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Stability in Delay Equations with Perturbed Time Lags

I. Győri
Computer Center of
A. Szent-Györgyi Medical University
H-6720 Szeged, Hungary

F. Hartung and J. Turi
Programs in Mathematical Sciences
University of Texas at Dallas
Richardson, TX 75083

1. Introduction

In this paper we study the effects of perturbations of time delays to the stability of a class of delay equations. Our goal is to obtain a "practical" condition, i.e., a norm bound on the perturbations corresponding to the particular system under consideration, which guarantees the preservation of stability under perturbations. It turns out that such condition can be formulated assuming that we know the fundamental solution of the unperturbed system (see Theorem 2.2 below). Since stability of the unperturbed system implies that the components of its fundamental solution go to zero at infinity, it is possible to get "good" numerical estimates of these components, and consequently obtain norm bounds on the allowable perturbations.

We present our main results in Section 2 and in Section 3 we consider numerical examples. Example 3.3 demonstrates how our results can be used to obtain an estimation on the maximum allowable sampling interval in the satability of a hybrid system with feedback delay. (Note that this problem was studied in [1] in the case when the plant is described by an ordinary differential equation.)

2. Main Results

Consider the delay differential system

$$\dot{x}(t) = Ax(t)+Bx(t-\tau-\varepsilon(t))+Cx(t-\sigma-\eta(t)),\ t\geq 0, \quad (2.1)$$

where $x(t)\in \mathrm{R}^n$, A, B and C are constant $n\times n$ matrices, with initial condition

$$x(t)=\varphi(t), \qquad -r\leq t\leq 0, \qquad (2.2)$$

where $\varphi : [-r,0]\to \mathrm{R}^n$ is a continuous function.

We have the following assumptions on the delays:

(H1) $\tau>0$, $\sigma>0$ and $r\geq \tau,\sigma$.

(H2) $\varepsilon(\cdot)$ and $\eta(\cdot)$ are piecewise continuous functions.

(H3) $t-r\leq t-\tau-\varepsilon(t)\leq t$ and $t-r\leq t-\sigma-\eta(t)\leq t$ for $t\geq 0$.

Under these assumptions the initial value problem (2.1)-(2.2) is a delay differential equation and has a unique solution (see e.g. [3]).

Consider the corresponding delay system with constant delays

$$\dot{y}(t) = Ay(t)+By(t-\tau)+Cy(t-\sigma), \quad t\geq 0, \quad (2.3)$$

where $y(t)\in\mathrm{R}^n$. We assume that

(H4) the null solution of (2.3) is asymptotically stable.

We can rewrite (2.1) in the form

$$\dot{x}(t) = Ax(t)+Bx(t-\tau)+Cx(t-\sigma)+f(t), \quad (2.4)$$

where $f(t)\equiv B\cdot\big(x(t-\tau-\varepsilon(t))-x(t-\tau)\big)+C\cdot\big(x(t-\sigma-\eta(t))-x(t-\sigma)\big)$. In this setting (2.3) can be considered as the homogeneous equation corresponding to (2.4). The variation-of-constants formula (see e.g. [3]) gives the following expression for the solution of (2.1) corresponding to the initial function $x(t)=\varphi(t)$ for $t\leq 0$ using the solution and the fundamental solution, $V(t)$, of the homogeneous equation:

$$x(t)=y(t)+\int_T^t V(t-s)f(s)\,ds, \qquad t\geq T, \quad (2.5)$$

where y is the solution of (2.3) with the initial function $y(t)=x(t)$ for $t\leq T$ and $T>0$.

For notational convenience, we introduce the ˜ operation on matrices, which means taking the absolute value of the matrix componentwise, i.e., if $A=(a_{ij})_{n\times n}$, then $\tilde{A}\equiv(|a_{ij}|)_{n\times n}$.

Remark 2.1 *Hypothesis (H4) implies (see e.g. [3]) that there exist constants $K>0$ and $\alpha>0$, such that $\|V(t)\|\leq Ke^{-\alpha t}$ for $t\geq 0$, (where $\|\cdot\|$ is the matrix norm induced by the vector norm $\|(x_1,x_2,\ldots,x_n)\|\equiv max\{|x_1|,|x_2|,\ldots,|x_n|\}$), and then every element of the matrix $\int_0^\infty \tilde{V}(s)\,ds$ is finite.*

The next theorem shows, that if the perturbations of the delays in (2.1) are small enough for large t, then the equation remains asymptotically stable.

Theorem 2.2 *Assume (H1)-(H4) and that the matrix $M\equiv\int_0^\infty\tilde{V}(s)\,ds\big(\limsup_{t\to\infty}|\varepsilon(t)|\cdot\tilde{B}+\limsup_{t\to\infty}|\eta(t)|\cdot\tilde{C}\big)(\tilde{A}+\tilde{B}+\tilde{C})$ has spectral radius less than 1, i.e., $\rho(M)<1$. Then the null solution of (2.1) is asymptotically stable.*

The proof of the theorem uses the so called "M-matrix" technique (see [2]) and relation (2.5). The following corollary is an easy consequence of the theorem.

