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Preservation of Stability in a Linear Neutral Differential Equation under Delay Perturbations

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ABSTRACT: In this paper we study preservation of stability under delay perturbation in a class of linear neutral delay differential equations.
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1. INTRODUCTION

Preservation of stability under perturbation, uncertain equations, robust stability has been studied by many authors for several classes of functional differential equations (see, e.g., [1],[3]–[6], [10],[11], [15]–[18] and [21]). Many of these papers deal with perturbation of the delays in retarded differential equations, however, to the best of our knowledge, there is no similar investigation done in this direction for neutral functional differential equations (NFDEs).

In this paper we study preservation of stability under delay perturbations of the vector NFDE

$$\frac{d}{dt}\Big(x(t) - Cx(t-\tau-\sigma(t))\Big) = \sum_{i=0}^{m} A_i x(t-r_i-\eta_i(t)). \tag{1.1}$$

We prove (see Theorem 4 below) that if the trivial solution of the "unperturbed" equation

$$\frac{d}{dt}\Big(x(t) - Cx(t-\tau)\Big) = \sum_{i=0}^{m} A_i x(t-r_i) \tag{1.2}$$

is asymptotically stable, then the same remains true for the trivial solution of (1.1), assuming that the delay perturbations are "small". This theorem extends our results of [10] and [11] where similar questions were studied for delay differential equations. As a special case of Theorem 4 we get that if $\lim_{t\to\infty}\sigma(t) = 0$ and $\lim_{t\to\infty}\eta_i(t) = 0$ for $i = 0, \dots, m$, then the asymptotic stability of the trivial solution of (1.2) implies that of Equation (1.1). This generalizes a result of Ladas et al. [14] from delay differential equations to NFDEs.

In the scalar case two standard conditions are given in the literature for the asymptotic stability of (1.2). Either

$$|C| + \sum_{i=0}^{m} |A_i| r_i < 1, \qquad \sum_{i=0}^{m} A_i > 0, \tag{1.3}$$

or, for $m = 0$,

$$2C(2 - C) + A_0 r_0 < \frac{3}{2}, \qquad C \in [0, 1/2), \quad A_0 > 0 \tag{1.4}$$

implies the asymptotic stability of the trivial solution of (1.2) (see, e.g., [7] and [19], respectively, and see [2] and [20] for the generalization of these results for different classes of NFDEs). In both cases the asymptotic stability is independent of the delay τ of the neutral term of (1.2). But there are equations (see, e..g, Example 3.1 below) where the stability of (1.2) depends on τ, as well. In this paper we investigate this more general case where we perturb the delay τ, as well. This introduces considerable technical difficulties to the problem.

Section 2 contains our perturbation theorems, and in Section 3 examples and applications are given. In particular, as an application of Theorem 4 we obtain stability theorems for NFDEs with time-dependent delays.

2. MAIN RESULTS

Consider the vector neutral differential equation

$$\frac{d}{dt}\Big(x(t) - Cx(t - \tau - \sigma(t))\Big) = \sum_{i=0}^{m} A_i x(t - r_i - \eta_i(t)), \qquad t \geq 0 \tag{2.1}$$

with initial condition

$$x(t) = \varphi(t), \qquad -r \leq t \leq 0, \tag{2.2}$$

where $x(\cdot) \in \mathbb{R}^q$, and $C, A_i \in \mathbb{R}^{q \times q}$. Let $|\cdot|$ denote a fixed vector norm on $\mathbb{R}^q$. The corresponding induced matrix norm will be denoted by $|\cdot|$, as well. Assume

(H1) $|C| < 1$;

(H2) $0 < \tau, \quad 0 \leq r_0 \leq r_1 \leq \ldots \leq r_m, \quad \max(\tau, r_m) < r$;

(H3) $\sigma\colon\ [0, \infty) \to \mathbb{R}$ and $\eta_i\colon\ [0, \infty) \to \mathbb{R}$ are continuous, and $0 \leq \tau + \sigma(t) \leq r$, $0 \leq r_i + \eta_i(t) \leq r$ for $t \geq 0$ $(i = 0, \ldots, m)$;

(H4) $\varphi\colon [-r, 0] \to \mathbb{R}^q$ is continuous.

We consider the corresponding unperturbed system with constant delays, i.e.,

$$\frac{d}{dt}\Big(y(t) - Cy(t - \tau)\Big) = \sum_{i=0}^{m} A_i y(t - r_i), \tag{2.3}$$

and we assume that

(H5) the trivial solution of (2.3) is asymptotically stable.

We can rewrite (2.1) in the form

$$\frac{d}{dt}\Big(x(t) - Cx(t-\tau) - g(t)\Big) = \sum_{i=0}^{m} A_i x(t-r_i) + f(t), \tag{2.4}$$

where

$$f(t) \equiv \sum_{i=0}^{m} A_i\Big(x(t-r_i-\eta_i(t)) - x(t-r_i)\Big) \tag{2.5}$$

and

$$g(t) \equiv C\Big(x(t-\tau-\sigma(t)) - x(t-\tau)\Big). \tag{2.6}$$

In this setting (2.4) can be considered as the homogeneous equation corresponding to (2.3). Let $T>0$, x be a solution of initial value problem (IVP) (2.1)-(2.2), and y_T be the solution of the homogeneous equation

$$\frac{d}{dt}\Big(y_T(t) - Cy_T(t-\tau)\Big) = \sum_{i=0}^{m} A_i y_T(t-r_i), \qquad t \geq T, \tag{2.7}$$

corresponding to the initial condition

$$y_T(t) = x(t), \qquad \text{for} \quad -r \leq t \leq T. \tag{2.8}$$

Assumption (H5) implies that $\lim_{t\to\infty} y_T(t) = 0$ for any $T>0$, since (2.7) is an autonomous equation. The variation-of-constants formula (see, e.g., [12]) gives the following expression for the solution of IVP (2.1)-(2.2):

