Qualitative properties of the solutions of a delay differential equation with impulses: I. Stability

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Abstract

In this paper we establish some sufficient conditions and also certain necessary condition for asymptotic stability in a class of delay differential equations with impulse effects.

1. Introduction

The study of certain ordinary differential equations with impulses was initiated in the 1960's by a seminar paper by Milman and Myshkis ([12]). After a period of active research, mostly in Eastern Europe, from 1960-1970, culminating with the monograph by Halanay and Wexler ([9]), the subject seemed to have received

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very little attention during the seventies. It regained interest in the beginning of the 80's (see the monograph by Pandit and Deo [13]) and has recently attracted a considerable audience. Decisive progress has been made during the last 20 years, in the case of ordinary differential equations with impulses. Two recent books by Bainov and Simeonov [5] and Lakshmikantham, Bainov and Simeonov [11] present the state of the art in the theory of such systems. Comparatively, not much has been done in the study of functional differential equations with impulses. Most of the efforts seem to have been devoted to understanding the initial value problem associated with such a system (see, e.g. Anokhin [1] and Anokhin and Braverman [2]). Nonetheless, in Gopalsamy and Zhang [7], preliminary stability and oscillation results are presented for the case of a single delay (see also [11] and [6]), under the strong condition that the delay is smaller than the length of the inter-impulse time intervals.

The purpose of our work, which we present in two separate papers, is to investigate as closely as possible the question of stability and oscillations for delay differential equations with impulse effects. The stability results are included in the present paper and we deal with oscillations in the second one ([3]).

In the present paper, we concentrate on the following problem: if the trivial solution of a delay equation is asymptotically stable without impulse effects, under what conditions can such effects maintain asymptotic stability. While some of our results can be extended to the general linear delay case and also to the case when the delay equation is nonautonomous, our main interest is in the methods and focusing the reader's attention on the role of impulses. In this respect, our equation is used as a prototype. The paper is organized as follows. Section 2 is general and important in investigations of both stability and oscillation questions. The initial value problem is stated there. Two fundamental formulae are derived for the solution of a delay differential equation with impulse effects, and its jumps via the solutions of the delay equation without impulses. Analogous formulae can be found in [1].

Section 3 deals with stability. Namely, sufficient conditions and also a necessary condition are given for asymptotic stability of the considered delay differential equation with impulse effects. Our results are given in the several delay case, and we drop the strong assumption in [7] that the delay is smaller than the length of the inter-impulse time intervals. We believe that the methods we present can be extended to more general cases.

2. Preliminaries

Consider the system of equations

$$\dot{x}(t) = \sum_{j=1}^{N} a_j x(t-\tau_j) \tag{2.1}$$

$$x(t_k+) - x(t_k-) = \beta_k x(t_k-) \tag{2.2}$$

where Eq. (2.1) is a standard delay-differential equation and Eq. (2.2) takes into account impulses. Throughout the paper, we assume that the delays $0 \leq \tau_1 < \tau_2 < \ldots < \tau_N = \tau$ and the coefficients $a_j \quad (1 \leq j \leq N)$ are constant. If $N = 1$, we denote $a_N = a$. Eq. (2.1) is associated with a pair $(\overline{a}, \overline{\tau})$, where $\overline{a} = (a_1, \ldots, a_N) \in R^N$ and $\overline{\tau} = (\tau_1, \ldots, \tau_N) \in R_+^N$. We also assume that the impulses are produced at given times $t_1 < t_2 < \ldots, t_k \to +\infty$, as $k \to +\infty$, with the $\beta_k \neq 0$ impulse rate at time t_k.

We call Eq. (2.2) the impulse equation where, as usual, $x(t_k-) \quad (x(t_k+))$ denotes the limit from the left (from the right) of $x(t)$ as t tends to t_k. From (2.2) it is immediate that $x(t_k+)$ can be computed in term of $x(t_k-)$:

$$x(t_k+) = (1+\beta_k)x(t_k-).$$

We set the initial value problem in the following way: Let $\sigma \in R$, throughout the paper, σ will denote the initial time. Let ϕ be an element of the set C of the continuous functions on $[-\tau, 0]$. We want to find a function x defined on $[\sigma - \tau, \infty)$ such that it satisfies (2.1) and (2.2) and the initial condition

$$x(t) = \phi(t-\sigma), \qquad \sigma - \tau \leq t \leq \sigma. \tag{2.3}$$

Definition 1. *We say that a function $x : [\sigma-\tau, \infty) \to R$ is a solution of problem (2.1), (2.2) and (2.3) if x is piecewise continuous on $[\sigma-\tau, \infty)$, x is differentiable on the complement of a denumerable subset of $[\sigma, \infty)$ and verifies Eq. (2.1) whenever $\dot{x}$ and the right hand side of Eq. (2.1) are defined, on $[\sigma, \infty)$. Moreover, x should be absolutely continuous with respect to the Lebesgue measure in each interval where it is continuous. Finally, x has to verify the impulse equation (2.2) at each point t_k, $t_k \geq \sigma$, $k \geq 1$, and the initial condition (2.3).*

It is not difficult to show that the problem, as posed, has for each $\phi \in C$ one and only one solution. This can be readily checked by integration of Eq. (2.1) from σ to the first t_k on right, then from this point to the next one and so on. The unique solution of the initial value problem for the impulse equation (2.1)-(2.3) is denoted by $x(\sigma, \phi)$.