Corollary 2.3 *Define* $M_0 \equiv \int_0^\infty \tilde{V}(s)\,ds(\tilde{B}+\tilde{C})(\tilde{A}+\tilde{B}+\tilde{C})$*. If* $\limsup_{t\to\infty}|\varepsilon(t)| < 1/\rho(M_0)$ *and* $\limsup_{t\to\infty}|\eta(t)| < 1/\rho(M_0)$*, then the null solution of* (2.1) *is asymptotically stable.*

For the scalar case of (2.1), (2.3) we have the following version of Theorem 2.2.

Corollary 2.4 *Assume that (H1)-(H4) hold and the functions* $\varepsilon(\cdot)$ *and* $\eta(\cdot)$ *satisfy* $|B|\limsup_{t\to\infty}|\varepsilon(t)| + |C|\limsup_{t\to\infty}|\eta(t)|$
$< 1/\big((|A|+|B|+|C|)\int_0^\infty |V(t)|\,dt\big)$*. Then the null solution of the scalar version of* (2.1) *is asymptotically stable.*

If the fundamental solution of the scalar version of (2.3) is positive, then it is easy to compute the integral in Corollary 2.4. In particular, we have the following result.

Proposition 2.5 *If the null solution of the scalar version of* (2.3) *is asymptotically stable, then the fundamental solution of* (2.3) *satisfies* $\int_0^\infty V(t)\,dt = -1/(A+B+C)$*.*

We close this section by noting that an obvious generalization of Theorem 2.2 applies for the multiple delay case (i.e., (2.1) with more than two delays).

3. Examples

Example 3.1 Consider the equation

$$\dot{x}(t) = -0.1x(t) + 2x(t-1) - 2x\left(\left[\frac{t-1.3}{h}\right]h\right), \quad (3.1)$$

where $[\cdot]$ denotes the greatest integer function and $h > 0$ is the sampling period. The piecewise constant delay in the last term can be considered as a perturbation of $t-1.3$ with $\eta(t) = t - 1.3 - [(t-1.3)/h]h$. Then we have that $|\eta(t)| \le h$ for all $t \ge 0$. The corresponding unperturbed delay equation is

$$\dot{x}(t) = -0.1x(t) + 2x(t-1) - 2x(t-1.3). \quad (3.2)$$

We show a numerical approximation of the fundamental solution of Equation (3.2) on Figure 1. The picture indicates that the fundamental solution exponentially tends to zero, i.e., the null solution of (3.2) is asymptotically stable. Numerical approximation gives that $\int_0^\infty |V(t)|\,ds = 10.5914$. Therefore using Corollary 2.4 if $h < \frac{1}{10.5914\cdot 8.2} = 0.0115$ then the null solution of (3.1) is asymptotically stable.

Example 3.2 Consider the following system

$$\dot{x}(t) = Ax(t) + Bx(t-1-\varepsilon(t)) + Cx(t-1.2-\eta(t)), \quad (3.3)$$

where $x(t) \in \mathrm{R}^2$, $A = \begin{pmatrix} -0.1 & 0.1 \\ 0.1 & -0.2 \end{pmatrix}$, $B = \begin{pmatrix} 2.0 & -0.5 \\ 0.5 & 2.0 \end{pmatrix}$ and $C = \begin{pmatrix} -2.0 & 0.0 \\ -0.5 & -2.1 \end{pmatrix}$. The corresponding unperturbed equation is

$$\dot{x}(t) = Ax(t) + Bx(t-1) + Cx(t-1.2). \quad (3.4)$$

On Figure 2 we display the components of the numerical solutions of the fundamental matrix solution. This picture indicates that every component function tends to zero exponentially as $t \to \infty$, therefore the null solution of (3.4) is asymptotically stable. Numerical approximation of the components of $\int_0^\infty \tilde{V}(t)\,dt$ gives the following numerical values for the matrix M_0

$$M_0 = \begin{pmatrix} 126.453 & 3.226 \\ 3.522 & 128.389 \end{pmatrix},$$

therefore $\rho(M_0) = 130.929$, hence using Corollary 2.3, if the perturbations of the delays satisfy $\limsup_{t\to\infty}|\varepsilon(t)| < 0.0076$ and $\limsup_{t\to\infty}|\eta(t)| < 0.0076$ then the null solution of (3.3) is asymptotically stable.

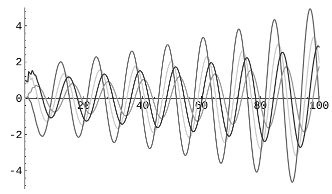

Figure 1

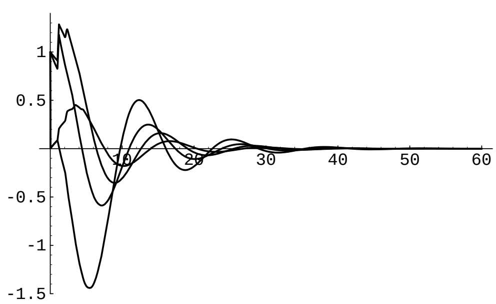

Figure 2

4. References

[1] K. L. Cooke, J. Turi and G. Turner, *Spectral Conditions and an Explicit Expression for the Stabilization of Hybrid Systems in the Presence of Feedback Delays*, Quarterly J. on Applied Mathematics, v.LI, 1993, 147–159.

[2] A. Berman and R. J. Plemmons, "Nonnegative Matrices in the Mathematical Sciences", Academic Press, New York, 1979.

[3] J. K. Hale, "Theory of Functional Differential Equations", Spingler-Verlag, New York, 1977.