$$x(t) = y_T(t) + g(t) - V(t-T)g(T) - \int_T^t g(s)\, d_sV(t-s) + \int_T^t V(t-s)f(s)\, ds, \quad t \geq T, \tag{2.9}$$

where $V(\cdot) \in \mathbb{R}^{q\times q}$ is the fundamental solution of (2.7), i.e., the solution of the following IVP

$$\frac{d}{dt}\Big(V(t) - CV(t-\tau)\Big) = \sum_{i=0}^{m} A_i V(t-r_i), \qquad t \geq 0, \tag{2.10}$$

and

$$V(t) = \begin{cases} I, & t = 0, \\ 0, & t < 0. \end{cases} \tag{2.11}$$

Here I and 0 denote the identity and the zero matrix, respectively. It is known (see, e.g., [12]) that V is absolutely continuous on the intervals $(k\tau,(k+1)\tau)$, $k=0,1,\ldots$, the right- and left-sided limit of V exist at each points $k\tau$, and $V(k\tau+) - V(k\tau-) = C^k$. Therefore (2.9) can be rewritten as

$$\begin{aligned} x(t) \;&=\; y_T(t) - V(t-T)g(T) + \sum_{k=0}^{\left[\frac{t-T}{\tau}\right]} C^k g(t-k\tau) + \int_T^t \dot V(t-s)g(s)\, ds \\ &\quad + \int_T^t V(t-s)f(s)\, ds, \qquad t \geq T. \end{aligned} \tag{2.12}$$

It is known (see, e.g., [12]) that if (H5) holds then V tends to zero exponentially, therefore $\int_0^\infty |V(s)|\,ds < \infty$. But then, by the next proposition, $\int_0^\infty |\dot V(s)|\,ds$ is finite, as well.

Proposition 1 *The fundamental solution of* (2.3) *satisfies*

$$\int_0^\infty |\dot V(s)|\,ds \le \frac{\sum_{i=0}^m |A_i|}{1-|C|}\int_0^\infty |V(s)|\,ds.$$

Proof The statement follows from the inequality

$$|\dot V(t)| \le |C||\dot V(t-\tau)| + \sum_{i=0}^m |A_i||V(t-r_i)|, \qquad \text{a.e. } t \ge 0$$

by integrating it from 0 to t, changing variables, and using that $V(t)=0$ and $\dot V(t)=0$ for $t<0$. □

For simplicity of the presentation we extend $\sigma(t)$ to $(-\infty,0)$ by $\sigma(t)=\sigma(0)$. Introduce the following sequence of functions

$$\alpha_0(t)\equiv t, \quad \alpha_1(t)\equiv t-\tau-\sigma(t), \quad \alpha_{j+1}(t)\equiv \alpha_1(\alpha_j(t)) \quad \text{for} \quad j=1,2,\ldots. \tag{2.13}$$

It is easy to see that

$$\alpha_j(t) = t - j\tau - \sum_{k=0}^{j-1}\sigma(\alpha_k(t)), \qquad j=1,2,\ldots. \tag{2.14}$$

Assumption (H3) yields that $0 \le \tau+\sigma(t) \le r$ for all t, therefore

$$t - jr \le \alpha_j(t) \le t \qquad \text{for} \quad t\ge 0 \quad \text{and} \quad j=0,1,\ldots.$$

In particular, if $t \ge T$, and

$$n = n(t) \equiv \left[\frac{t-T}{r}\right], \tag{2.15}$$

where $[\cdot]$ is the greatest integer part function, then for all $s \ge t$

$$T \le \alpha_j(s) \le s, \qquad j=0,1,\ldots,n, \qquad \text{and} \qquad T-r \le \alpha_{n+1}(s) \le s. \tag{2.16}$$

Assumption (H3) implies that $-\tau \le \sigma(t) \le r-\tau$, hence $|\sigma(t)| \le \max(r-\tau,\tau)$ for $t \ge 0$. Suppose

$$|\sigma(t)| \le b, \qquad \text{for} \quad t \ge T \tag{2.17}$$

for some nonnegative constants T and b. Let $T \le t_1 \le t_2$, $n=n(t_1)$ be defined by (2.15). Then it follows from (2.14) and (2.17) that

$$|\alpha_j(t_2)-\alpha_j(t_1)| \le |t_2-t_1| + j2b, \qquad j=0,1,\ldots,n. \tag{2.18}$$

The proof of our main result will be based on the following proposition.

Proposition 2 *Assume (H1)–(H4) and suppose* (2.17) *holds for some* $T \geq 0$ *and* $0 \leq b \leq r$. *Let* x *be the solution of IVP* (2.1)-(2.2), *and* $T \leq t_1 \leq t_2$, $t_2 - t_1 \leq r$. *Then there exists a monotone decreasing function* $h : [0, \infty) \to [0, \infty)$ *satisfying* $\lim_{u\to\infty} h(u) = 0$, *such that*

$$\begin{aligned} &|x(t_2) - x(t_1)| \\ &\quad \leq \left(2|C|^{\frac{t_1-T}{r}} + \frac{1}{1-|C|}\sum_{i=0}^{m}|A_i||t_2 - t_1| + \frac{2|C|b}{(1-|C|)^2}\sum_{i=0}^{m}|A_i|\right)\max_{T\leq u\leq t_2}|x(u)| \\ &\qquad + h(t_1 - T)\cdot \max_{T-r\leq u\leq T}|x(u)|. \end{aligned} \tag{2.19}$$

Proof Let $n = n(t_1)$ be defined by (2.15). Integrating (2.1) from t_1 to t_2, and applying the resulting relation n times, we get

$$\begin{aligned} &x(t_2) - x(t_1) \\ &\quad = C\Big(x(\alpha_1(t_2)) - x(\alpha_1(t_1))\Big) + \sum_{i=0}^{m} A_i \int_{t_1}^{t_2} x(s - r_i - \eta_i(s))\, ds \\ &\quad = C^{n+1}\Big(x(\alpha_{n+1}(t_2)) - x(\alpha_{n+1}(t_1))\Big) + \sum_{j=0}^{n} C^j \sum_{i=0}^{m} A_i \int_{\alpha_j(t_1)}^{\alpha_j(t_2)} x(s - r_i - \eta_i(s))\, ds. \end{aligned}$$

