$). In establishing this result a version of the Uniform Contraction Principle for quasi-Banach spaces was used. In many applications (e.g., in parameter identification problems, see, e.g., [2] and [3]) this sort of differentiability (i.e., differentiability in a $W^{1,p}$ norm) is too weak. In this paper we establish sufficient conditions implying "pointwise" differentiability of the parameter map, i.e., differentiability of $\Gamma \to \mathbb{R}^n$, $\gamma \mapsto x(t;\gamma)$, and the stronger property, differentiability of the map $\Gamma \to W^{1,\infty}$, $\gamma \mapsto x(\cdot;\gamma)_t$.

Our main results are contained in Section 3. In Section 2 we list our assumptions on IVP (1)-(2), introduce our notations, and give some necessary preliminary results.

2. Notations, assumptions and preliminaries.

Throughout this paper a norm on $\mathbb{R}^n$ and the corresponding matrix norm on $\mathbb{R}^{n\times n}$ are denoted by $|\cdot|$ and $\|\cdot\|$, respectively.

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The notation $f\,:\,\Big(A\subset X\Big)\to Y$ will be used to denote that the function maps the subset A of the normed linear space X to Y. This notation emphasizes that the topology on A is defined by the norm of X.

We denote the open ball around a point x_0 with radius R in a normed linear space $(X,|\cdot|_X)$ by $\mathcal{G}_X(x_0;\,R)$, i.e., $\mathcal{G}_X(x_0;\,R)\equiv\{x\in X\,:\,|x-x_0|_X<R\}$, and the corresponding closed ball by $\overline{\mathcal{G}}_X(x_0;\,R)$. Similarly, a neighborhood of a set $M\subset X$ with radius R is denoted by $\mathcal{G}_X(M;\,R)$, i.e., $\mathcal{G}_X(M;\,R)\equiv\{x\in X\,:\,\text{there exists } y\in M \text{ such that } |x-y|_X<R\}$. The closure of this neighborhood is denoted by $\overline{\mathcal{G}}_X(M;\,R)$.

The space of continuous functions from $[-r,0]$ to $\mathbb{R}^n$ and the usual supremum norm on it are denoted by C and $|\cdot|_C$, respectively. The space of absolutely continuous functions from $[-r,0]$ to $\mathbb{R}^n$ with essentially bounded derivatives is denoted by $W^{1,\infty}$. The corresponding norm on $W^{1,\infty}$ is $|\psi|_{W^{1,\infty}}\equiv\max\{|\psi|_C,\operatorname{ess\,sup}\{|\dot\psi(s)|\,:\,s\in[-r,0]\}\}$.

The partial derivatives of a function $g(t,x_2,\ldots,x_n)$ wrt its second, third, etc. arguments are denoted by D_2g, D_3g, etc, and the derivative wrt t is denoted by $\dot g$. Note that all derivatives we use in this paper are Frechét-derivatives.

Next we consider a set of technical conditions, guaranteeing well-posedness and differentiability of solutions wrt parameters, for the state-dependent delay differential equation (1) with initial condition (2).

Let $\Omega_1\subset\mathbb{R}^n$, $\Omega_2\subset\mathbb{R}^n$, $\Omega_3\subset\Theta$, $\Omega_4\subset C$, and $\Omega_5\subset\Sigma$ be open subsets of the respective spaces. $T>0$ is finite or $T=\infty$, in which case $[0,T]$ denotes the interval $[0,\infty)$.

(A1) (i) $f\,:\,[0,T]\times\Omega_1\times\Omega_2\times\Omega_3\to\mathbb{R}^n$ is continuous,

(ii) $f(t,v,w,\theta)$ is locally Lipschitz-continuous in v, w and θ in the following sense: for every $\alpha>0$, $M_1\subset\Omega_1$, $M_2\subset\Omega_2$, $M_3\subset\Omega_3$, where M_1 and M_2 are compact subsets of $\mathbb{R}^n$ and M_3 is a closed, bounded subset of Θ, there exists a constant $L_1=L_1(\alpha,M_1,M_2,M_3)$ such that

$$|f(t,v,w,\theta)-f(t,\bar v,\bar w,\bar\theta)|\le L_1\Big(|v-\bar v|+|w-\bar w|+|\theta-\bar\theta|_\Theta\Big),$$

for $t\in[0,\alpha]$, $v,\bar v\in M_1$, $w,\bar w\in M_2$, and $\theta,\bar\theta\in M_3$,

(iii) $f\,:\,\Big([0,T]\times\Omega_1\times\Omega_2\times\Omega_3\subset\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\times\Theta\Big)\to\mathbb{R}^n$ is continuously differentiable wrt its second, third and fourth arguments,

(A2) (i) $\tau\,:\,[0,T]\times\Omega_4\times\Omega_5\to[0,\infty)$ is continuous, and

$$t-\tau(t,\psi,\sigma)\ge -r,\qquad \text{for } t\in[0,T],\ \psi\in\Omega_4,\ \text{and } \sigma\in\Omega_5,$$

(ii) $\tau(t,\psi,\sigma)$ is locally Lipschitz-continuous in ψ and σ in the following sense: for every $\alpha>0$, $M_4\subset\Omega_4$ and $M_5\subset\Omega_5$, where M_4 is a compact subset of C, and M_5 is a closed, bounded subset of Σ, there exists a constant $L_2=L_2(\alpha,M_4,M_5)$ such that

$$|\tau(t,\psi,\sigma)-\tau(t,\bar\psi,\bar\sigma)|\le L_2\Big(|\psi-\bar\psi|_C+|\sigma-\bar\sigma|_\Sigma\Big)$$

for $t\in[0,\alpha]$, $\psi,\bar\psi\in M_4$, and $\sigma,\bar\sigma\in M_5$,

(iii) $\tau : \Big([0,T]\times\Omega_4\times\Omega_5 \subset [0,\alpha]\times C\times\Sigma\Big) \to \mathbb{R}$ is continuously differentiable wrt its second and third arguments.

Note that (A1) (i), (ii) and (A2) (i), (ii) together with $\varphi \in W^{1,\infty}$ are standard assumptions in state-dependent delay equations guaranteeing the existence and uniqueness of the solution (see, e.g., [4] or [8]). If the parameter spaces Θ and Σ are finite dimensional, then (A1) (ii) and (A2) (ii) follow from (A1) (iii) and (A2) (iii), respectively. We refer to [8] for further comments on the particular definition of local Lipschitz-continuity we use in (A1) (ii) and (A2) (ii).

We will use the following function to simplify the notation:

$$\Lambda : \Big([0,T]\times\Omega_4\times\Omega_5 \subset \mathbb{R}\times W^{1,\infty}\times\Sigma\Big) \to \mathbb{R}^n, \quad \Lambda(t,\psi,\sigma) \equiv \psi(-\tau(t,\psi,\sigma)). \tag{3}$$

With this notation we can rewrite (1) simply as:

$$\dot{x}(t) = f(t, x(t), \Lambda(t, x_t, \sigma), \theta), \qquad t\in[0,T].$$

It follows from the definition of Λ, (A2) (ii) and the Mean Value Theorem that

$$\begin{aligned} &|\Lambda(t,\psi,\sigma)-\Lambda(t,\bar\psi,\bar\sigma)| \\ &\quad\le |\bar\psi(-\tau(t,\psi,\sigma))-\bar\psi(-\tau(t,\bar\psi,\bar\sigma))| + |\psi(-\tau(t,\psi,\sigma))-\bar\psi(-\tau(t,\psi,\sigma))| \\ &\quad\le L_2|\bar\psi|_{W^{1,\infty}}(|\psi-\bar\psi|_C+|\sigma-\bar\sigma|_\Sigma)+|\psi-\bar\psi|_C \end{aligned} \tag{4}$$

for $t\in[0,\alpha]$, $\psi,\bar\psi\in M_4$, $\bar\psi\in W^{1,\infty}$, and $\sigma,\bar\sigma\in M_5$.

