Qualitative properties of the solutions of a delay differential equation with impulses: II. Oscillations

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1. Introduction and preliminaries

Consider the system of equations

$$\dot{x}(t) = \sum_{j=1}^{N} a_j x(t - \tau_j) \tag{1.1}$$

$$x(t_k+) - x(t_k-) = \beta_k x(t_k-) \tag{1.2}$$

where Eq. (1.1) is a standard delay–differential equation and Eq. (1.2) takes impulses into account. Throughout the paper, we assume the delays

$$0 \le \tau_1 < \tau_2 < \ldots < \tau_N = \tau$$

* Research supported partially by Hungarian National Foundation for Scientific Research Grant No. T019846 and Hungarian Ministry of Culture and Education Grant No. 1024/97.

and the coefficients

$$a_j \leq 0, \qquad 1 \leq j \leq N,$$

are constants. If $N = 1$, we denote $a_N = a$. In general, Eq. (1.1) is associated with a pair $(\bar{a}, \bar{\tau})$, where $\bar{a} = (a_1, \ldots, a_N) \in R_-^N$ and $\bar{\tau} = (\tau_1, \ldots, \tau_N) \in R_+^N$. We also assume that the impulses are produced at given times

$$t_1 < t_2 < \ldots < t_k < \ldots \quad , \quad t_k \to +\infty \quad (k \to +\infty),$$

with $\beta_k \neq 0$, the impulse rate at time t_k.

We call Eq.(1.2) the impulse equation where, as usual, $x(t_k-) \quad (x(t_k+))$ denotes the limit from the left (from the right) of x, as t tends to t_k.

We define the solution of (1.1) and (1.2) in the following way:

Definition 1. *We say that a function x is a solution of system (1.1) and (1.2), if there exists $\sigma \in R$ such that x is defined on $[\sigma - \tau, \infty)$, piecewise continuous, and x is continuous on $[\sigma - \tau, \sigma]$. The function x is differentiable on the complement of a denumerable subset of $[\sigma, \infty)$ and verifies Eq. (1.1) at each point of $[\sigma, \infty)$ where both $\dot{x}$ and the right-hand side of Eq. (1.1) are defined. Moreover, x should be absolutely continuous with respect to the Lebesgue measure in each interval where it is continuous. Finally, x has to verify the impulse equation (1.2) at each point t_k, $t_k \geq \sigma$, $k \geq 1$.*

As it was made clear in the stability part of our paper [9] system (1.1) – (1.2) has solutions on any interval $[\sigma, \infty)$, $\sigma \in R$, and for any initial function $\varphi \in C([-\tau, 0], R)$ system (1.1) and (1.2) has exactly one solution with initial condition

$$x(s) = \varphi(s - \sigma), \qquad \sigma - \tau \leq s \leq \sigma.$$

By Theorem 2.1 in the stability section of our paper [9] we have that any solution $x : [\sigma - \tau, \infty) \to R, \quad \sigma \in R,$ of system (1.1) and (1.2) can be written as

$$x(t) = y(t - \sigma) + \sum_{\sigma \leq t_j < t} v(t - t_j) u_j \qquad t \geq 0, \quad t \notin \{t_k\}_{k \geq 1} \tag{1.3}$$

where $y : [-\tau, \infty) \to R$ is the continuous solution of the delay equation (without impulses) with initial condition

$$y(s) = x(s + \sigma), \quad -\tau \leq s \leq \sigma,$$

and

$$u_j = x(t_j+) - x(t_j-), \quad j \geq k(\sigma), \tag{1.4}$$

$k(\sigma)$ denotes the first index such that $t_i \geq \sigma, \quad i \geq k(\sigma)$. The jumps u_j are determined by the following nonautonomous recurrence equation

$$\frac{u_k}{\beta_k} = y(t_k - \sigma) + \sum_{\sigma \leq t_j < t_k} v(t_k - t_j) u_j \quad , k \geq k(\sigma), \tag{1.5}$$

starting from

$$u_{k(\sigma)} = \beta_{k(\sigma)} y(t_{k(\sigma)} - \sigma). \tag{1.6}$$

Our purpose here is to investigate as closely as possible the question of oscillation for system (1.1) and (1.2).