Therefore relations (2.16) and $T + r \leq \alpha_j(s)$ for $t_1 \leq s$ and $j = 0, 1, \ldots, n-1$ imply

$$\begin{aligned} &|x(t_2) - x(t_1)| \\ &\quad \leq 2|C|^{n+1}\max_{T-r\leq u\leq t_2}|x(u)| + \sum_{j=0}^{n-1}|C|^j\sum_{i=0}^{m}|A_i||\alpha_j(t_2) - \alpha_j(t_1)|\max_{T\leq u\leq t_2}|x(u)| \\ &\qquad + |C|^n\sum_{i=0}^{m}|A_i||\alpha_n(t_2) - \alpha_n(t_1)|\Big(\max_{T-r\leq u\leq T}|x(u)| + \max_{T\leq u\leq t_2}|x(u)|\Big). \end{aligned}$$

Then (2.17), (2.18), $\sum_{j=0}^{\infty}|C|^j = \frac{1}{1-|C|}$ and $\sum_{j=1}^{\infty} j|C|^j = \frac{|C|}{(1-|C|)^2}$ yield

$$\begin{aligned} &|x(t_2) - x(t_1)| \\ &\quad \leq 2|C|^{n+1}\max_{T-r\leq u\leq t_2}|x(u)| + \sum_{j=0}^{n}|C|^j\sum_{i=0}^{m}|A_i|(|t_2 - t_1| + j2b)\max_{T\leq u\leq t_2}|x(u)| \\ &\qquad + |C|^n\sum_{i=0}^{m}|A_i|(|t_2 - t_1| + n2b)\max_{T-r\leq u\leq T}|x(u)| \\ &\quad \leq \left(2|C|^{n+1} + |C|^n\sum_{i=0}^{m}|A_i|(|t_2 - t_1| + n2b)\right)\max_{T-r\leq u\leq T}|x(u)| \\ &\qquad + \left(2|C|^{n+1} + \frac{1}{1-|C|}\sum_{i=0}^{m}|A_i||t_2 - t_1| + \frac{2|C|b}{(1-|C|)^2}\sum_{i=0}^{m}|A_i|\right)\max_{T\leq u\leq t_2}|x(u)|. \end{aligned}$$

Hence the statement of the proposition follows from the inequality $\frac{t_1-T}{r} - 1 < n \leq \frac{t_1-T}{r}$ using the function

$$h(u) \equiv 2|C|^{\frac{u}{r}} + |C|^{\frac{u-r}{r}}\sum_{i=0}^{m}|A_i|(d + 2u), \tag{2.20}$$

where $d \geq r$ is selected such that h be monotone decreasing. $\square$

Proposition 3 *The solution, x, of (2.1)-(2.2) satisfies*

$$|x(t)| \leq \frac{1+|C|}{1-|C|}\|\varphi\| \exp\left(\frac{\sum_{i=0}^m |A_i|}{1-|C|}t\right), \qquad t \geq 0,$$

where $\|\varphi\| = \max\{|\varphi(t)|\colon\ t \in [-r, 0]\}$.

Proof Integrating (2.1) from 0 to t and applying simple estimates we get

$$|x(t)| \leq |C||x(t-\tau-\sigma(t))|+|\varphi(0)|+|C||\varphi(-\tau-\sigma(0))|+\sum_{i=0}^m |A_i| \int_0^t |x(s-r_i-\eta_i(s)|\,ds.$$

Therefore

$$|x(t)| \leq |C|w(t) + (1+|C|)\|\varphi\| + \sum_{i=0}^m |A_i| \int_0^t w(s)\,ds,$$

where $w(t) \equiv \max\{|x(u)| :\ -r \leq u \leq t\}$. The right-hand-side is monotone in t, therefore it implies

$$w(t) \leq |C|w(t) + (1+|C|)\|\varphi\| + \sum_{i=0}^m |A_i| \int_0^t w(s)\,ds,$$

and hence

$$w(t) \leq \frac{1+|C|}{1-|C|}\|\varphi\| + \frac{\sum_{i=0}^m |A_i|}{1-|C|} \int_0^t w(s)\,ds.$$

The statement of the proposition follows from Gronwall's inequality. $\square$

Theorem 4 *Assume (H1)–(H5), and suppose the delay perturbations satisfy*

$$K \overline{\lim_{u\to\infty}} |\sigma(u)| + \frac{a}{1-|C|} \int_0^\infty |V(s)|\,ds \left(\sum_{i=0}^m |A_i| \overline{\lim_{u\to\infty}} |\eta_i(u)|\right) < 1, \qquad (2.21)$$

where

$$K \equiv \frac{(1+|C|)|C|a}{(1-|C|)^3} + \frac{(1+|C|)|C|a}{(1-|C|)^2} \int_0^\infty |\dot{V}(s)|\,ds + \frac{2|C|a^2}{(1-|C|)^2} \int_0^\infty |V(s)|\,ds,$$

$a \equiv \sum_{i=0}^m |A_i|$, *and V is the fundamental solution of* (2.3). *Then the trivial solution of* (2.1) *is asymptotically stable.*

Proof It follows from the assumptions that there exists $\delta > 0$ such that

$$K(\overline{\lim_{u\to\infty}} |\sigma(u)| + \delta) + \frac{a}{1-|C|} \int_0^\infty |V(s)|\,ds \left(\sum_{i=0}^m |A_i|(\overline{\lim_{u\to\infty}} |\eta_i(u)| + \delta)\right) < 1, \quad (2.22)$$

and $\overline{\lim}_{u\to\infty}|\sigma(u)|+\delta \le r$. The last relation follows from the inequality $|\sigma(t)| < \max(r-\tau,\tau) < r$ for $t \ge 0$. To this δ we can choose $T \ge 0$ such that

$$|\sigma(t)| < \overline{\lim_{u\to\infty}}|\sigma(u)| + \delta, \quad |\eta_i(t)| < \overline{\lim_{u\to\infty}}|\eta_i(u)| + \delta, \qquad \text{for} \quad t \ge T, \quad i = 0,\dots,m. \tag{2.23}$$