Lemma 1. *Assume (A2), and let Λ be defined by (3). Then $D_2\Lambda(t,\psi,\sigma)$ and $D_3\Lambda(t,\psi,\sigma)$ exist for $t\in[0,T]$, $\psi\in\Omega_4\cap C^1$, $\sigma\in\Omega_5$, and*

$$D_2\Lambda(t,\psi,\sigma)h = -\dot\psi(-\tau(t,\psi,\sigma))D_2\tau(t,\psi,\sigma)h + h(-\tau(t,\psi,\sigma)), \ \ h\in W^{1,\infty}, \tag{5}$$

$$D_3\Lambda(t,\psi,\sigma) = -\dot\psi(-\tau(t,\psi,\sigma))D_3\tau(t,\psi,\sigma). \tag{6}$$

Moreover, $D_2\Lambda(t,\cdot,\cdot)$ and $D_3\Lambda(t,\cdot,\cdot)$ are continuous on $(\Omega_4\cap C^1)\times\Omega_5$ for $t\in[0,T]$.

Proof. Let $\psi\in\Omega_4\cap C^1$, and introduce $\omega^\psi(\bar s;s)\equiv\psi(s)-\psi(\bar s)-\dot\psi(\bar s)(s-\bar s)$ for $\bar s, s\in[-r,0]$, and $\omega^\tau(t,\psi,\sigma;\psi+h)\equiv\tau(t,\psi+h,\sigma)-\tau(t,\psi,\sigma)-D_2\tau(t,\psi,\sigma)h$ for $t\in[0,T]$, $\psi,\psi+h\in\Omega_4$, and $\sigma\in\Omega_5$. Let $t\in[0,T]$, $\psi+h\in\Omega_4$, and $\sigma\in\Omega_5$, and consider

$$\begin{aligned} &\Lambda(t,\psi+h,\sigma)-\Lambda(t,\psi,\sigma) \\ &\quad= \psi(-\tau(t,\psi+h,\sigma))-\psi(-\tau(t,\psi,\sigma))+h(-\tau(t,\psi+h,\sigma)) \\ &\quad= -\dot\psi(-\tau(t,\psi,\sigma))(\tau(t,\psi+h,\sigma)-\tau(t,\psi,\sigma))+h(-\tau(t,\psi,\sigma)) \\ &\qquad +\omega^\psi(-\tau(t,\psi,\sigma);-\tau(t,\psi+h,\sigma))+h(-\tau(t,\psi+h,\sigma))-h(-\tau(t,\psi,\sigma)) \\ &\quad= -\dot\psi(-\tau(t,\psi,\sigma))D_2\tau(t,\psi,\sigma)h+h(-\tau(t,\psi,\sigma)) \\ &\qquad -\dot\psi(-\tau(t,\psi,\sigma))\omega^\tau(t,\psi,\sigma;\psi+h) \\ &\qquad +\omega^\psi(-\tau(t,\psi,\sigma);-\tau(t,\psi+h,\sigma))+h(-\tau(t,\psi+h,\sigma))-h(-\tau(t,\psi,\sigma)). \end{aligned}$$

Relation (5) follows from the last equation, using the continuity of τ, the inequality

$$|h(-\tau(t,\psi+h,\sigma))-h(-\tau(t,\psi,\sigma))|\le|h|_{W^{1,\infty}}|\tau(t,\psi+h,\sigma)-\tau(t,\psi,\sigma)|$$

guaranteed by the Mean Value Theorem, $|\omega^{\psi}(\bar{s};s)|/|s-\bar{s}| \to 0$ as $s \to \bar{s}$, and $|\omega^{\tau}(t,\psi,\sigma;\psi+h)|/|h|_{W^{1,\infty}} \to 0$ as $|h|_{W^{1,\infty}} \to 0$. Note that the last relation follows from $|\omega^{\tau}(t,\psi,\sigma;\psi+h)|/|h|_C \to 0$ as $|h|_C \to 0$. Relation (6) is an immediate consequence of the Chain-rule. The continuity of $D_2\Lambda(t,\cdot,\cdot)$ and $D_3\Lambda(t,\cdot,\cdot)$ follows readily from (5) and (6) and from the assumed continuity of τ, $D_2\tau$ and $D_3\tau$. □

We introduce the function

$$\omega^{\Lambda}(t,\bar{\psi},\bar{\sigma};\psi,\sigma) \equiv \Lambda(t,\psi,\sigma) - \Lambda(t,\bar{\psi},\bar{\sigma}) - D_2\Lambda(t,\bar{\psi},\bar{\sigma})(\psi-\bar{\psi}) - D_3\Lambda(t,\bar{\psi},\bar{\sigma})(\sigma-\bar{\sigma})$$

for $t\in[0,T]$, $\bar{\psi},\psi\in\Omega_4$, $\bar{\psi}\in C^1$, and $\bar{\sigma},\sigma\in\Omega_5$.

Let $\alpha>0$, $M_4\subset\Omega_4$ be a compact subset of C, $M_5\subset\Omega_5$ be a closed and bounded subset of Σ. It is easy to prove, using the definition of ω^{Λ}, (A2) (ii), (iii), (4), (5), and (6), that there exists a constant $K=K(\alpha,M_4,M_5)$ such that

$$\|D_2\Lambda(t,\bar{\psi},\bar{\sigma})\|_{\mathcal{L}(W^{1,\infty},\mathbb{R}^n)} \le K, \qquad \|D_3\Lambda(t,\bar{\psi},\bar{\sigma})\|_{\mathcal{L}(\Sigma,\mathbb{R}^n)} \le K, \tag{7}$$

and

$$|\omega^{\Lambda}(t,\bar{\psi},\bar{\sigma};\psi,\sigma)| \le 2K(|\psi-\bar{\psi}|_C + |\sigma-\bar{\sigma}|_{\Sigma}) \tag{8}$$

for $t\in[0,\alpha]$, $\psi,\bar{\psi}\in M_4$, $\bar{\psi}\in C^1$, and $\sigma,\bar{\sigma}\in M_5$.

Similarly to ω^{Λ}, we define

$$\begin{aligned}\omega^{f}(t,\bar{x},\bar{y},\bar{\theta};x,y,\theta) \;\equiv\; & f(t,x,y,\theta) - f(t,\bar{x},\bar{y},\bar{\theta}) - D_2 f(t,\bar{x},\bar{y},\bar{\theta})(x-\bar{x}) \\ & - D_3 f(t,\bar{x},\bar{y},\bar{\theta})(y-\bar{y}) - D_4 f(t,\bar{x},\bar{y},\bar{\theta})(\theta-\bar{\theta})\end{aligned}$$

for $t\in[0,T]$, $\bar{x},x\in\Omega_1$, $\bar{y},y\in\Omega_2$, and $\bar{\theta},\theta\in\Omega_3$. Assumption (A1) (iii) implies, that

$$\frac{|\omega^{f}(t,\bar{x},\bar{y},\bar{\theta};x,y,\theta)|}{|x-\bar{x}|+|y-\bar{y}|+|\theta-\bar{\theta}|_{\Theta}} \to 0, \qquad \text{as } |x-\bar{x}|+|y-\bar{y}|+|\theta-\bar{\theta}|_{\Theta} \to 0. \tag{9}$$

Let $\alpha>0$ be fixed, $M_i\subset\Omega_i$ $(i=1,2,3)$ be such that M_1 and M_2 be compact subsets of $\mathbb{R}^n$ and M_3 be a closed and bounded subset of Θ, and let $L_1=L_1(\alpha,M_1,M_2,M_3)$ be the constant from (A1) (ii). Then assumptions (A1) (ii) and (iii) yield that

$$\|D_2 f(t,\bar{x},\bar{y},\bar{\theta})\| \le L_1,\ \|D_3 f(t,\bar{x},\bar{y},\bar{\theta})\| \le L_1,\ \|D_4 f(t,\bar{x},\bar{y},\bar{\theta})\|_{\mathcal{L}(\Theta,\mathbb{R}^n)} \le L_1 \tag{10}$$

and

$$|\omega^{f}(t,\bar{x},\bar{y},\bar{\theta};x,y,\theta)| \le 2L_1(|x-\bar{x}|+|y-\bar{y}|+|\theta-\bar{\theta}|_{\Theta}) \tag{11}$$

for $t\in[0,\alpha]$, $x,\bar{x}\in M_1$, $y,\bar{y}\in M_2$, and $\theta,\bar{\theta}\in M_3$.