Remark 1. *It is clear from the definition of $x(\sigma,\phi)$ that*

- *$x(\sigma,\phi)$ is in fact differentiable on $[\sigma,\infty)$ at each point t such that neither t nor any of the $t-\tau_j$'s belong to the union of $\{t_k\}_{k\geq 1}$ and the set of discontinuities of the initial function ϕ;*
- *the derivative of the solution $x(\sigma,\phi)$ is also piecewise continuous on each interval $[t_k,t_{k+1}]$, a discontinuity at t originating from that of $x(\sigma,\phi)$ at $t-\tau_j$ for some j;*
- *we do not require that x verifies equation (2.2) at the points t_k which belong to $[\sigma-\tau,\sigma)$.*

Throughout in this paper, the solution of Eq. (2.1) with initial condition

$$x(s)=\phi(s), \qquad -\tau\leq\sigma\leq 0,$$

is denoted by $y(\phi)$. The fundamental solution of Eq. (2.1) is denoted by v, that is v is defined on $[-\tau,\infty)$ such that v verifies Eq. (2.1) on $]0,\infty)$, moreover $v(0)=1$ and $v(s)=0 \quad \text{for} \quad -\tau\leq s<0$. Since Eq. (2.1) is autonomous, it is clear that for any $(\sigma,\phi)\in R\times C$ the function $y(\sigma,\phi)(t)=y(\phi)(t-\sigma)$, $t\geq\sigma-\tau$, is the unique solution of problem (2.1) and (2.3). For any $\sigma\in R$, $k(\sigma)$ denotes the first index such that $t_i\geq\sigma, \quad i\geq k(\sigma)$.

The next result gives a key representation formula for the solutions of the impulse equation (2.1)-(2.2)-(2.3) in terms of the solutions of Eq. (2.1) (without impulses).

Theorem 2.1. *Let $(\sigma,\phi)\in R\times C$. Then, the solution $x(\sigma,\phi)$ of (2.1)-(2.3) can be written as*

$$x(\sigma,\phi)(t)=y(\phi)(t-\sigma)+\sum_{\sigma\leq t_j<t}v(t-t_j)u_j(\sigma,\phi), \quad t>\sigma,\ t\notin\{t_k\}_{k\geq k(\sigma)} \tag{2.4}$$

and the sequence

$$u_k(\sigma,\phi)=x(\sigma,\phi)(t_k+)-x(\sigma,\phi)(t_k-), \quad k\geq k(\sigma), \tag{2.5}$$

is determined by the following nonautonomous recurrence equation

$$\frac{u_k(\sigma,\phi)}{\beta_k}=y(\phi)(t_k-\sigma)+\sum_{\sigma\leq t_j<t_k}v(t_k-t_j)u_j(\sigma,\phi), \quad k\geq k(\sigma), \tag{2.6}$$

starting from

$$u_{k(\sigma)}(\sigma,\phi)=\beta_{k(\sigma)}y(\phi)(t_{k(\sigma)}-\sigma). \tag{2.7}$$

Proof. Let $x = x(\sigma, \phi)$ be the solution of *(2.1)-(2.3)*. We will show that x satisfies (2.4) by rewriting equation (2.1) in terms of a new variable. Namely, x can be expressed on a unique way as a sum of a continuous function $z : [\sigma, \infty) \to R$ and a stepfunction whose jumps are at the points t_k and are equal to the jumps of x at these points.

In terms of z and the $u_k = u_k(\sigma, \phi)$'s, x reads as

$$\left.\begin{array}{rcll} x(t) & = & z(t) + \sum\limits_{\sigma \le t_j < t} H(t - t_j)u_j, & t \ge \sigma, \\ x(t) & = & z(t), & t \le \sigma, \end{array}\right\} \tag{2.8}$$

where $H : R \to R$ denotes the Heaviside function, that is

$$H(t) = 1, \quad t > 0, \qquad \text{and} \qquad H(t) = 0, \quad t < 0.$$

The function z introduced above is continuous on $[\sigma - \tau, \infty)$ and differentiable whenever x is differentiable. Moreover, z satisfies

$$\left.\begin{array}{rcll} \dot{z}(t) & = & L(z_t) + \sum\limits_{\sigma \le t_j < t} L(T_{t_j} H_t)u_j, & t > \sigma, \\ z(t) & = & \phi(t - \sigma), & \sigma - \tau \le t \le \sigma, \end{array}\right\} \tag{2.9}$$

Here, the notation z_t has the same meaning as it is customary in delay equations, that is $z_t(s) = z(t+s)$, $-\tau \le s \le 0$. L denotes the map defined by the right hand side of Eq. (2.1), that is

$$L(z_t) = \sum_{j=1}^{N} a_j z_t(-\tau_j) = \sum_{j=1}^{N} a_j z(t - \tau_j),$$

moreover T_a is the shift operator ($a \in R$), that is for a function $f : R \to R$, $(T_a f)(t) = f(t - a)$, $t \in R$. Equation (2.9) is an inhomogeneous equation which can be written in the form

$$\dot{z}(t) = L(z_t) + f(t), \qquad t > \sigma,$$

where the inhomogeneous term $f(t) = \sum\limits_{\sigma \le t_j < t} L(T_{t_j} H_t)u_j$ is locally integrable on $[\sigma, \infty)$.