Proposition 2 with $b \equiv \overline{\lim}_{u\to\infty}|\sigma(u)| + \delta \le r$, $t_2 \equiv \max(t-\tau-\sigma(t), t-\tau)$ and $t_1 \equiv \min(t-\tau-\sigma(t), t-\tau)$ implies for $t \ge T$ that

$$\begin{aligned}
|g(t)| &\le |C||x(t-\tau-\sigma(t)) - x(t-\tau)| \\
&\le |C|h(t_1 - T)\cdot \max_{T-r\le u\le T}|x(u)| \\
&\quad + \left(2|C|^{\frac{t_1-T}{r}+1} + \frac{(1+|C|)|C|}{(1-|C|)^2}a(\overline{\lim_{u\to\infty}}|\sigma(u)|+\delta)\right)\max_{T\le u\le t_2}|x(u)|.
\end{aligned}$$

For simplicity, we extend the function h to $t < 0$ by $h(t) = h(0)$. Then the monotonicity of h and the inequalities $t - r \le t_1 \le t_2 \le t$ imply

$$\begin{aligned}
|g(t)| &\le |C|h(t-T-r)\cdot \max_{T-r\le u\le T}|x(u)| \\
&\quad + \left(2|C|^{\frac{t-T}{r}} + \frac{(1+|C|)|C|}{(1-|C|)^2}a(\overline{\lim_{u\to\infty}}|\sigma(u)|+\delta)\right)\max_{T\le u\le t}|x(u)|
\end{aligned} \tag{2.24}$$

for $t \ge T$. Similarly,

$$\begin{aligned}
|f(t)| &\le ah(t-T-r)\max_{T-r\le u\le T}|x(u)| + 2a|C|^{\frac{t-T-r}{r}}\max_{T\le u\le t}|x(u)| \\
&\quad + \frac{a}{1-|C|}\sum_{i=0}^{m}|A_i|(\overline{\lim_{u\to\infty}}|\eta_i(u)|+\delta)\max_{T\le u\le t}|x(u)| \\
&\quad + \frac{2|C|a^2}{(1-|C|)^2}(\overline{\lim_{u\to\infty}}|\sigma(u)|+\delta)\max_{T\le u\le t}|x(u)|, \qquad t\ge T.
\end{aligned} \tag{2.25}$$

Next we show that the trivial solution of (2.1) is stable. For this it is enough to show that the solution x of (2.1)-(2.2) corresponding to an initial function satisfying $\|\varphi\| \le 1$ is bounded on $[0,\infty)$ by a constant independent of φ.

Let $t \ge T$ be fixed and let $p = p(t) \equiv \left[\frac{t-T}{\tau}\right]$. It follows from (2.12) that

$$\begin{aligned}
|x(t)| &\le |y_T(t)| + |V(t-T)||g(T)| + \sum_{k=0}^{p}|C|^k|g(t-k\tau)| \\
&\quad + \int_T^t |\dot V(t-s)||g(s)|\,ds + \int_T^t |V(t-s)||f(s)|\,ds.
\end{aligned} \tag{2.26}$$

We estimate the last three terms of (2.26) separately. An application of (2.24) yields

$$\begin{aligned}
&\sum_{k=0}^{p}|C|^k|g(t-k\tau)| \\
&\quad\le \sum_{k=0}^{p}|C|^{k+1}h(t-k\tau-T-r)\cdot\max_{T-r\le u\le T}|x(u)| + 2\sum_{k=0}^{p}|C|^{\frac{t-k\tau-T}{r}+k}\max_{T\le u\le t}|x(u)|
\end{aligned}$$

$$
\begin{aligned}
&+\ \frac{(1+|C|)|C|}{(1-|C|)^2}a(\overline{\lim_{u\to\infty}}|\sigma(u)|+\delta)\sum_{k=0}^{p}|C|^k\max_{T\le u\le t-k\tau}|x(u)| \\
\le&\ \sum_{k=0}^{p}|C|^{k+1}h(t-k\tau-T-r)\cdot\max_{T-r\le u\le T}|x(u)|+2\frac{|C|^{\frac{t-T}{r}}}{1-|C|^{1-\frac{\tau}{r}}}\max_{T\le u\le t}|x(u)| \\
&+\ \frac{(1+|C|)|C|}{(1-|C|)^3}a(\overline{\lim_{u\to\infty}}|\sigma(u)|+\delta)\max_{T\le u\le t}|x(u)|. \qquad (2.27)
\end{aligned}
$$

To estimate the first term of (2.27) we use the definition of h given by (2.20). We have

$$
\begin{aligned}
&\sum_{k=0}^{p}|C|^{k+1}h(t-k\tau-T-r) \\
&=\ 2\sum_{k=0}^{p}|C|^{\frac{t-k\tau-T}{r}+k}+a(d+2t-2r-2T)\sum_{k=0}^{p}|C|^{\frac{t-k\tau-T-r}{r}+k} \\
&\quad -\ 2a\tau\sum_{k=0}^{p}k|C|^{\frac{t-k\tau-T-r}{r}+k} \\
&\le\ \frac{2}{1-|C|^{1-\frac{\tau}{r}}}|C|^{\frac{t-T}{r}}+\frac{a(d+2t-2r-2T)}{1-|C|^{1-\frac{\tau}{r}}}|C|^{\frac{t-T-r}{r}}-\frac{2a\tau|C|^{1-\frac{\tau}{r}}}{(1-|C|^{1-\frac{\tau}{r}})^2}|C|^{\frac{t-T-r}{r}}.
\end{aligned}
$$