We define the parameter space $\Gamma=W^{1,\infty}\times\Sigma\times\Theta$, and use the notation $\gamma=(\varphi,\sigma,\theta)$ (or $\gamma=(\gamma^{\varphi},\gamma^{\sigma},\gamma^{\theta})$) for the components of $\gamma\in\Gamma$, and $|\gamma|_{\Gamma}\equiv|\varphi|_{W^{1,\infty}}+|\sigma|_{\Sigma}+|\theta|_{\Theta}$ for the norm on

Γ. The solution of IVP (1)-(2) corresponding to a parameter γ and its segment function at t are denoted by $x(t;\gamma)$ and $x(\cdot;\gamma)_t$, respectively.

Introduce

$$\Pi \equiv \Big\{\gamma = (\varphi,\sigma,\theta) \in \Omega_4\times\Omega_5\times\Omega_3 : \quad \varphi\in W^{1,\infty}, \quad \varphi(0)\in\Omega_1, \quad \Lambda(0,\varphi,\sigma)\in\Omega_2\Big\}$$

and

$$\mathcal{M} \equiv \Big\{\gamma = (\varphi,\sigma,\theta)\in\Pi : \quad \varphi\in C^1, \quad \dot\varphi(0-) = f(0,\varphi(0),\Lambda(0,\varphi,\sigma),\theta)\Big\}.$$

THEOREM 1. *Assume (A1) (i), (ii), (A2) (i), (ii), and let $\bar\gamma\in\Pi$. Then there exist $\delta>0$ and $0<\alpha\le T$ such that*

(i) $\mathcal{G}_\Gamma(\bar\gamma;\,\delta)\subset\Pi$,

(ii) IVP (1)-(2) has a unique solution, $x(t;\gamma)$, on $[0,\alpha]$ for all $\gamma\in\mathcal{G}_\Gamma(\bar\gamma;\,\delta)$,

(iii) there exist $M_1\subset\Omega_1$, $M_2\subset\Omega_2$ and $M_4\subset\Omega_4$ compact subsets of $\mathbb{R}^n$ and C, respectively, such that

$$x(t;\gamma)\in M_1, \quad \Lambda(t,x(\cdot;\gamma)_t,\gamma^\sigma)\in M_2, \quad \textit{and} \quad x(\cdot;\gamma)_t\in M_4, \tag{12}$$

for $t\in[0,\alpha]$, $\gamma\in\mathcal{G}_\Gamma(\bar\gamma;\,\delta)$,

(iv) $x(\cdot;\gamma)_t\in W^{1,\infty}$ for $t\in[0,\alpha]$, $\gamma\in\mathcal{G}_\Gamma(\bar\gamma;\,\delta)$, and there exists $L=L(\alpha,\delta)$, such that

$$|x(\cdot;\gamma)_t - x(\cdot;\bar\gamma)_t|_{W^{1,\infty}} \le L|\gamma-\bar\gamma|_\Gamma \qquad \textit{for } t\in[0,\alpha],\ \gamma\in\mathcal{G}_\Gamma(\bar\gamma;\,\delta), \tag{13}$$

(v) the function $x(\cdot;\gamma) : [-r,\alpha]\to\mathbb{R}^n$ is continuously differentiable for $\gamma\in\mathcal{M}\cap\mathcal{G}_\Gamma(\bar\gamma;\,\delta)$.

Proof. Part (i) and (v) are obvious (see also [7]). For the proof of (ii) we refer to [8], [7] or [4]. Part (iii) and (iv) will be essential in our proofs in the next section, therefore we prove them here. Let $\delta^1>0$ and $\alpha>0$ be such that they satisfy (i) and (ii). We will show that $0<\delta\le\delta^1$ can be selected so that (iii) and (iv) are also satisfied.

Let $\bar\gamma=(\bar\varphi,\bar\sigma,\bar\theta)\in\Pi$, and define $M_1^*\equiv\{x(t;\bar\gamma) : t\in[0,\alpha]\}$, $M_2^*\equiv\{\Lambda(t,x(\cdot;\bar\gamma)_t,\bar\sigma),: t\in[0,\alpha]\}$, and $M_4^*\equiv\{x(\cdot;\bar\gamma)_t : t\in[0,\alpha]\}$. It follows from part (ii) of the theorem that $M_i^*\subset\Omega_i$ $(i=1,2,4)$. Moreover, M_1^* and M_2^* are compact subsets of $\mathbb{R}^n$ since $t\mapsto x(t;\bar\gamma)$ and $t\mapsto\Lambda(t,x(\cdot;\bar\gamma)_t,\bar\sigma)$ are continuous functions on $[0,\alpha]$. M_4^* is also compact in C since $t\mapsto x(\cdot;\bar\gamma)_t$ is continuous on $[0,\alpha]$. Therefore there exist $\varepsilon^i>0$ $(i=1,2,4)$ such that $M_1\equiv\overline{\mathcal{G}}_{\mathbb{R}^n}\big(M_1^*;\,\varepsilon^1\big)\subset\Omega_1$, $M_2\equiv\overline{\mathcal{G}}_{\mathbb{R}^n}\big(M_2^*;\,\varepsilon^2\big)\subset\Omega_2$, and $\overline{\mathcal{G}}_C\big(M_4^*;\,\varepsilon^4\big)\subset\Omega_4$ since Ω_i $(i=1,2,4)$ are open sets in $\mathbb{R}^n$ and C, respectively. Let $M_4\equiv\overline{\mathcal{G}}_{W^{1,\infty}}\big(M_4^*;\,\varepsilon^4\big)$. Clearly, M_1 and M_2 are compact subsets of $\mathbb{R}^n$. We have $M_4\subset\Omega_4$, and it is compact in C by Arsela-Ascoli's Theorem, since it is a bounded subset of $W^{1,\infty}$.

Let $\delta^2\equiv\min\{\delta^1,\varepsilon^1,\varepsilon^2/(L_2|\bar\varphi|_{W^{1,\infty}}+1),\varepsilon^4\}$. Let $\gamma=(\varphi,\sigma,\theta)\in\mathcal{G}_\Gamma\big(\bar\gamma;\,\delta^2\big)$. We have from (4) and the definition of $|\cdot|_\Gamma$ that $|\varphi(0)-\bar\varphi(0)|<\varepsilon^1$, $|\Lambda(0,\varphi,\sigma)-\Lambda(0,\bar\varphi,\bar\sigma)|\le L_2|\bar\varphi|_{W^{1,\infty}}(|\varphi-\bar\varphi|_C+|\sigma-\bar\sigma|_\Sigma)+|\varphi-\bar\varphi|_C<\varepsilon^2$, and $|\varphi-\bar\varphi|_C<\varepsilon^4$. Therefore there exists $0<\alpha^\gamma\le\alpha$ such that

$$|x(t;\gamma)-x(t;\bar\gamma)|<\varepsilon^1, \qquad |\Lambda(t,x(\cdot;\gamma)_t,\sigma)-\Lambda(t,x(\cdot;\bar\gamma)_t,\bar\sigma)|<\varepsilon^2, \tag{14}$$

and

$$|x(\cdot;\gamma)_t - x(\cdot;\bar\gamma)_t|_C < \varepsilon^4 \tag{15}$$

for $t \in [0, \alpha^\gamma]$.