Recall ([10]) that the solution of the nonhomogeneous equation can be expressed in terms of the solution of the homogeneous equation and the inhomogeneous term:

$$z(t) = y(\phi)(t - \sigma) + \int_{\sigma}^{t} v(t - s) \sum_{\sigma \le t_j < s} L(T_{t_j} H_s)u_j \, ds, \qquad t \ge \sigma \tag{2.10}$$

We remark that in our case the inhomogeneous term is just locally integrable. But, it is known that the inhomogeneous equation makes sense and formula (2.10) is still valid in that case ([10]). By slightly rearranging the integrals, we obtain

$$z(t) = y(\phi)(t-\sigma) + \sum_{\sigma \le t_j < t} u_j \int_\sigma^t v(t-s) L(T_{t_j} H_s) ds, \qquad t > \sigma. \tag{2.11}$$

Substituting the right hand side of the above relation for z in formula (2.8) (part $t > \sigma$), we arrive at a first expression for x:

$$x(t) = y(\phi)(t-\sigma) + \sum_{\sigma \le t_j < t} \left[1 + \int_\sigma^t v(t-s) L(T_{t_j} H_s)\, ds \right] u_j, \qquad t > \sigma. \tag{2.12}$$

We will now get a relation involving the u_k's. For this, note that taking the limit from the left of formula (2.8) as t tends to $t_k > \sigma$, we have

$$x(t_k-) = z(t_k) + \sum_{\sigma \le t_j < t_k} u_j.$$

Taking formula (2.11) at $t = t_k$, we obtain

$$z(t_k) = y(\phi)(t_k-\sigma) + \sum_{\sigma \le t_j < t_k} u_j \int_\sigma^{t_k} v(t_k-s) L(T_{t_j} H_s)\, ds.$$

Substituting the right hand side of the latter expression for $z(t_k)$ in the expression of $x(t_k-)$, and using equation (2.2), we arrive at

$$\frac{u_k}{\beta_k} = y(\phi)(t_k-\sigma) + \sum_{\sigma \le t_j < t_k} u_j \left[1 + \int_\sigma^{t_k} v(t_k-s) L(T_{t_j} H_s)\, ds \right] \tag{2.13}$$

for $t_k > \sigma$. In particular, if $k = k(\sigma)$, that is the first index for which $t_k > \sigma$, then $\{j : \sigma \le t_j < t_{k(\sigma)}\} = \emptyset$, therefore

$$u_{k(\sigma)} = \beta_{k(\sigma)} y(\phi)(t_{k(\sigma)} - \sigma),$$

which yields formula (2.7). Now, we simplify both formulae (2.12) and (2.13). In fact

$$\int_\sigma^t v(t-s) L(T_{t_j} H_s) ds = \sum_{i=1}^N a_i \int_\sigma^t v(t-s) H(s - t_j - \tau_i) ds.$$

Let us consider any one of the integrals in the right hand side of the above relation:

$$\int_\sigma^t v(t-s) H(s-t_j-\tau_i) ds = \int_{t_j-\sigma+\tau_i}^{t-\sigma} v(t-\sigma-s) ds = \int_0^{t-t_j-\tau_i} v(s) ds = \int_0^{t-t_j} v(s-\tau_i) ds.$$

Note that the above formulae have been obtained from the ones before by changing the integration variables and using the facts that $v(t) = 0$ for $t < 0$ and H is the Heaviside function. Thus, we have

$$\int_{\sigma}^{t} v(t-s)L(T_{t_j}H_s)ds = \int_{0}^{t-t_j} L(v_s)ds. \tag{2.14}$$

But from the definition of the fundamental solution v one can easily see that

$$1 + \int_{0}^{t-t_j} L(v_s)ds = v(t-t_j), \qquad t > t_j > \sigma.$$

Substituting the above expression for the coefficients of the u_j's in (2.12) and (2.13), we arrive at formula (2.4) and (2.5). The proof of the theorem is complete.