Define the function $\tilde{h}\colon [0,\infty)\to[0,\infty)$ by

$$
\tilde{h}(u)\equiv\frac{2}{1-|C|^{1-\frac{\tau}{r}}}|C|^{\frac{u}{r}}+\frac{a(\tilde{d}+2u-2r)}{1-|C|^{1-\frac{\tau}{r}}}|C|^{\frac{u-r}{r}}-\frac{2a\tau|C|^{1-\frac{\tau}{r}}}{(1-|C|^{1-\frac{\tau}{r}})^2}|C|^{\frac{u-r}{r}},
$$

where $\tilde{d}\ge d$ is selected so that $\tilde{h}$ be monotone decreasing. Then $\lim_{u\to\infty}\tilde{h}(u)=0$, and (2.27) yields

$$
\begin{aligned}
\sum_{k=0}^{p}|C|^k|g(t-k\tau)|\ \le&\ \tilde{h}(t-T)\cdot\max_{T-r\le u\le T}|x(u)|+2\frac{|C|^{\frac{t-T}{r}}}{1-|C|^{1-\frac{\tau}{r}}}\max_{T\le u\le t}|x(u)| \\
&+\ \frac{(1+|C|)|C|}{(1-|C|)^3}a(\overline{\lim_{u\to\infty}}|\sigma(u)|+\delta)\max_{T\le u\le t}|x(u)|. \qquad (2.28)
\end{aligned}
$$

Let $T<S$, where S will be specified later. It follows from Proposition 3 that there exist constants $X_S>0$ and $G_S>0$ such that and $|x(s)|\le X_S\|\varphi\|\le X_S$ and $g(s)\le G_S\|\varphi\|\le G_S$ for $s\le S$. Then (2.24) implies for $t\ge S$

$$
\begin{aligned}
&\int_T^t|\dot{V}(t-s)||g(s)|\,ds \\
&=\ \int_T^S|\dot{V}(t-s)||g(s)|\,ds+\int_S^t|\dot{V}(t-s)||g(s)|\,ds \\
&\le\ G_S\int_{t-S}^{t-T}|\dot{V}(s)|\,ds+|C|X_Sh(S-T-r)\int_S^t|\dot{V}(t-s)|\,ds \\
&\quad+\left(2|C|^{\frac{S-T}{r}}+\frac{(1+|C|)|C|}{(1-|C|)^2}a(\overline{\lim_{u\to\infty}}|\sigma(u)|+\delta)\right)\max_{T\le u\le t}|x(u)|\int_S^t|\dot{V}(t-s)|\,ds. \qquad (2.29)
\end{aligned}
$$

Similarly, there exists $F_S > 0$ such that $|f(s)| \le F_S\|\varphi\| \le F_S$ for $s \le S$. Then (2.25) implies for $t \ge S$

$$\begin{aligned}
&\int_T^t |V(t-s)||f(s)|\,ds \\
&\quad\le\ F_S \int_{t-S}^{t-T} |V(s)|\,ds + aX_S h(S-T-r)\int_S^t |V(t-s)|\,ds \\
&\qquad+\ \Bigg(2a|C|^{\frac{S-T-r}{r}} + \frac{a}{1-|C|}\sum_{i=0}^m |A_i|(\overline{\lim_{u\to\infty}}|\eta_i(u)|+\delta) \\
&\qquad\quad+\ \frac{2|C|a^2}{(1-|C|)^2}(\overline{\lim_{u\to\infty}}|\sigma(u)|+\delta)\Bigg)\cdot \max_{T\le u\le t}|x(u)|\int_S^t |V(t-s)|\,ds. \qquad (2.30)
\end{aligned}$$

Combining inequalities (2.26), (2.28), (2.29) and (2.30), we get for $T < S \le t$

$$\begin{aligned}
&|x(t)| \\
&\le\ |y_T(t)| + |V(t-T)||g(T)| + G_S\int_{t-S}^{t-T}|\dot V(s)|\,ds + F_S\int_{t-S}^{t-T}|V(s)|\,ds \\
&\quad+\Big(\tilde h(S-T) + h(S-T-r)\Big(|C|\int_0^\infty|\dot V(s)|\,ds + a\int_0^\infty |V(s)|\,ds\Big)\Big)X_S \\
&\quad+\Bigg(2\frac{|C|^{\frac{S-T}{r}}}{1-|C|^{1-\frac{\tau}{r}}} + 2|C|^{\frac{S-T}{r}}\int_0^\infty|\dot V(s)|\,ds + 2a|C|^{\frac{S-T-r}{r}}\int_0^\infty|V(s)|\,ds\Bigg)\max_{T\le u\le t}|x(u)| \\
&\quad+\ (\overline{\lim_{u\to\infty}}|\sigma(u)|+\delta)\Big(\frac{(1+|C|)|C|a}{(1-|C|)^3} + \frac{(1+|C|)|C|a}{(1-|C|)^2}\int_0^\infty|\dot V(s)|\,ds \\
&\qquad+\ \frac{2|C|a^2}{(1-|C|)^2}\int_0^\infty|V(s)|\,ds\Big)\max_{T\le u\le t}|x(u)| \\
&\quad+\ \frac{a}{1-|C|}\sum_{i=0}^m|A_i|(\overline{\lim_{u\to\infty}}|\eta_i(u)|+\delta)\max_{T\le u\le t}|x(u)|\int_0^\infty|V(s)|\,ds. \qquad (2.31)
\end{aligned}$$

Let M_δ denote the left-hand-side of inequality (2.22), and define the functions

$$\begin{aligned}
\alpha(t) &\equiv |y_T(t)| + |V(t-T)||g(T)| + G_S\int_{t-S}^{t-T}|\dot V(s)|\,ds + F_S\int_{t-S}^{t-T}|V(s)|\,ds, \\
\beta(u) &\equiv \tilde h(u-T) + |C|h(u-T-r)\int_0^\infty|\dot V(s)|\,ds + ah(u-T-r)\int_0^\infty|V(s)|\,ds,
\end{aligned}$$

and

$$\gamma(u) \equiv 2\frac{|C|^{\frac{u-T}{r}}}{1-|C|^{1-\frac{\tau}{r}}} + 2|C|^{\frac{u-T}{r}}\int_0^\infty|\dot V(s)|\,ds + 2a|C|^{\frac{u-T-r}{r}}\int_0^\infty|V(s)|\,ds.$$