Let $L_1 = L_1(\alpha, M_1, M_2, M_3)$ and $L_2 = L_2(\alpha, M_4, M_5)$ be the constants from (A1) (ii) and (A2) (ii), respectively. We have for $t \in [0, \alpha^\gamma]$:

$$\begin{aligned} &|x(t;\gamma) - x(t;\bar\gamma)| \\ &\quad \le |\varphi(0) - \bar\varphi(0)| + \int_0^t \Big| f(s, x(s;\gamma), \Lambda(s, x(\cdot;\gamma)_s, \sigma), \theta) \\ &\qquad - f(s, x(s;\bar\gamma), \Lambda(s, x(\cdot;\bar\gamma)_s, \bar\sigma), \bar\theta) \Big|\, ds \\ &\quad \le |\gamma - \bar\gamma|_\Gamma + L_1 \int_0^t \Big(|x(s;\gamma) - x(s;\bar\gamma)| + |\Lambda(s, x(\cdot;\gamma)_s, \sigma) - \Lambda(s, x(\cdot;\bar\gamma)_s, \bar\sigma)| \\ &\qquad + |\theta - \bar\theta|_\Theta \Big)\, ds. \end{aligned}$$

Let $N \equiv \max\{\max\{|x(t;\bar\gamma)| : t \in [-r, \alpha]\}, \operatorname{ess\,sup}\{|\dot x(t;\bar\gamma)| : t \in [-r, \alpha]\}\}$. Then (4) yields

$$\begin{aligned} |x(t;\gamma) - x(t;\bar\gamma)| \le\ & |\gamma - \bar\gamma|_\Gamma + L_1 \int_0^t \Big(|x(s;\gamma) - x(s;\bar\gamma)| + L_2 N(|x(\cdot;\gamma)_s - x(\cdot;\bar\gamma)_s|_C \\ & + |\sigma - \bar\sigma|_\Sigma) + |x(\cdot;\gamma)_s - x(\cdot;\bar\gamma)_s|_C + |\gamma - \bar\gamma|_\Gamma \Big)\, ds. \end{aligned}$$

Introduce $\eta(t;\bar\gamma,\gamma) \equiv \sup\{|x(s;\gamma) - x(s;\bar\gamma)| : s \in [-r, t]\}$. With this notation we get

$$|x(t;\gamma) - x(t;\bar\gamma)| \le (1 + L_1 + L_1 L_2 N)|\gamma - \bar\gamma|_\Gamma + L_1(2 + L_2 N) \int_0^t \eta(s;\bar\gamma,\gamma)\, ds,$$

for $t \in [0, \alpha^\gamma]$. The monotonicity of the right-hand side in t and $\eta(t;\bar\gamma,\gamma) \le |\gamma - \bar\gamma|_\Gamma$ for $t \in [-r, 0]$ yield

$$\eta(t;\bar\gamma,\gamma) \le (1 + L_1 + L_1 L_2 N)|\gamma - \bar\gamma|_\Gamma + L_1(2 + L_2 N) \int_0^t \eta(s;\bar\gamma,\gamma)\, ds, \qquad t \in [0, \alpha^\gamma].$$

Applying the Gronwall-Bellmann inequality we get

$$|x(t;\gamma) - x(t;\bar\gamma)| \le \eta(t;\bar\gamma,\gamma) \le L^* |\gamma - \bar\gamma|_\Gamma, \qquad t \in [-r, \alpha^\gamma], \tag{16}$$

where $L^* \equiv (1 + L_1 + L_1 L_2 N) e^{L_1(2 + L_2 N)\alpha}$. Let $\delta \equiv \min\{\delta^2, \varepsilon^1/L^*, \varepsilon^2/(L_2 N(L^* + 1) + L^*), \varepsilon^4/L^*\}$. Then it is easy to show, using (16), that $\alpha^\gamma = \alpha$ can be used in (14) and (15) for $\gamma \in \mathcal{G}_\Gamma(\bar\gamma;\, \delta)$. This proves (12) as well.

It follows from (1), (16), (A1) (ii) and (A2) (ii) that

$$\begin{aligned} &|\dot x(t;\gamma) - \dot x(t;\bar\gamma)| \\ &\quad = |f(t, x(t;\gamma), \Lambda(t, x(\cdot;\gamma)_t, \sigma), \theta) - f(t, x(t;\bar\gamma), \Lambda(t, x(\cdot;\bar\gamma)_t, \bar\sigma), \bar\theta)| \\ &\quad \le L_1 \Big(|x(t;\gamma) - x(t;\bar\gamma)| + L_2 N(|x(\cdot;\gamma)_t - x(\cdot;\bar\gamma)_t|_C + |\sigma - \bar\sigma|_\Sigma) \\ &\qquad + |x(\cdot;\gamma)_t - x(\cdot;\bar\gamma)_t|_C + |\theta - \bar\theta|_\Theta \Big) \\ &\quad \le L^{**} |\gamma - \bar\gamma|_\Gamma, \qquad t \in [0, \alpha], \end{aligned} \tag{17}$$

where $L^{**} \equiv L_1(2 + L_2 N)L^* + L_1(L_2 N + 1)$. Therefore (13) follows from (16), (17) and from $|\dot\varphi(t) - \dot{\bar\varphi}(t)| \le |\gamma - \bar\gamma|_\Gamma$ for almost every $t \in [-r, 0]$ with $L \equiv \max\{L^*, L^{**}\}$. □

3. Differentiability wrt parameters. In this section we study differentiability of solutions of IVP (1)-(2) wrt the initial function, φ, the parameter σ of the delay function τ, and the parameter θ of the function f.

Let $\bar\gamma = (\bar\varphi, \bar\sigma, \bar\theta) \in \mathcal{M}$, and $x(\cdot;\bar\gamma)$ be the corresponding solution of IVP (1)-(2) on $[0,\alpha]$. Fix $h = (h^\varphi, h^\sigma, h^\theta) \in \Gamma$ and consider the variational equation

$$\begin{aligned}
(18)\qquad \dot z(t;\bar\gamma,h) &= D_2 f(t, x(t;\bar\gamma), \Lambda(t, x(\cdot;\bar\gamma)_t, \bar\sigma), \bar\theta) z(t;\bar\gamma,h)\\
&\quad + D_3 f(t, x(t;\bar\gamma), \Lambda(t, x(\cdot;\bar\gamma)_t, \bar\sigma), \bar\theta)\Big(D_2\Lambda(t, x(\cdot;\bar\gamma)_t, \bar\sigma) z(\cdot;\bar\gamma,h)_t\\
&\quad + D_3\Lambda(t, x(\cdot;\bar\gamma)_t, \bar\sigma) h^\sigma\Big) + D_4 f(t, x(t;\bar\gamma), \Lambda(t, x(\cdot;\bar\gamma)_t, \bar\sigma), \bar\theta) h^\theta,\\
&\quad t \in [0,\alpha],\\
(19)\qquad z(t;\bar\gamma,h) &= h^\varphi(t), \qquad t \in [-r, 0].
\end{aligned}$$

This is a linear state-independent delay equation for $z(\cdot;\bar\gamma,h)$, and the right-hand side of (18) depends continuously on t and $z(\cdot;\bar\gamma,h)_t$ since $x(\cdot;\bar\gamma)_t \in C^1$ by Theorem 1 (v). Therefore this IVP has a unique solution, $z(\cdot;\bar\gamma,h)$, which depends linearly on h.