We recall that a continuous function is said to be oscillatory if and only if it has zeros on every interval $[t, \infty)$. For a function with discontinuities of the first kind, we have to extend slightly the definition.

Definition 2. *Let x be a function defined on interval $[\sigma, +\infty)$ such that x is piecewise continuous on its domain. Then x is said to be oscillatory, if for every $t \geq \sigma$ one can find $t' > t$ such that $x(t'-)x(t'+) \leq 0$.*

Of course this definition reduces to the usual one if x is continuous. On the other hand, if x is discontinuous, it covers situations when x does not take value zero at any point, but has zero as a left (or, right) limit or at certain points it jumps from a positive value to a negative one (or, the opposite).

We say that Eq. (1.1) (system (1.1)-(1.2)) is oscillatory if any solution of Eq. (1.1) (system (1.1) and (1.2)) is oscillatory.

Oscillation theory for linear delay differential equations has well established methods and results. In this direction we refer to the monographs [6], [3] and to the references cited here. A basic result for Eq. (1.1) (without impulses) is that Eq. (1.1) is oscillatory if and only if its characteristic equation

$$\Phi(\lambda) := \lambda - \sum_{j=1}^{N} a_j \, e^{-\lambda \tau_j} = 0 \tag{1.7}$$

has no real root (see, for example [3], page 33.). In the case $N = 1$ this is equivalent to

$$|a| \, \tau > \frac{1}{e}. \tag{1.8}$$

Since the function $\Phi(\lambda) - \lambda, \quad \lambda \in R$, is a convex function, Eq. (1.7) has at most two real roots. It can be easily seen that if (1.7) has two real roots, then they are simple and if it has one real root then it should be double. For our next considerations, the parameters $(\bar{a}, \bar{\tau})$ in Eq. (1.1) are assumed to be in the admissible parameter space

$$\mathcal{M} := \{(\bar{a}, \bar{\tau}) : 0 \leq \tau_1 < \tau_2 < \ldots < \tau_N, \quad a_i < 0 \ , 1 \leq i \leq N\}.$$

In that case $\mathcal{M}$ is the sum of the following three distinct subsets:

$$\mathcal{N}_- := \{(\bar{a}, \bar{\tau}) \in \mathcal{M} : \text{Eq. (1.7) has two (single) real roots}\},$$

$$\mathcal{N}_0 := \{(\bar{a}, \bar{\tau}) \in \mathcal{M} : \text{Eq. (1.7) has exactly one (double) real roots}\},$$

$$\mathcal{N}_+ := \{(\bar{a}, \bar{\tau}) \in \mathcal{M} : \text{Eq. (1.7) has no real roots}\}.$$

On occasion, for a set $U \subset \mathcal{M}$, we write that Eq. (1.1) is in region U if the admissible parameters $(\bar{a}, \bar{\tau})$ defining Eq. (1.1) are in U. From the above definitions it can be easily seen that Eq. (1.1) is not oscillatory if and only if Eq. (1.1) is in $\mathcal{N} := \mathcal{N}_- \cup \mathcal{N}_0$. Moreover, by using the continuous dependence of the roots of Eq. (1.1) on $(\bar{a}, \bar{\tau})$, one can easily conclude that the sets $\mathcal{N}_-$ and $\mathcal{N}_+$ are relatively open with respect to the admissible parameter space $\mathcal{M}$, while $\mathcal{N}_0$ is not open with respect $\mathcal{M}$. We call $\mathcal{N}_0$ the critical region for Eq. (1.1), and we will refer to Eq. (1.1) with parameters in $\mathcal{N}_0$ as a critical equation.

Let us now turn to the impulse system (1.1)-(1.2). Our main objective is to analyze the effect of impulses on oscillations: results range from the case when impulses equation to the case when they tend to slow down or even eliminate oscillations of an otherwise oscillatory equation.

In the case, when Eq. (1.1) is not critical and β_k 's tend to zero, system (1.1)-(1.2) appears essentially as a perturbation of Eq. (1.1) and they have the same oscillatory properties. The critical case is still not completely understood. A partial study of it is undertaken in the case $N = 1$, that is, for a single delay.