Since $\lim_{u\to\infty}\gamma(u) = 0$, there exists $S > T$ such that $\gamma(S) < (1-M_\delta)/2$. With this S and the notations introduced above, (2.31) simplifies to

$$|x(t)| \ \le\ \alpha(t) + \beta(S)X_S + \frac{1+M_\delta}{2}\max_{T\le u\le t}|x(u)|, \qquad t \ge S. \qquad (2.32)$$

The assumptions imply that there exists a constant $\alpha_0 > 0$ such that $\alpha(t) \le \alpha_0$ for all $t \ge 0$, therefore (2.32) yields that

$$\max_{0\le u\le t} |x(u)| \le X_S + \alpha_0 + \beta(S)X_S + \frac{1+M_\delta}{2} \max_{0\le u\le t} |x(u)|, \qquad t \ge 0,$$

i.e.,

$$\max_{0\le u\le t} |x(u)| \le \frac{2(X_S + \alpha_0 + \beta(S)X_S)}{1-M_\delta}, \qquad t \ge 0.$$

This proves that the solution corresponding to any initial function $\|\varphi\| \le 1$ is bounded, i.e., the trivial solution of (2.1) is stable. In particular, we get that the constants F_S, G_S and X_S we used above can be selected independently of S.

Finally, we show that $\lim_{t\to\infty} x(t) = 0$ for any φ. We may assume that, in addition to (2.23), T satisfies

$$|x(t)| \le \overline{\lim_{u\to\infty}} |x(u)| + \delta, \qquad t \ge T. \tag{2.33}$$

Then using that $\lim_{t\to\infty} \alpha(t) = 0$, (2.32) implies

$$\overline{\lim_{u\to\infty}} |x(u)| \le \beta(S)X_S + \frac{1+M_\delta}{2}\Big(\overline{\lim_{u\to\infty}} |x(u)| + \delta\Big),$$

therefore

$$0 \le \overline{\lim_{u\to\infty}} |x(u)| \le \frac{2\beta(S)X_S}{1-M_\delta} + \delta\frac{1+M_\delta}{1-M_\delta}.$$

This implies $\overline{\lim\limits_{u\to\infty}} |x(u)| = 0$, since the right-hand-side can be arbitrary small, since δ and S can be chosen arbitrary small and arbitrary large, respectively. $\square$

In the special case when the delay perturbations tend to 0, the theorem has the following corollary, which extends a result of Ladas et at. [14] for neutral equations.

Corollary 5 *Assume (H1)–(H5), and*

$$\lim_{t\to\infty} \sigma(t) = 0, \qquad \text{and} \qquad \lim_{t\to\infty} \eta_i(t) = 0 \quad \text{for } i = 0, \dots, m.$$

Then the trivial solution of (2.1) *is asymptotically stable.*

Theorem 4 and Proposition 1 imply:

Corollary 6 *Assume (H1)–(H5), and the delay perturbations satisfy*

$$\Big(\frac{(1+|C|)|C|a}{(1-|C|)^3} + \frac{(3-|C|)|C|a^2}{(1-|C|)^3} \int_0^\infty |V(s)|\,ds\Big) \overline{\lim_{u\to\infty}} |\sigma(u)|$$
$$+ \frac{a}{1-|C|} \int_0^\infty |V(s)|\,ds \left(\sum_{i=0}^m |A_i| \overline{\lim_{u\to\infty}} |\eta_i(u)|\right) < 1,$$

where $a \equiv \sum_{i=0}^m |A_i|$, and V is the fundamental solution of (2.3). *Then the trivial solution of* (2.1) *is asymptotically stable.*

Proposition 7 *If the trivial solution of* (2.3) *is asymptotically stable, then the fundamental solution of* (2.3) *satisfies*

$$\left(\sum_{i=0}^{m} A_i\right)\int_0^\infty V(s)\,ds = -I \qquad \textit{and} \qquad (I-C)\int_0^\infty \dot V(s)\,ds = -I.$$

Proof By integrating (2.10) from 0 to $t>0$ we get

$$V(t) - CV(t-\tau) - V(0) + CV(-\tau) = \sum_{i=0}^{m} A_i \int_0^t V(s-r_i)\,ds.$$

A change of variables in the integrals and the assumed initial conditions $V(0)=I$ and $V(t)=0$ for $t<0$ yield

$$V(t) - CV(t-\tau) - I = \sum_{i=0}^{m} A_i \int_0^{t-r_i} V(s)\,ds,$$

which implies the first statement by taking the limit $t\to\infty$. The second relation follows from the identity

$$\int_0^T \dot V(s)\,ds = C\int_0^T \dot V(s-\tau)\,ds + \sum_{i=0}^{m} A_i \int_0^T V(s-r_i)\,ds.$$

$\square$

For a special class of scalar equations the previous result gives an explicit condition for stability. Consider the scalar neutral equation

$$\frac{d}{dt}\Big(x(t) - cx(t-\tau-\sigma(t))\Big) = -\sum_{i=0}^{m} a_i x(t-r_i-\eta_i(t)), \qquad t\ge 0 \tag{2.34}$$

and the corresponding unperturbed equation

$$\frac{d}{dt}\Big(y(t) - cy(t-\tau)\Big) = -\sum_{i=0}^{m} a_i x(t-r_i), \qquad t\ge 0. \tag{2.35}$$

Note that we used negative sign of the coefficients on the right-hand-side of the equations. If the fundamental solution of (2.35) is nonnegative, then Proposition 7 gives explicite value of the integral of the absolute value of the fundamental solution we used in our conditions before. The following result from [8] gives a condition guaranteeing the positiveness of the fundamental solution.