First we study differentiability of the function $x(t;\gamma) = x(t;(\varphi,\sigma,\theta))$ wrt φ and θ only. We denote this differentiation by $D_{(\varphi,\theta)}x$. Let

$$(20)\qquad G^{\varphi,\theta}(\delta,\bar\gamma) \equiv \{(\varphi,\theta) \in W^{1,\infty} \times \Theta : (\varphi,\bar\sigma,\theta) \in \mathcal{G}_\Gamma(\bar\gamma;\delta)\}.$$

Theorem 2. *Assume (A1), (A2), and let $\bar\gamma \in \mathcal{M}$ be fixed. Let $\delta > 0$ and $\alpha > 0$ be defined by Theorem 1, and $x(t;\gamma)$ be the solution of IVP (1)-(2) on $[0,\alpha]$ for $\gamma \in \mathcal{G}_\Gamma(\bar\gamma;\delta)$, and $G^{\varphi,\theta}(\bar\gamma,\delta)$ be defined by (20). Then the function $x(t;(\cdot,\bar\sigma,\cdot)) : G^{\varphi,\theta}(\bar\gamma,\delta) \to \mathbb{R}^n$ is differentiable at $(\bar\varphi,\bar\theta)$ for $t \in [0,\alpha]$, and*

$$D_{(\varphi,\theta)} x(t;(\bar\varphi,\bar\sigma,\bar\theta))(h^\varphi, h^\sigma) = z(t;\bar\gamma,(h^\varphi, 0, h^\theta)),$$

where z is the solution of IVP (18)-(19), and $(h^\varphi,h^\theta) \in W^{1,\infty} \times \Theta$.

Proof. Let $\bar\gamma \in \mathcal{M}$, $\delta > 0$, α, and $G^{\varphi,\theta}(\bar\gamma,\delta)$ be as in the assumption of the theorem. We can and do assume that δ is such that $M_3 \equiv \overline{\mathcal{G}}_\Theta\big(\bar\theta;\delta\big) \subset \Omega_3$ and $M_5 \equiv \overline{\mathcal{G}}_\Sigma(\bar\sigma;\delta) \subset \Omega_5$. Let $h = (h^\varphi, h^\sigma, h^\theta) \in \Gamma$ such that $|h|_\Gamma < \delta$. (Here, for our future purposes, we do not assume yet that $h^\sigma = 0$.) Note that $z(t;\bar\gamma,h)$ is well-defined since, by our assumptions, $x(\cdot;\bar\gamma)_s \in C^1$. Integrating (1) and (18), and using the definition of ω^f and ω^Λ we get

$$\begin{aligned}
&x(t;\bar\gamma+h) - x(t;\bar\gamma) - z(t;\bar\gamma,h)\\
&= \int_0^t \Big(f(s, x(s;\bar\gamma+h), \Lambda(s, x(\cdot;\bar\gamma+h)_s, \bar\sigma+h^\sigma), \bar\theta+h^\theta)\\
&\quad - f(s, x(s;\bar\gamma), \Lambda(s, x(\cdot;\bar\gamma)_s, \bar\sigma), \bar\theta) - D_2 f(s, x(s;\bar\gamma), \Lambda(s, x(\cdot;\bar\gamma)_s, \bar\sigma), \bar\theta) z(s;\bar\gamma,h)\\
&\quad - D_3 f(s, x(s;\bar\gamma), \Lambda(s, x(\cdot;\bar\gamma)_s, \bar\sigma), \bar\theta)\Big(D_2\Lambda(s, x(\cdot;\bar\gamma)_s, \bar\sigma) z(\cdot;\bar\gamma,h)_s\\
&\quad + D_3\Lambda(s, x(\cdot;\bar\gamma)_s, \bar\sigma) h^\sigma\Big) - D_4 f(s, x(s;\bar\gamma), \Lambda(s, x(\cdot;\bar\gamma)_s, \bar\sigma), \bar\theta) h^\theta\Big) ds\\
&= \int_0^t \Big(\omega^f(s, x(s;\bar\gamma), \Lambda(s, x(\cdot;\bar\gamma)_s, \bar\sigma), \bar\theta; x(s;\bar\gamma+h), \Lambda(s, x(\cdot;\bar\gamma+h)_s, \bar\sigma+h^\sigma), \bar\theta+h^\theta)
\end{aligned}$$

$$+ D_2 f(s, x(s;\bar\gamma), \Lambda(s, x(\cdot;\bar\gamma)_s, \bar\sigma), \bar\theta)\Big(x(s;\bar\gamma+h) - x(s;\bar\gamma) - z(s;\bar\gamma,h)\Big)$$
$$+ D_3 f(s, x(s;\bar\gamma), \Lambda(s, x(\cdot;\bar\gamma)_s, \bar\sigma), \bar\theta)\Big(\omega^\Lambda(s, x(\cdot;\bar\gamma)_s, \bar\sigma; x(\cdot;\bar\gamma+h)_s, \bar\sigma + h^\sigma)$$
$$+ D_2\Lambda(s, x(\cdot;\bar\gamma)_s, \bar\sigma)(x(\cdot;\bar\gamma+h)_s - x(\cdot;\bar\gamma)_s - z(\cdot;\bar\gamma,h)_s)\Big)\Big)ds.$$

Let M_i $(i = 1, 2, 4)$ be defined by Theorem 1. Let $L_1 = L_1(\alpha, M_1, M_2, M_3)$ and $L_2 = L_2(\alpha, M_4, M_5)$ be the constants from (A1) (ii) and (A2) (ii), respectively, and $K = K(\alpha, M_4, M_5)$ be the constant from (7)-(8). Then (10) yields

$$\begin{aligned}(21)\qquad & |x(t;\bar\gamma+h) - x(t;\bar\gamma) - z(t;\bar\gamma,h)| \\ & \le \int_0^t \Big(G^f(s;\bar\gamma,h) + L_1\Big|x(s;\bar\gamma+h) - x(s;\bar\gamma) - z(s;\bar\gamma,h)\Big| + L_1 G^\Lambda(s;\bar\gamma,h) \\ & \quad + L_1 K|x(\cdot;\bar\gamma+h)_s - x(\cdot;\bar\gamma)_s - z(\cdot;\bar\gamma,h)_s|_C\Big)ds, \qquad t\in[0,\alpha],\end{aligned}$$

where $G^f(s;\bar\gamma,h) \equiv |\omega^f(s, x(s;\bar\gamma), \Lambda(s, x(\cdot;\bar\gamma)_s, \bar\sigma), \bar\theta; x(s;\bar\gamma+h), \Lambda(s, x(\cdot;\bar\gamma+h)_s, \bar\sigma+h^\sigma), \bar\theta+h^\theta)|$ and $G^\Lambda(s;\bar\gamma,h) \equiv |\omega^\Lambda(s, x(\cdot;\bar\gamma)_s, \bar\sigma; x(\cdot;\bar\gamma+h)_s, \bar\sigma+h^\sigma)|$. Introduce $\eta(t;\bar\gamma,h) \equiv \sup_{-r\le s\le t}|x(s;\bar\gamma+h) - x(s;\bar\gamma) - z(s;\bar\gamma,h)|$. Inequality (21) implies

$$\begin{aligned}(22)\qquad & |x(t;\bar\gamma+h) - x(t;\bar\gamma) - z(t;\bar\gamma,h)| \\ & \le \int_0^\alpha \Big(G^f(s;\bar\gamma,h) + L_1 G^\Lambda(s;\bar\gamma,h)\Big)ds + L_1(1+K)\int_0^t \eta(s;\bar\gamma,h)\,ds.\end{aligned}$$