Remark 2. *Although we restricted our study to the case of discrete delays, we believe that what we have done can be extended to the case of a general delay equation. In fact, the derivation of formulae (2.11) and (2.12) and (2.13) works out without any change. The only problem is in extending formula (2.14) from discrete to distributed delays.*

3. Stability

The stability issue for the impulse system (2.1) and (2.2) is examined in this section from the point of view of Eq. (2.1) being perturbed by the impulse equation (2.2). Assuming that Eq. (2.1) is asymptotically stable, that is, all the solutions of Eq. (2.1) tend to zero at $+\infty$, we set the following question: Under which additional condition is the impulse system (2.1)-(2.2) asymptotically stable, too? The partial answers given in this paper depend heavily upon somehow fine properties regarding equation (2.1). Let $B(t,\sigma)$ be defined by

$$B(t,\sigma) = \sum_{\sigma \le t_j < t} |v(t-t_j)|\,|\beta_j|, \qquad t > \sigma, \tag{3.1}$$

where v is the fundamental solution of Eq. (2.1).

Now, we state and prove our basic stability result.

Theorem 3.1. *Assume that Eq. (2.1) is asymptotically stable, that is all solutions of it tend to zero at $+\infty$.*

(a) If

$$\limsup_{t\to+\infty} B(t,0) < \infty \quad \textit{and} \quad \limsup_{k\to+\infty} B(t_k,0) < 1 \tag{3.2}$$

then for any $(\sigma, \phi) \in R \times C$ *the solution* $x(\sigma, \phi)$ *of problem (2.1)-(2.3) tends to zero at* $+\infty$.

(b) If

$$v(t) > 0, \quad \textit{and} \quad \beta_k > 0, \quad k \geq 1, \tag{3.3}$$

and there exists $T_0 \geq 0$ *and* $\sigma_0 \in R$ *such that*

$$B(t_k, \sigma_0) = \sum_{\sigma_0 \leq t_j \leq t_k} v(t_k - t_j)\beta_j \geq 1, \quad t_k \geq \sigma_0 + T_0, \tag{3.4}$$

then there exists at least one initial function $\phi_0 \in C$ *such that the solution* $x(\sigma_0, \phi_0)$ *of system (2.1)-(2.2) does not tend to zero at* $+\infty$.

We need the following lemma.

Lemma 1. *Assume that Eq. (2.1) is asymptotically stable.*

(α) If there exists a $\sigma_1 \in R$, *such that*

$$\sup_{t \geq \sigma_1} B(t, \sigma_1) < \infty,$$

then

$$A(\sigma) := \sup_{t \geq \sigma} B(t, \sigma) < \infty, \qquad \sigma \in R.$$

(β) For any $\sigma \in R$, *one has*

$$\overline{A}_0 := \limsup_{k \to +\infty} B(t_k, 0) = \limsup_{k \to +\infty} B(t_k, \sigma),$$

and

$$\underline{A}_0 := \liminf_{k \to +\infty} B(t_k, 0) = \liminf_{k \to +\infty} B(t_k, \sigma).$$

Proof. By the definition (3.1) of $B(t, \sigma)$, it is clear that

$$B(t, \sigma_2) = \sum_{\sigma_2 \leq t_j < \sigma_3} |v(t - t_j)|\,|\beta_j| + B(t, \sigma_3)$$

for any $-\infty < \sigma_2 < \sigma_3 < t < \infty$. But Eq. (2.1) is asymptotically stable and hence v is bounded on $[0, \infty)$ and it tends to zero at $+\infty$. Thus

$$\sup_{t \geq \sigma_2} \sum_{\sigma_2 \leq t_j < \sigma_3} |v(t - t_j)|\,|\beta_j| < \infty, \quad \text{and} \quad \sum_{\sigma_2 \leq t_j < \sigma_3} |v(t - t_j)|\,|\beta_j| \to 0 \text{ as } t \to +\infty,$$

which imply the statements of the lemma.

Proof of Theorem 3.1. Let $(\sigma, \phi) \in R \times C$ be fixed and consider the solution $x = x(\sigma, \phi)$ of problem (2.1)-(2.3). Then, we have that x satisfies (2.4) with (2.6), where $y(\phi)(t-\sigma)$ and $v(t)$ are the solutions of Eq. (2.1). But Eq. (2.1) is autonomous and asymptotically stable and, hence, there exists a constant $c_1 > 0$ independent on (σ, ϕ), such that

$$|y(\phi)(t-\sigma)| \leq c_1 \|\phi\|, \qquad t \geq \sigma, \tag{3.5}$$

where $\|\phi\| = \sup_{-\tau \leq s \leq 0} |\phi(s)|$. Moreover,

$$\lim_{t \to +\infty} |y(\phi)(t-\sigma)| = 0 \qquad \text{and} \qquad c_2 = \sup_{t \geq 0} |v(t)| < \infty.$$

On the other hand, by using (3.5) and Lemma 1, we may (and do) fix a constant $c \geq 0$ such that

$$A_0 := \sup_{u \in R} \left\{ \sup_{k \geq k(T(u))} B(t_k, u) \right\} < 1,$$

where $T(u) = max\{u, c\}$, $u \in R$, and $k(T(u))$ denotes the first index for which $t_k \geq T(u)$, for all $k \geq k(T(u))$. Define

$$M = \max \left\{ \frac{1}{1-A_0} \left(c_1 \|\phi\| + c_2 \sum_{k(\sigma) \leq j \leq k(T(\sigma))} |u_j| \right), \max_{k(\sigma) \leq k \leq k(T(\sigma))} \left| \frac{u_k}{\beta_k} \right| \right\}. \tag{3.6}$$