Note, if for an infinite number of integers k, $\beta_k \leq -1$, then system (1.1) and (1.2) is oscillatory. Immediate, since for each such k, we have $x(t_k+)x(t_k-) \leq 0$.

From now on, in this paper, we assume that

$$\beta_k \text{ is eventually } > \text{ -1.}$$

The paper is organized as follows. Section 2 gives two important lemmas related to the exponential representation of the nonoscillatory solutions of the impulse system (1.1) and (1.2). Section 3 deals with the oscillation problem in impulsive delay differential system (1.1) and (1.2) by using appropriate generalization of methods developed in the oscillation theory of delay differential equations without impulses (see, e.g. [5], [6], [4], [3]). Our interest in this subject was partially motivated by the preliminary oscillation result of Gopalsamy and Zhang [1] (see also [2]) which is given for a single delay equation under the strong condition that the delay is smaller than the length of the inter-impulse time intervals. In our approach, we developed a technique for several delay case which allows us to drop the above strong assumption on the length of the inter-impulse time intervals. We hope that our approach could give a starting point to the investigations of more general cases.

2. Exponential representation of positive solutions

In this section we give two lemmas related to the exponential representation of the positive solutions of system (1.1) and (1.2) which will play an important role in our considerations.

Let x be a solution of system (1.1) and (1.2) on an interval $[\sigma, \infty)$; x is continuous except possibly at points t_k and x is also absolutely continuous with respect to the Lebesgue measure on each interval $(t_k, t_{k+1}), \quad k \geq k(\sigma)$, where $k(\sigma)$ denotes the first index for which $t_k > \sigma, \quad k \geq k(\sigma)$.

Now, consider a nonoscillatory solution x of system (1.1) and (1.2). In that case, we may (and do) assume that for some $T \left(\notin \{t_k\}_{k\geq 1} \right), \quad x(t)$ is positive for $t \geq T$, $t \notin \{t_k\}_{k\geq 1}$, and x has positive limit from right and from left everywhere on $[t, +\infty)$. So, the function $\alpha : [T, \infty) \backslash \{t_k\}_{k \geq k(\sigma)} \to R$ defined by

$$\alpha(t) = -\frac{\dot{x}(t)}{x(t)}, \quad t_k < t < t_{k+1}, \quad k \geq k(T), \tag{2.1}$$

gives an exponential representation of x on the interval $(t_k, t_{k+1}), \quad k \geq k(T)$. For every $t \in \{t_k\}_{k\geq 1}, \quad t > t_1$, we denote $K(t)$ the integer such that

$$t_{l-1} < t < t_l \quad \text{for} \quad l = K(t). \tag{2.2}$$

Moreover, for each $s, t, \quad T < s \leq t, \quad s, t \notin \{t_k\}_{k \geq k(T)}$, we denote

$$C(s,t) = \begin{cases} \prod\limits_{i=K(s)}^{K(t)-1} \frac{1}{1+\beta_i}, & \text{if} \quad K(t) - 1 \geq K(s) \\ \\ 1, & \text{otherwise} \end{cases} \tag{2.3}$$

Note, since β_i is eventually greater than $-1, T$ can be chosen such that $C(t,s)$ is well defined for every $s, t, \quad T \leq s \leq t, \quad s, t \notin \{t_k\}_{k \geq k(T)}$.

Now, we state and prove the next lemma which gives the exponential representation for the eventually positive solutions of system (1.1)-(1.2).

Lemma 1. *Suppose x is a nonoscillatory solution of system (1.1)-(1.2) on an interval $[\sigma, \infty)$, say x is positive on an interval $[T, \infty)$, where $T \geq \sigma, \quad T \notin \{t_k\}_{k\geq 1}$ and for $i \geq k(T), \quad \beta_i > -1$. Then, we have*

$$\frac{x(s)}{x(t)} = \exp\left(\int_s^t \alpha(u) du \right) C(s,t), \tag{2.4}$$

for $T \leq s \leq t, \quad s, t \notin \{t_k\}_{k\geq 1}$.