Using that $\eta(0;\bar\gamma,h) = 0$, and the right-hand side of (22) is monotone in t, we get from (22)

$$\eta(t;\bar\gamma,h) \le \int_0^\alpha \Big(G^f(s;\bar\gamma,h) + L_1 G^\Lambda(s;\bar\gamma,h)\Big)ds + L_1(1+K)\int_0^t \eta(s;\bar\gamma,h)\,ds,$$

which, by the Gronwall-Bellman inequality, implies

$$(23)\qquad \eta(t;\bar\gamma,h) \le \int_0^\alpha \Big(G^f(s;\bar\gamma,h) + L_1 G^\Lambda(s;\bar\gamma,h)\Big)ds\, e^{L_1(1+K)\alpha}, \qquad t\in[0,\alpha].$$

Applying (23) we get

$$\begin{aligned} & |x(t;\bar\gamma+h) - x(t;\bar\gamma) - z(t;\bar\gamma,h)|/|h|_\Gamma \\ & \le \eta(t;\bar\gamma,h)/|h|_\Gamma \\ & \le \int_0^\alpha \Big(G^f(s;\bar\gamma,h)/|h|_\Gamma + L_1 G^\Lambda(s;\bar\gamma,h)/|h|_\Gamma\Big)ds\, e^{L_1(1+K)\alpha}, \quad t\in[-r,\alpha].\end{aligned}$$

Here we used that $x(t;\bar\gamma+h) - x(t;\bar\gamma) - z(t;\bar\gamma,h) = 0$ for $t\in[-r,0]$. We will show that $\int_0^\alpha G^f(s;\bar\gamma,h)/|h|_\Gamma\,ds \to 0$ and $\int_0^\alpha G^\Lambda(s;\bar\gamma,h)/|h|_\Gamma\,ds \to 0$ as $|h|_\Gamma \to 0$.

Using (4) and (13), we get that there exists $K^* = K^*(\alpha, M_4, M_5)$ such that

$$(24)\qquad |\Lambda(s, x(\cdot;\bar\gamma+h)_s, \bar\sigma+h^\sigma) - \Lambda(s, x(\cdot;\bar\gamma)_s, \bar\sigma)| \le K^*|h|_\Gamma, \quad |h|_\Gamma < \delta, \quad s\in[0,\alpha].$$

Using the obvious relation

$$
\begin{aligned}
(25)\qquad & \frac{G^f(s;\bar\gamma,h)}{|h|_\Gamma}\\
&= \frac{\omega^f(s,x(s;\bar\gamma),\Lambda(s,x(\cdot;\bar\gamma)_s,\bar\sigma),\bar\theta;x(s;\bar\gamma+h),\Lambda(s,x(\cdot;\bar\gamma+h)_s,\bar\sigma+h^\sigma),\bar\theta+h^\theta)|}{|x(s;\bar\gamma+h)-x(s;\bar\gamma)|+|\Lambda(s,x(\cdot;\bar\gamma+h)_s,\bar\sigma+h^\sigma)-\Lambda(s,x(\cdot;\bar\gamma)_s,\bar\sigma)|+|h^\theta|_\Theta}\\
&\quad\cdot\frac{|x(s;\bar\gamma+h)-x(s;\bar\gamma)|+|\Lambda(s,x(\cdot;\bar\gamma+h)_s,\bar\sigma+h^\sigma)-\Lambda(s,x(\cdot;\bar\gamma)_s,\bar\sigma)|+|h^\theta|_\Theta}{|h|_\Gamma},
\end{aligned}
$$

(11), (12), (13), (24) and (25) yield $G^f(s;\bar\gamma,h)/|h|_\Gamma \le 2L_1(L+K^*+1)$. On the other hand, (9) and (25) imply $G^f(s;\bar\gamma,h)/|h|_\Gamma \to 0$ as $|h|_\Gamma \to 0$ for $s\in[0,\alpha]$. Therefore $\int_0^\alpha G^f(s;\bar\gamma,h)/|h|_\Gamma\,ds \to 0$ as $|h|_\Gamma\to 0$ by the Lebesgue's Dominated Convergence Theorem.

Similarly, inequalities (8) and (13) imply $G^\Lambda(s;\bar\gamma,h)/|h|_\Gamma \le 2K(L+1)$. To show that $G^\Lambda(s;\bar\gamma,h)/|h|_\Gamma \to 0$ we now assume that $h^\sigma=0$. Lemma 1 implies $G^\Lambda(s;\bar\gamma,h)/|h|_\Gamma = |\Lambda(s,x(\cdot;\bar\gamma+h)_s,\bar\sigma)-\Lambda(s,x(\cdot;\bar\gamma)_s,\bar\sigma)-D_2\Lambda(s,x(\cdot;\bar\gamma)_s,\bar\sigma)(x(\cdot;\bar\gamma+h)_s-x(\cdot;\bar\gamma)_s)|/|h|_\Gamma \to 0$ as $|h|_\Gamma\to 0$ for $s\in[0,\alpha]$, since, by (13), $|x(\cdot;\bar\gamma+h)_s-x(\cdot;\bar\gamma)_s|_{W^{1,\infty}}\to 0$ as $|h|_\Gamma\to 0$. Therefore $\int_0^\alpha G^\Lambda(s;\bar\gamma,h)/|h|_\Gamma\,ds \to 0$ as $|h|_\Gamma\to 0$.

We conclude that $|x(t;\bar\gamma+h)-x(t;\bar\gamma)-z(t;\bar\gamma,h)|/|h|_\Gamma \to 0$ as $|h|_\Gamma\to 0$, which proves the theorem. □

The proof of the previous theorem implies immediately:

Corollary 1. *Assume the conditions of Theorem 2. Then the function* $G^{\varphi,\theta}(\bar\gamma,\delta)\to C$, $(\varphi,\theta)\mapsto x(\cdot;(\varphi,\bar\sigma,\theta))_t$ *is differentiable at* $(\bar\varphi,\bar\theta)$ *for* $t\in[0,\alpha]$, *and its derivative is given by* $D_{(\varphi,\theta)}x(\cdot;(\bar\varphi,\bar\sigma,\bar\theta))_t(h^\varphi,h^\theta) = z(\cdot;\bar\gamma,(h^\varphi,0,h^\theta))_t$, $(h^\varphi,h^\theta)\in W^{1,\infty}\times\Theta$.

Next we study differentiability wrt σ as well. We will need the following definition.

Definition 1. *Let X and Y be normed linear spaces, $M\subset X$, and $x_0\in M$ be an accumulation point of M. We say that $f:\Big(M\subset X\Big)\to Y$ is differentiable at the point x_0 with respect to the set M if there exists $L\in\mathcal{L}(X,Y)$ such that*

$$
\lim_{\substack{x\to x_0\\ x\in M}} \frac{|f(x)-f(x_0)-L(x-x_0)|_Y}{|x-x_0|_X} = 0.
$$

We have the following result.