Then

$$\left| \frac{u_{k(\sigma)}}{\beta_{k(\sigma)}} \right| \leq M,$$

and by induction we show that

$$\left| \frac{u_j}{\beta_j} \right| \leq M \tag{3.7}$$

for all $j \geq k(\sigma)$. Indeed if (3.7) is satisfied for $k(\sigma) \leq j \leq i$, then (2.6) yields

$$\left| \frac{u_{i+1}}{\beta_{i+1}} \right| \leq c_1 \|\phi\| + \sum_{k(\sigma) \leq j \leq k(T(\sigma))} |v(t_{i+1} - t_j)||u_j| + \sum_{k(T(\sigma)) < j \leq i} |v(t_{i+1} - t_j)||\beta_j| \left| \frac{u_j}{\beta_j} \right| \leq$$

$$c_1 \|\phi\| + c_2 \sum_{k(\sigma) \leq j \leq k(T(\sigma))} |u_j| + A_0 M \leq M,$$

therefore (3.7) holds for all $j \geq k(\sigma)$. Thus

$$K = \limsup_{k \to +\infty} \left| \frac{u_k}{\beta_k} \right| < \infty.$$

Let $\varepsilon > 0$ be an arbitrary fixed number. Then there exists a $T_1 = T_1(\varepsilon) \geq 0$ such that

$$\left|\frac{u_j}{\beta_j}\right| < K + \varepsilon, \qquad t_j \geq \sigma + T_1.$$

On the other hand, under condition (3.2), by Lemma 1 we have

$$\overline{A}_0 = \limsup_{k\to+\infty} \sum_{T_1+\sigma\leq t_j<t_k} |v(t_k - t_j)|\,|\beta_j| < 1.$$

Thus, (2.6) yields

$$K = \limsup_{k\to+\infty} \left|\frac{u_k}{\beta_k}\right| \leq \limsup_{k\to+\infty} |y(\phi)(t_k - \sigma)| +$$

$$\limsup_{k\to+\infty} \sum_{\sigma\leq t_j<\sigma+T_1} |v(t_k - t_j)|\,|u_j| + \limsup_{k\to+\infty} \sum_{T_1+\sigma\leq t_j<t_k} |v(t_k - t_j)|\,|\beta_j| \left|\frac{u_j}{\beta_j}\right| \leq$$

$$(K+\varepsilon) \limsup_{k\to+\infty} \sum_{T_1+\sigma\leq t_j<t_k} |v(t_k - t_j)|\,|\beta_j| = \overline{A}_0\, K + \overline{A}_0\, \varepsilon.$$

Therefore

$$0 \leq K \leq \frac{\overline{A}_0}{1 - \overline{A}_0}\, \varepsilon,$$

where $\varepsilon > 0$ is an arbitrarily fixed number. This yields that

$$\lim_{k\to+\infty} \frac{u_k}{\beta_k} = 0.$$

Now we show that $\lim\limits_{t\to+\infty} x(t) = 0$. In fact, for a fixed $\varepsilon > 0$ there exists a $T_2 = T_2(\varepsilon) \geq 0$, such that

$$\left|\frac{u_j}{\beta_j}\right| < \varepsilon, \qquad t_j \geq \sigma + T_2.$$

In that case, using (2.4), we find

$$\limsup_{t\to+\infty} |x(t)| \leq \limsup_{t\to+\infty} |y(\phi)(t - \sigma)| +$$

$$\limsup_{t\to+\infty} \sum_{\sigma\leq t_j<\sigma+T_2} |v(t - t_j)|\,|u_j| + \limsup_{t\to+\infty} \sum_{\sigma+T_2\leq t_j<t} |v(t - t_j)|\,|\beta_j| \left|\frac{u_j}{\beta_j}\right| \leq$$

$$\varepsilon \limsup_{t\to+\infty} \sum_{\sigma+T_2\leq t_j<t} |v(t - t_j)|\,|\beta_j| \leq A(\sigma + T_2)\, \varepsilon,$$

where we used condition (3.2) and Lemma 1. Since $\varepsilon > 0$ was arbitrarily fixed, this yields $\lim_{t\to+\infty} x(t) = 0$ and the proof of statement (a) is complete.