Proposition 8 (see Lemma 2.1 of [8]) *If* $0<c<1$, $a_i\ge 0$ $(i=0,\dots,m)$, *and the characteristic equation*

$$\lambda(1-ce^{-\lambda\tau}) = -\sum_{i=0}^{m} a_i e^{-\lambda r_i} \tag{2.36}$$

of (1.2) *has a real solution, then the fundamental solution* $v(t)$ *of* (2.35) *satisfies* $v(t)>0$ *for* $t\ge 0$, $\lim_{t\to\infty} v(t)=0$, *and* $\dot v(t)\le 0$ *for a.e.* $t>0$.

In the next proposition a simple explicit condition is given guaranteeing the existence of a real root of (2.36).

Proposition 9 *Assume* $0 < c < 1$, $a_i \geq 0$ $(i = 0, \ldots, m)$, *and*

$$ce^{\frac{\tau}{d}-1} + d\sum_{i=0}^{m} a_i \leq \frac{1}{e}, \qquad \text{where } d \equiv \max\{r_0, \ldots, r_m\}. \tag{2.37}$$

Then the fundamental solution $v(t)$ *of* (2.35) *satisfies* $v(t) > 0$ *for* $t \geq 0$, $\lim_{t\to\infty} v(t) = 0$, *and* $\dot{v}(t) \leq 0$ *for a.e.* $t > 0$.

Proof Let $p(\lambda) \equiv \lambda(1-ce^{-\lambda\tau})+\sum_{i=0}^{m} a_i e^{-\lambda r_i}$, then $p(0) > 0$. For $d \equiv \max\{r_0, \ldots, r_m\}$ we have

$$p\left(-\frac{1}{d}\right) \leq -\frac{1}{d}\left(1 - ce^{\frac{\tau}{d}}\right) + \left(\sum_{i=0}^{m} a_i\right) e.$$

Hence, if (2.37) holds, then the characteristic equation (2.36) has a root in the interval $[-1/d, 0)$. Therefore Proposition 8 implies the statement. □

Theorem 4, Propositions 7 and 9 imply immediately the next result.

Corollary 10 *Assume*

(i) $0 < c < 1$, $a_i \geq 0$ $(i = 0, \ldots, m)$,

(ii) $ce^{\frac{\tau}{d}-1} + d\sum_{i=0}^{m} a_i \leq \frac{1}{e}$, *where* $d \equiv \max\{r_0, \ldots, r_m\}$,

(iii) $$\frac{4c\sum_{i=0}^{m} a_i}{(1-c)^3} \overline{\lim_{u\to\infty}} |\sigma(u)| + \frac{1}{1-c}\sum_{i=0}^{m} a_i \overline{\lim_{u\to\infty}} |\eta_i(u)| < 1.$$

Then the trivial solution of (2.34) *is asymptotically stable.*

3. APPLICATIONS

In the case when $\int_0^\infty |v(t)|\, ds$ and $\int_0^\infty |\dot{v}(t)|\, ds$ can not be computed analytically the condition of Theorem 4 can be checked numerically, since it is easy to obtain good numerical approximations of these integrals. The next example illustrates this case.

Example 3.1 Consider the NFDE

$$\frac{d}{dt}\Big(x(t) - 0.5x(t - 2.4 - \sigma(t))\Big) = -x(t - 1 - \eta(t)), \qquad t \geq 0, \tag{3.1}$$

and its "unperturbed" equation

$$\frac{d}{dt}\Big(y(t) - 0.5y(t-2.4)\Big) = -y(t-1), \qquad t \geq 0. \tag{3.2}$$

We computed the numerical approximation of the fundamental solution v of (3.2) using the following numerical scheme introduced in [9] and [13]:

$$\begin{aligned} z(n+1) &= z(n) - 0.5z(n+1-[2.4/h]) + 0.5z(n-[2.4/h]) \\ &\quad - 0.5hz(n-[1/h]), \qquad n \geq 0 \\ z(0) &= 1, \\ z(n) &= 0, \qquad n < 0, \end{aligned}$$

where h is the stepsize of the numerical approximation, $[\cdot]$ is the greatest integer function, and $z(n)$ is the numerical approximation of the value of the fundamental solution at the mesh point nh. Figure 1 contains the corresponding result for $h = 0.01$. We can observe from this graph that the trivial solution of (3.2) is asymptotically stable. Note that numerical studies show that the asymptotic stability of (3.2) is lost if we increase the delay 2.4 of the neutral term to 3.3 or decrease it to 1. Therefore in this example the stability of the trivial solution of the equation depends on the delay of the neutral term.

We define a sequence $w(n)$ by the same definition as $z(n)$, except that $w(0) = 0$. Then this sequence approximates the left-sided limit of the value of the fundamental solution at mesh points. Using this two sequences and the trapesoidal-rule, we approximated

$$\int_0^\infty |v(t)|\, ds \approx \frac{h}{2} \sum_{i=0}^{N} (|z(i)| + |w(i+1)|),$$

where N is sufficiently large. We got $\int_0^\infty |v(t)|\, ds \approx 3.0306077$. Define the sequences $z'(n)$ and $w'(n)$ by

$$\begin{aligned} z'(n) &= -0.5z'(n-[2.4/h]) - z(n-[1/h]), \quad n \geq 0, \\ z'(n) &= 0, \qquad n < 0 \end{aligned}$$

and

$$\begin{aligned} w'(n) &= -0.5w'(n-[2.4/h]) - w(n-[1/h]), \quad n \geq 0, \\ w'(n) &= 0, \qquad n < 0, \end{aligned}$$

respectively. Then $z'(n)$ and $w'(n)$ approximate the right- and the left-sided derivatives of v at the mesh point nh. Then, similarly to the approximation of $\int_0^\infty |v(t)|\, ds$, we can get, using the sequences $z'(n)$ and $w'(n)$, that $\int_0^\infty |\dot{v}(t)|\, ds \approx 3.9210814$. Then Theorem 4 yields that if the delay perturbations σ and η satisfy

$$29.885675 \cdot \overline{\lim_{t\to\infty}} |\sigma(t)| + 6.0612154 \cdot \overline{\lim_{t\to\infty}} |\eta(t)| < 1,$$

then the trivial solution of (3.1) is asymptotically stable. Figure 2 contains the numerical approximation of (3.1) corresponding to the initial function of the fundamental solution and to delay perturbations $\sigma(t) = 0.1\sin(5t)$ and $\eta(t) = 0.4 + 3/(t+1)$.