Theorem 3. *Assume (A1), (A2), and let $\bar\gamma\in\mathcal{M}$ be an accumulation point of $\mathcal{M}$. Let $\delta>0$ and $\alpha>0$ be defined by Theorem 1, and $x(t;\gamma)$ be the solution of IVP (1)-(2) on $[0,\alpha]$ for $\gamma\in\mathcal{G}_\Gamma(\bar\gamma;\delta)$. Then the function $x(t;\cdot):\Big((\mathcal{G}_\Gamma(\bar\gamma;\delta)\cap\mathcal{M})\subset\Gamma\Big)\to\mathbb{R}^n$ is differentiable at $\bar\gamma$ wrt $\mathcal{G}_\Gamma(\bar\gamma;\delta)\cap\mathcal{M}$ for $t\in[0,\alpha]$, and its derivative is $D_\gamma x(t;\bar\gamma)h = z(t;\bar\gamma,h)$, where z is the solution of IVP (18)-(19), $h\in\Gamma$ is such that $\bar\gamma+h\in\mathcal{M}$.*

Proof. We proceed as in the proof of Theorem 2. The only step needs a different argument here is the last one, to show that $G^\Lambda(s;\bar\gamma,h)/|h|_\Gamma\to 0$ as $|h|_\Gamma\to 0$. We have $G^\Lambda(s;\bar\gamma,h) = |\Lambda(s,x(\cdot;\bar\gamma+h)_s,\bar\sigma+h^\sigma)-\Lambda(s,x(\cdot;\bar\gamma)_s,\bar\sigma)-D_2\Lambda(s,x(\cdot;\bar\gamma)_s,\bar\sigma)(x(\cdot;\bar\gamma+h)_s-x(\cdot;\bar\gamma)_s)-D_3\Lambda(s,x(\cdot;\bar\gamma)_s,\bar\sigma)h^\sigma|/|h|_\Gamma$. Let h be such that $\bar\gamma+h\in\mathcal{M}$. Then, using that $\Lambda(t,\cdot,\cdot)$ is continuously differentiable on

$\Omega_4 \cap C^1 \times \Omega_5$, and $x(\cdot;\bar\gamma+h)_s \in C^1$ for $s \in [0,\alpha]$, we get

$$
\begin{aligned}
(26)\qquad & G^\Lambda(s;\bar\gamma,h)\\
&\le \sup_{0<\nu<1}\Big\| D_2\Lambda(s,(1-\nu)x(\cdot;\bar\gamma)_s+\nu x(\cdot;\bar\gamma+h)_s,\bar\sigma+\nu h^\sigma)\\
&\qquad - D_2\Lambda(s,x(\cdot;\bar\gamma)_s,\bar\sigma)\Big\|_{\mathcal{L}(W^{1,\infty},\mathbb{R}^n)}\cdot|x(\cdot;\bar\gamma+h)_s - x(\cdot;\bar\gamma)_s|_{W^{1,\infty}}\\
&\quad + \sup_{0<\nu<1}\Big\| D_3\Lambda(s,(1-\nu)x(\cdot;\bar\gamma)_s+\nu x(\cdot;\bar\gamma+h)_s,\bar\sigma+\nu h^\sigma)\\
&\qquad - D_3\Lambda(s,x(\cdot;\bar\gamma)_s,\bar\sigma)\Big\|_{\mathcal{L}(\Sigma,\mathbb{R}^n)}\cdot|h^\sigma|_\Sigma.
\end{aligned}
$$

Therefore the continuity of $D_2\Lambda(s,\cdot,\cdot)$ and $D_3\Lambda(s,\cdot,\cdot)$ (see Lemma 1), and (13) imply $G^\Lambda(s;\bar\gamma,h)/|h|_\Gamma \to 0$ as $|h|_\Gamma \to 0$. □

Next we show that, under the assumptions of the previous theorem, $x(\cdot;\gamma)_t$ is differentiable wrt γ (in the sense of Definition 1) if we use $W^{1,\infty}$ as the state-space of the solutions.

THEOREM 4. *Assume (A1), (A2), and let $\bar\gamma \in \mathcal{M}$ be an accumulation point of $\mathcal{M}$. Let $\delta>0$ and $\alpha>0$ be defined by Theorem 1, and $x(t;\gamma)$ be the solution of IVP (1)-(2) on $[0,\alpha]$ for $\gamma \in \mathcal{G}_\Gamma(\bar\gamma;\delta)$. Then the function $\Big((\mathcal{G}_\Gamma(\bar\gamma;\delta)\cap\mathcal{M}) \subset \Gamma\Big) \to W^{1,\infty}$, $\gamma \mapsto x(\cdot;\gamma)_t$ is differentiable at $\bar\gamma$ wrt $\mathcal{G}_\Gamma(\bar\gamma;\delta)\cap\mathcal{M}$ for $t\in[0,\alpha]$, and $D_\gamma x(\cdot;\bar\gamma)_t h = z(\cdot;\bar\gamma,h)_t$, where z is the solution of IVP (18)-(19), and $h\in\Gamma$ is such that $\bar\gamma+h\in\mathcal{M}$.*

Proof. We use all the notations introduced in the proof of Theorem 2. It follows from the proofs of Theorems 2 and 3 that $|x(\cdot;\bar\gamma+h)_t - x(\cdot;\bar\gamma)_t - z(\cdot;\bar\gamma,h)_t|_C/|h|_\Gamma \to 0$ as $\bar\gamma+h\in\mathcal{M}$ and $|h|_\Gamma\to 0$. Similarly to (22) we get

$$
\begin{aligned}
(27)\qquad & |\dot x(t;\bar\gamma+h) - \dot x(t;\bar\gamma) - \dot z(t;\bar\gamma,h)|\\
&\le G^f(t;\bar\gamma,h) + L_1 G^\Lambda(t;\bar\gamma,h) + L_1(1+K)\eta(t;\bar\gamma,h), \qquad t\in[0,\alpha].
\end{aligned}
$$

Clearly, $\dot x(t;\bar\gamma+h)-\dot x(t;\bar\gamma)-\dot z(t;\bar\gamma,h)=0$ for $t\in[-r,0]$. Therefore, in view of (23), it suffices to show that $G^f(t;\bar\gamma,h)/|h|_\Gamma\to 0$ and $G^\Lambda(t;\bar\gamma,h)/|h|_\Gamma\to 0$ as $\bar\gamma+h\in\mathcal{M}$ and $|h|_\Gamma\to 0$ uniformly in $t\in[0,\alpha]$. Consider a sequence $h^k=(h^{k,\varphi},h^{k,\sigma},h^{k,\theta})\in\Gamma$ such that $\bar\gamma+h^k\in\mathcal{M}$ for $k\in\mathbb{N}$ and $|h^k|_\Gamma\to 0$ as $k\to\infty$. We have

$$
\begin{aligned}
(28)\qquad & G^f(t;\bar\gamma,h^k)\\
&\le \sup_{0<\nu<1}\Big\| D_2 f(t,(1-\nu)x(t;\bar\gamma)+\nu x(t;\bar\gamma+h^k),\\
&\qquad (1-\nu)\Lambda(t,x(\cdot;\bar\gamma)_t,\bar\sigma)+\nu\Lambda(t,x(\cdot;\bar\gamma+h^k)_t,\bar\sigma+h^{k,\sigma}),\bar\theta+\nu h^{k,\theta})\\
&\quad - D_2 f(t,x(t;\bar\gamma),\Lambda(t,x(\cdot;\bar\gamma)_t,\bar\sigma),\bar\theta)\Big\| |x(t;\bar\gamma+h^k)-x(t;\bar\gamma)|\\
&\quad + \sup_{0<\nu<1}\Big\| D_3 f(t,(1-\nu)x(t;\bar\gamma)+\nu x(t;\bar\gamma+h^k),\\
&\qquad (1-\nu)\Lambda(t,x(\cdot;\bar\gamma)_t,\bar\sigma)+\nu\Lambda(t,x(\cdot;\bar\gamma+h^k)_t,\bar\sigma+h^{k,\sigma}),\bar\theta+\nu h^{k,\theta})\\
&\quad - D_3 f(t,x(t;\bar\gamma),\Lambda(t,x(\cdot;\bar\gamma)_t,\bar\sigma),\bar\theta)\Big\|\\
&\qquad \cdot|\Lambda(t,x(\cdot;\bar\gamma+h^k)_t,\bar\sigma+h^{k,\sigma})-\Lambda(t,x(\cdot;\bar\gamma)_t,\bar\sigma)|
\end{aligned}
$$

$$+\sup_{0<\nu<1}\Big\|D_4f(t,(1-\nu)x(t;\bar\gamma)+\nu x(t;\bar\gamma+h^k),$$
$$(1-\nu)\Lambda(t,x(\cdot;\bar\gamma)_t,\bar\sigma)+\nu\Lambda(t,x(\cdot;\bar\gamma+h^k)_t,\bar\sigma+h^{k,\sigma}),\bar\theta+\nu h^{k,\theta})$$
$$-D_4f(t,x(t;\bar\gamma),\Lambda(t,x(\cdot;\bar\gamma)_t,\bar\sigma),\bar\theta)\Big\|_{\mathcal{L}(\Theta,\mathbb{R}^n)}|h^{k,\theta}|_\Theta.$$