(b) Let $\phi : [-\tau, 0] \to R$ be defined through the fundamental solution v of Eq. (2.1) with the relation

$$\phi(s) = v(s+\tau), \qquad -\tau \le s \le 0.$$

Then, ϕ is a continuous and positive initial function and $y(\phi)(t) = v(t+\tau)$, $t \ge 0$, is the unique solution of Eq. (2.1) through the initial point $(0, \phi)$. In virtue of (2.4) and (2.6), the solution $x = x(\sigma_0, \phi)$ of problem (2.1)-(2.3) can be written in the form

$$x(t) = v(t - \sigma_0 + \tau) + \sum_{\sigma_0 \le t_j < t} v(t - t_j) u_j, \qquad t > \sigma_0,\ t \notin \{t_k\}_{k \ge k(\sigma)}, \tag{3.8}$$

where the sequence $\{u_k\}$ satisfies the recurrence equation

$$\frac{u_k}{\beta_k} = v(t_k - \sigma_0 + \tau) + \sum_{\sigma_0 \le t_j < t_k} v(t_k - t_j)\, u_j, \qquad k \ge k(\sigma_0),$$

with initial value

$$u_{k(\sigma_0)} = \beta_{k(\sigma_0)}\, v(t_{k(\sigma_0)} - \sigma_0 + \tau) > 0.$$

Since $v(t) > 0$ for $t \ge 0$, from the recurrence equation it can be easily concluded that $u_k > 0$ for $k \ge k(\sigma_0)$, and hence (3.8) yields that $x(t) > 0$ for $t \ge \sigma_0 - \tau$. Define

$$m = \min_{k(\sigma_0) \le j \le k(\sigma_0 + T_0)} \left\{ \frac{u_j}{\beta_j} \right\},$$

where T_0 is given in condition (3.4). Then, $m > 0$, and an easy induction shows that $u_k \ge m\, \beta_k$, $k \ge k(\sigma_0)$. Thus,

$$x(t) \ge \sum_{\sigma_0 \le t_j < t} v(t - t_j)\, \beta_j \cdot m,$$

and hence by condition (3.4) we obtain $\limsup_{t\to+\infty} x(t) \ge m > 0$, which proves statement (b). The proof of Theorem 3.1 is complete.

The next result shows that under the conditions of statement (a) in Theorem 3.1, the zero solution of system (2.1)-(2.2) is uniformly stable on $[0, \infty)$, that is for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any $(\sigma, \phi) \in R_+ \times C$ the solution $x(\sigma, \phi)$ of problem (2.1)-(2.3) satisfies

$$|x(\sigma, \phi)(t)| < \varepsilon \quad \text{for} \quad t \ge \sigma, \quad \text{whenever} \quad \|\phi\| = \sup_{-\tau \le s \le 0} |\phi(s)| < \delta.$$

Proposition 1. *Assume that Eq. (2.1) is asymptotically stable and (3.2) is satisfied. Then, there exists a positive constant L such that for any $(\sigma, \phi) \in R_+ \times C$ the solution $x(\sigma, \phi)$ of problem (2.1)-(2.3) satisfies*

$$|x(\sigma, \phi)(t)| \leq L \cdot \|\phi\|, \qquad t \geq \sigma, \tag{3.9}$$

and, hence, the zero solution of system (2.1)-(2.2) is uniformly stable on $[0, \infty)$.

Proof. Let $(\sigma, \phi) \in R_+ \times C$ be fixed and consider the solution $x = x(\sigma, \phi)$ of problem (2.1)-(2.3). Then, using (2.7) and (3.5), we obtain

$$|u_{k(\sigma)}(\sigma, \phi)| \leq \beta_{k(\sigma)} c_1 |\phi\|,$$

that is, $u_{k(\sigma)} = u_{k(\sigma)}(\sigma, \phi)$ satisfies

$$\left| \frac{u_{k(\sigma)}}{\beta_{k(\sigma)}} \right| \leq c_1 \|\phi\|.$$

Combining (2.6), (3.5) and the latest estimation, it can be easily seen that

$$\left| \frac{u_k}{\beta_k} \right| \leq c_1 \|\phi\| + c_2 \sum_{k(\sigma) \leq j < k} |u_j| \leq c_3 \|\phi\|,$$

for $k(\sigma) \leq k \leq k(T(\sigma))$, where c_3 does not depend on σ. Thus, (3.6) and (3.7) yield

$$\left| \frac{u_j}{\beta_j} \right| \leq c_4 \|\phi\|, \qquad j \geq k(\sigma),$$

where the constant c_4 does not depend on σ. But, in that case, from (2.4) it follows

$$|x(\sigma, \phi)(t)| \leq c_1 \|\phi\| + c_2 c_4 \|\phi\| = L \|\phi\|, \quad \sigma \geq 0,$$

where $L = c_1 + c_2 c_4$ is independent of σ. The proof of the proposition is complete.

In our consideration, Eq. (2.1) is asymptotically stable. This implies that the exponential growth rate of the fundamental solution v of Eq. (2.1) is negative ([10]), more precisely, there exist constants $M \geq 1$ and $\omega > 0$, such that

$$|v(t)| \leq M e^{-\omega(t-\sigma)}, \qquad t \geq \sigma \geq 0. \tag{3.10}$$

By using this inequality, from Theorem 3.1 it follows

Theorem 3.2. *Assume that (3.10) is satisfied and*

$$\limsup_{j \to +\infty} |\beta_j| < \frac{1}{M} \left(\exp\left(\omega \liminf_{j \to +\infty} (t_{j+1} - t_j) \right) - 1 \right). \tag{3.11}$$

Then, for any $(\sigma, \phi) \in R \times C$ the solution $x(\sigma, \phi)$ of problem (2.1)-(2.3) tends to zero at $+\infty$.