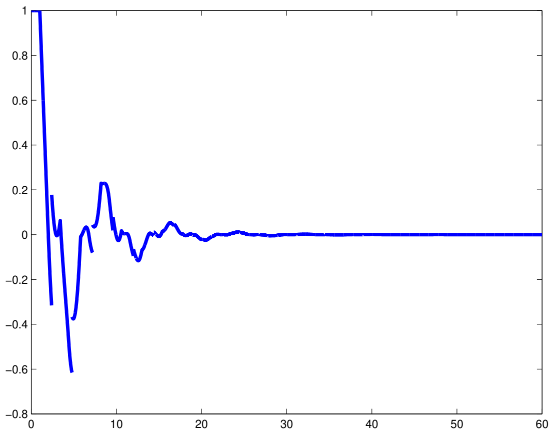

Figure 1: $v(t)$

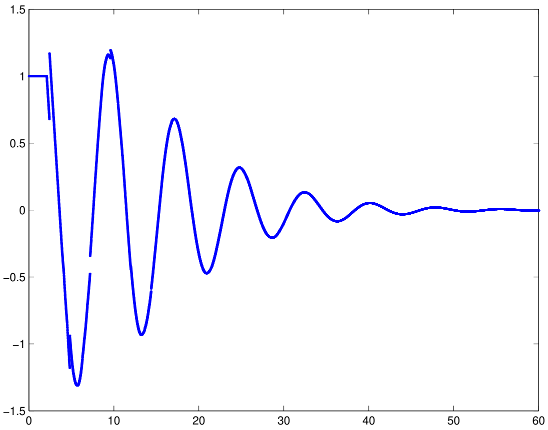

Figure 2: $x(t)$
$\sigma(t) = 0.01\sin(6t)$, $\eta(t) = 0.1 + 3/(t+1)$

As an application of the results of the previous section we present stability theorems for the scalar NFDE

$$\frac{d}{dt}\Big(x(t) - cx(t-\sigma(t))\Big) = -\sum_{i=0}^{m} a_i x(t-\eta_i(t)), \qquad t \geq 0, \tag{3.3}$$

where $0 \leq \sigma(t) \leq r$ and $0 \leq \eta_i(t) \leq r$ $(i = 0, \dots, m)$ for $t \geq 0$. Suppose $0 < c < 1$, $\sum_{i=0}^{m} a_i > 0$, let $1 < \alpha < \frac{1}{c}$ be arbitrary, and define

$$\tau \equiv \frac{(1-\alpha c)\ln\alpha}{\sum_{i=0}^{m} a_i}.$$

Then $\lambda \equiv \frac{-\sum_{i=0}^{m} a_i}{1-\alpha c} < 0$ is a root of the equation

$$\lambda(1 - ce^{-\lambda\tau}) = -\sum_{i=0}^{m} a_i,$$

therefore, by Proposition 8, the trivial solution of the neutral equation

$$\frac{d}{dt}\Big(y(t) - cy(t-\tau)\Big) = -\left(\sum_{i=0}^{m} a_i\right) y(t) \tag{3.4}$$

is asymptotically stable, and its fundamental solution, v, is positive and $\dot{v}(t) \leq 0$ for $t \geq 0$. Then if we consider (3.3) as the delay perturbed equation of (3.4), we get the following result.

Proposition 2 *Assume*

(i) $0 < c < 1, \quad \sum_{i=0}^{m} a_i > 0,\ 1 < \alpha < \frac{1}{c}$,

(ii) $\displaystyle \frac{4c}{(1-c)^3}\left(\sum_{i=0}^{m} a_i\right) \overline{\lim_{u\to\infty}} \left|\sigma(u) - \frac{(1-\alpha c)\ln\alpha}{\sum_{i=0}^{m} a_i}\right| + \frac{1}{1-c}\sum_{i=0}^{m} a_i \overline{\lim_{u\to\infty}} \eta_i(u) < 1.$

Then the trivial solution of (3.3) *is asymptotically stable.*

Finally we compare the stability of (3.3) to that of the equation

$$\frac{d}{dt}\Big(x(t) - cx(t-\tau)\Big) = -\left(\sum_{i=0}^{m} a_i\right) x(t-d), \qquad t \geq 0. \tag{3.5}$$

Corollary 10 implies immediately:

Corollary 3 *Assume*

(i) $0 < c < 1$, $a_i \geq 0$ $(i = 0, \ldots, m)$,

(ii) $ce^{\frac{\tau}{d}-1} + d\sum_{i=0}^{m} a_i \leq \frac{1}{e}$,

(iii) $$\frac{4c\sum_{i=0}^{m} a_i}{(1-c)^3} \overline{\lim_{u\to\infty}} |\sigma(u) - \tau| + \frac{1}{1-c}\sum_{i=0}^{m} a_i \overline{\lim_{u\to\infty}} |\eta_i(u) - d| < 1.$$

Then the trivial solution of (3.3) *is asymptotically stable.*

Example 3.4 Consider the NFDE

$$\frac{d}{dt}\Big(x(t) - 0.5x(t-0.05)\Big) = -0.5x(t-r), \qquad t \geq 0. \tag{3.6}$$

For $\tau = 0.05$ and $d = 0.25$ condition (ii) of Corollary 3 is satisfied. Therefore condition (iii) of the same corollary yields that if $|r - 0.25| < 1$ then (3.6) is asymptotically stable. This holds, e.g., for $r = 1$, as well, but the asymptotic stability of (3.6) in this case does not follow from conditions (1.3) or (1.4).

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