Let $M_3^*\equiv\{\bar\theta+\nu h^{k,\theta}\,:\,k\in\mathbb{N},\ \nu\in[0,1]\}$, and $A\equiv[0,\alpha]\times M_1\times M_2\times M_3^*$. The set A is a compact subset of $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\times\Theta$, since M_1 and M_2 are compact subsets of $\mathbb{R}^n$, and, it is easy to see that M_3^* is a compact subset of Θ. By (A1) (iii) D_2f, D_3f and D_4f are continuous, therefore uniformly continuous on A. Therefore (28), together with (13) and (24), yields $G^f(t;\bar\gamma,h^k)/|h^k|_\Gamma\to0$ as $k\to\infty$ uniformly in $t\in[0,\alpha]$.

Similarly, define $M_5^*\equiv\{\bar\sigma+\nu h^{k,\sigma}\,:\,k\in\mathbb{N},\ \nu\in[0,1]\}$, and $B\equiv[0,\alpha]\times M_4\times M_5^*$. Then B is a compact subset of $\mathbb{R}\times C\times\Sigma$, therefore (13) and (26) imply that $G^\Lambda(t;\bar\gamma,h^k)/|h^k|_\Gamma\to0$ as $k\to\infty$ uniformly in $t\in[0,\alpha]$. This concludes the proof of the theorem. □

The next two examples show cases when the differentiability property of the solution wrt some parameter guaranteed by Theorem 4 equals to the usual Frechét-differentiability of the solution wrt the parameter.

Example 1. Suppose f satisfies (A1) and has the form

$$f(t,x,y,\theta)=f^1(t,x,y)+f^2(t,x,y,\theta),$$

where $f^2(0,x,y,\theta)=0$ for all $x\in\Omega_1$, $y\in\Omega_2$ and $\theta\in\Omega_3$. Then if $\bar\gamma=(\bar\varphi,\bar\sigma,\bar\theta)\in\Pi$ satisfies $\bar\varphi\in C^1$ and $\dot{\bar\varphi}(0-)=f^1(0,\bar\varphi(0),\Lambda(0,\bar\varphi,\bar\sigma))$, then the solution of IVP (1)-(2), $x(\cdot;\theta)_t$, is differentiable wrt θ on Ω_3 for $t\in[0,\alpha]$ in the usual Frechét-sense as a function $\Big(\Omega_3\subset\Theta\Big)\to W^{1,\infty}$, $\theta\mapsto x(\cdot;\theta)_t$.

Example 2. Suppose the function τ satisfies (A2) and $\tau(t,\psi,\sigma)=\tau^1(t,\psi)+\tau^2(t,\psi,\sigma)$, where $\tau^2(0,\psi,\sigma)=0$ for all $\psi\in\Omega_4$ and $\sigma\in\Omega_5$. Then if $\bar\gamma=(\bar\varphi,\bar\sigma,\bar\theta)\in\Pi$ satisfies $\bar\varphi\in C^1$ and $\dot{\bar\varphi}(0-)=f(0,\bar\varphi(0),\bar\varphi(-\tau^1(0,\bar\varphi)),\bar\theta)$, then the solution, $x(\cdot;\sigma)_t$, is differentiable wrt σ on Ω_5 for $t\in[0,\alpha]$ (in Frechét-sense) as a function $\Big(\Omega_5\subset\Sigma\Big)\to W^{1,\infty}$, $\sigma\mapsto x(\cdot;\sigma)_t$.

Finally, we consider the state-independent version of IVP (1)-(2), i.e., we assume that $\tau(t,\psi,\sigma)$ is independent of ψ. Let $\bar\psi\in C^1$. First we note that (5) yields in this case that $D_2\Lambda(t,\bar\psi,\bar\sigma)h=h(-\tau(t,\bar\psi,\bar\sigma))$, therefore a simple calculation and (6) imply

$$\begin{aligned}
&|\omega^\Lambda(t,\bar\psi,\bar\sigma;\psi,\sigma)|\\
&\quad=|\bar\psi(-\tau(t,\psi,\sigma))-\bar\psi(-\tau(t,\bar\psi,\bar\sigma))-D_3\Lambda(t,\bar\psi,\bar\sigma)(\sigma-\bar\sigma)\\
&\qquad+\psi(-\tau(t,\psi,\sigma))-\bar\psi(-\tau(t,\psi,\sigma))-\psi(-\tau(t,\bar\psi,\bar\sigma))+\bar\psi(-\tau(t,\bar\psi,\bar\sigma))|\\
&\quad\le|\bar\psi(-\tau(t,\psi,\sigma))-\bar\psi(-\tau(t,\bar\psi,\bar\sigma))+\dot{\bar\psi}(-\tau(t,\bar\psi,\bar\sigma))D_3\tau(t,\bar\psi,\bar\sigma)(\sigma-\bar\sigma)|\\
&\qquad+|\psi-\bar\psi|_{W^{1,\infty}}|\tau(t,\psi,\sigma)-\tau(t,\bar\psi,\bar\sigma)|.
\end{aligned}$$

Therefore (A2) (iii), the Chain-rule and the Mean Value Theorem yield

$$\frac{|\omega^\Lambda(t,\bar\psi,\bar\sigma;\psi,\sigma)|}{|\psi-\bar\psi|_{W^{1,\infty}}+|\sigma-\bar\sigma|_\Sigma}\to0,\qquad\text{as }|\psi-\bar\psi|_{W^{1,\infty}}+|\sigma-\bar\sigma|_\Sigma\to0.$$

Consequently, $G^{\Lambda}(t;\bar\gamma,h)/|h|_\Gamma \to 0$ as $|h|_\Gamma \to 0$. Using this relation, it follows easily from the proof of Theorem 4:

COROLLARY 2. *Assume (A1), (A2), and let $\bar\gamma \in \mathcal{M}$ be fixed. Assume moreover that $\tau(t,\psi,\sigma)$ is independent of ψ. Let $\delta > 0$ and $\alpha > 0$ be defined by Theorem 1, and $x(t;\gamma)$ be the solution of IVP (1)-(2) on $[0,\alpha]$ for $\gamma \in \mathcal{G}_\Gamma(\bar\gamma;\delta)$. Then the function $\Big(\mathcal{G}_\Gamma(\bar\gamma;\delta) \subset \Gamma\Big) \to W^{1,\infty}$, $\gamma \mapsto x(\cdot;\gamma)_t$ is differentiable at $\bar\gamma$ for $t \in [0,\alpha]$, and $D_\gamma x(\cdot;\bar\gamma)_t h = z(\cdot;\bar\gamma,h)_t$, where z is the solution of IVP (18)-(19), and $h \in \Gamma$.*

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