Proof. From (3.11), it follows that there exist two constants $\overline{\beta}$ and T and an index j_0 such that

$$|\beta_j| \leq \overline{\beta} \qquad \text{and} \qquad t_{j+1} - t_j \geq T, \quad j \geq j_0,$$

and

$$\overline{\beta} < \frac{1}{M}(e^{\omega T} - 1).$$

Moreover, since $t_j \to +\infty$ $(j \to +\infty)$, we may (and do) fix a σ_0 such that $k(\sigma_0) \geq j_0$. Thus from (3.1) and (3.10), it follows

$$|B(t_k, \sigma_0)| \leq M \sum_{j=k(\sigma_0)}^{k-1} e^{-\omega(t_k - t_j)}|\beta_j| \leq M \sum_{j=k(\sigma_0)}^{k-1} \left(e^{-\omega T}\right)^{k-j} \overline{\beta} =$$

$$M\overline{\beta} \sum_{i=1}^{k-k(\sigma_0)} \left(e^{-\omega T}\right)^{i} = M\overline{\beta}e^{-\omega T}\left(\left(e^{-\omega T}\right)^{k-k(\sigma_0)} - 1\right)\frac{1}{1 - e^{-\omega T}},$$

for all $k > k(\sigma_0)$. Thus

$$\limsup_{k \to +\infty} |B(t_k, \sigma_0)| \leq M\overline{\beta}\frac{e^{-\omega T}}{e^{-\omega T} - 1} = M\overline{\beta}\frac{1}{e^{\omega T} - 1} < 1. \tag{3.12}$$

Moreover, for $t > \sigma_0$, one can find

$$|B(t, \sigma_0)| \leq M \sum_{\sigma_0 \leq t_j < t} e^{-\omega(t - t_j)}|\beta_j| \leq M \sum_{j=k(\sigma_0)}^{K(t)} e^{-\omega(t_{K(t)} - t_j)},$$

where $K(t)$ denotes the greatest index for which $t_k < t$, $k \geq K(t)$. Thus, (3.12) yields that $\limsup_{t \to +\infty} |B(t, \sigma_0)| < \infty$, and, hence, all the conditions of statement (a) in Theorem 3.1 are satisfied. The proof of the theorem is complete by using Theorem 3.1.

Remark 3. *Condition (3.11) is sharp in some special cases. Indeed, we show this considering the ordinary differential equation*

$$\dot{x}(t) = -ax(t), \qquad t \geq 0, \tag{3.13}$$

with impulse conditions

$$x(kT+) - x(kT-) = \beta x(kT-), \qquad k = 1, 2, \ldots, \tag{3.14}$$

where $a > 0$, $\beta > 0$ and $T > 0$ are given numbers. Now, the fundamental solution of (3.13) is

$$v(t) = e^{-at}, \qquad t \geq 0,$$

and hence condition (3.11) is equivalent to

$$\beta < e^{aT} - 1, \qquad (M = 1). \tag{3.15}$$

Therefore, if (3.15) is satisfied, then, any solution of system (3.13) and (3.14) tends to zero at $+\infty$. But, an easy computation shows that system (3.13) and (3.14) has a solution which does not tend to zero at $+\infty$, if (3.15) does not hold (compare this situation also to statement (b) of Theorem 2.1). This means that, in that case, condition (3.11) is sharp.

The main problem in verifying condition (3.11) is that we do not know precisely the values of M and ω, in general. Motivated by Remark 1, we now concentrate on the case when the fundamental solution v of Eq. (2.1) is positive on $[0, \infty)$. In that case, the characteristic equation of Eq. (2.1)

$$\lambda = \sum_{j=1}^{N} a_j e^{-\lambda \tau_j} \tag{3.16}$$

has a real root (see e.g. [8]). Moreover, by using the series representation of v ([4]), it can be easily concluded that Eq. (3.16) has a leading real root, say λ_1. This means that for any root $\overline{\lambda}$ of Eq. (3.16), one has $Re\overline{\lambda} \leq \lambda_1$. Moreover, Eq. (2.1) is asymptotically stable if and only if $\lambda_1 < 0$ (see e.g. [10]).

Lemma 2. *Assume that the fundamental solution v of Eq. (2.1) is positive on $[0, \infty)$ and the leading (real) root λ_1 of Eq. (3.16) satisfies*

$$q := \sum_{j=1}^{N} a_j^- \tau_j e^{-\lambda_1 \tau_j} < 1, \tag{3.17}$$

where $a_j^- = \max\{0, -a_j\}$. Then,

$$v(t) \leq \frac{1}{1-q} e^{\lambda_1 t}, \qquad t \geq 0. \tag{3.18}$$

Proof. Let

$$y(t) = v(t) e^{-\lambda_1 t}, \qquad t \geq -\tau.$$

Then, for any $t \geq 0$, we find

$$\dot{y}(t) = -\lambda_1 y(t) + \sum_{j=1}^{N} a_j e^{-\lambda_1 \tau_j} y(t - \tau_j) = -\sum_{j=1}^{N} a_j e^{-\lambda_1 \tau_j} [y(t) - y(t - \tau_j)]. \tag{3.19}$$

But, $y(t) = 0$, $-\tau \leq t < 0$, and hence, integrating both sides of the above equation we obtain

$$y(t) = 1 - \sum_{j=1}^{N} a_j e^{-\lambda_1 \tau_j} \int_0^t (y(s) - y(s - \tau_j))ds \leq 1 + \sum_{j=1}^{N} a_j^- e^{-\lambda_1 \tau_j} \int_{t-\tau_j}^t y(s)ds,$$

for $t \geq 0$. Moreover, $y(0) = 1$. If $q = 0$ then (3.18) holds. Let $q > 0$. Now, we show that

$$y(t) < \frac{1}{1-q},$$

for $t \geq 0$. Otherwise, since $y(t)$ is continuous, there exists a $T > 0$ such that

$$y(t) < \frac{1}{1-q}, \quad 0 \leq t < T, \quad \text{and} \quad y(T) = \frac{1}{1-q}.$$

In that case,

$$y(T) < 1 - \sum_{j=1}^{N} a_j^- e^{-\lambda_1 \tau_j} \tau_j \frac{1}{1-q} = \frac{1}{1-q},$$

which is a contradiction. Thus,

$$v(t)e^{-\lambda_1 t} = y(t) < \frac{1}{1-q}, \qquad t \geq 0,$$

and the proof of the lemma is complete.

The following result is an easy consequence of Theorem 3.2 and Lemma 2.

Theorem 3.3. *Assume that Eq. (2.1) is asymptotically stable, its fundamental solution v is positive on $[0, \infty)$ and (3.17) is satisfied. If*

$$\limsup_{j \to +\infty} |\beta_j| < (1-q) \left(\exp\left(-\lambda_1 \liminf_{j \to +\infty} (t_{j+1} - t_j) \right) - 1 \right)$$

then, the conclusion of Theorem 3.2 is valid.

Remark 4. *If $a_1 < 0$, $\tau_1 = 0$, $a_j \geq 0$, $2 \leq j \leq N$, then it can be easily seen that the fundamental solution v of Eq. (2.1) is positive on $[0, \infty)$. Moreover, Eq. (2.1) is asymptotically stable if and only if $\sum_{j=1}^{N} a_j < 0$. In that case $q = 0$ and the estimation (3.18) is the best possible one on $[0, \infty)$.*

Remark 5. *If $a_j \leq 0, 1 \leq j \leq N$, and $\sum_{j=1}^{N} a_j < 0$, then condition (3.17) is equivalent to that Eq. (3.16) does not have a double real root. Moreover, from (3.19) it can be easily concluded that $\dot{y}(t) > 0$, $t \geq 0$, and hence the limit $y(\infty) = \lim_{t\to+\infty} y(t)$ exists. In fact, $y(\infty)$ satisfies*

$$y(\infty) = 1 + \sum_{j=1}^{N} a_j^- e^{-\lambda_1 \tau_j} \tau_j y(\infty),$$

that is,

$$y(\infty) = \lim_{t\to+\infty} e^{-\lambda_1 t} v(t) = \frac{1}{1-q}.$$

This shows that in this case estimation (3.18) is sharp.

Theorems 3.2 and 3.3 can be used when the time elapsed between two consecutive impulses is uniformly taking away from zero, that is the impulse system (2.1)-(2.2) behaves as a perturbation of Eq. (2.1).

The next result gives condition for system (2.1) and (2.2) to be asymptotically stable under no extra assumption on $t_{j+1} - t_j$.

Proposition 2. *Assume that Eq. (2.1) is asymptotically stable and*

$$\sum_{j=1}^{\infty} |\beta_j| < \infty.$$

Then, for any $(\sigma, \phi) \in R \times C$ the solution $x(\sigma, \phi)$ of problem (2.1)-(2.3) tends to zero at $+\infty$.

Proof. Since Eq. (2.1) is asymptotically stable, we have

$$c = \sup_{0\leq t<\infty} |v(t)| < \infty.$$

Thus, there exists a $\sigma_0 \in R$ such that

$$B(t, \sigma_0) = \sum_{\sigma_0 \leq t_j < t} |v(t - t_j)|\, |\beta_j| \leq c \sum_{\sigma_0 \leq t_j < t} |\beta_j| \leq c \sum_{\sigma_0 \leq t_j} |\beta_j| < 1.$$

But, by using Lemma 1, we find that condition (3.1) is satisfied, in that case. Thus, our statement is an easy consequence of Theorem 3.1.

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