Proof. Since $T \le s \le t$, it follows $K(s) \le K(t)$. If we have $K(t) < K(s)+1$, then in fact $K(t) = K(s)$, that is, s and t are in the same interval (t_{k-1}, t_k), $\quad k = K(t)$. Therefore, in that case,

$$\frac{x(s)}{x(t)} = \exp\left(\int_s^t \alpha(u)du\right),$$

and $C(t,s) = 1$, in view of formula (2.3). On the other hand if $K(t) \ge K(s)+1$, then interval $[s,t]$ contains a finite number of points t_j, namely $t_j \in [s,t]$ if $K(s)+1 \le j \le K(t)$. In that case,

$$\frac{x(s)}{x(t_{K(s)+1}-)} = \lim_{v \to t_{K(s)+1}-} \exp\left(\int_s^v \alpha(u)du\right) = \exp\left(\int_s^{t_{K(s)+1}} \alpha(u)du\right),$$

$$\frac{x(t_j+)}{x(t_{j+1}-)} = \lim_{\substack{v \to t_{j+1}+ \\ \xi \to t_j+}} \exp\left(\int_\xi^v \alpha(u)du\right) = \exp\left(\int_{t_j}^{t_{j+1}} \alpha(u)du\right),$$

and

$$\frac{x(t_{K(t)}+)}{x(t)} = \lim_{v \to t_{K(t)}+} \exp\left(\int_v^t \alpha(u)du\right) = \exp\left(\int_{t_{K(t)}}^t \alpha(u)du\right).$$

Since $x(t_j+) = (1+\beta_j)x(t_j-)$ and $1+\beta_j > 0$, we find

$$\frac{x(t_j-)}{x(t_{j+1}-)} = \frac{x(t_j-)x(t_j+)}{x(t_j+)x(t_{j+1}-)} = \frac{1}{1+\beta_j}\exp\left(\int_{t_j}^{t_{j+1}} \alpha(u)du\right),$$

and hence

$$\frac{x(t_{K(s)+1}-)}{x(t_{K(t)}+)} = \prod_{j=K(s)+1}^{K(t)-1} \frac{x(t_j-)x(t_{K(t)}-)}{x(t_{j+1}-)x(t_{K(t)}+)} = \prod_{j=K(s)+1}^{K(t)-1} \frac{1}{1+\beta_j}\exp\left(\int_{t_j}^{t_{j+1}} \alpha(u)du\right).$$

From this, it follows

$$\frac{x(s)}{x(t)} = \frac{x(s)}{x(t_{K(s)+1}-)}\frac{x(t_{K(s)+1}-)}{x(t_{K(t)}+)}\frac{x(t_{K(t)}+)}{x(t)} =$$

$$\exp\left(\int_s^{t_{K(s)+1}} \alpha(u)du\right) \prod_{j=K(s)+1}^{K(t)} \frac{1}{1+\beta_j} \exp\left(\int_{t_{K(s)+1}}^{t_{K(t)}} \alpha(u)du\right) \exp\left(\int_{t_{K(t)}}^t \alpha(u)du\right) =$$

$$\exp\left(\int_s^t \alpha(u)du\right) C(s,t),$$

which completes the proof of Lemma.

The following lemma is a known result and is stated without proof. See [4] and also Lemma 1.6.3 in [3].

Lemma 2. *Suppose that* $a \in C([T_1,\infty), R_+), \tau \in C(T_1,\infty), R_+)$,

$$\lim_{t\to+\infty}(t-\tau(t)) = +\infty,$$

and

$$\lim_{t\to+\infty}\inf\int_{t-\tau(t)}^{t} a(s)ds > 0.$$

Set $T = \inf_{t\geq T_1}\{t-\tau(t)\}$. *Assume that* $\beta \in C([T_1,\infty), R_+)$ *satisfies the inequality*

$$\beta(t) \geq a(t)\exp\left(\int_{t-\tau(t)}^{t}\beta(s)ds\right), \quad t \geq T_1.$$

Then

$$\lim_{t\to+\infty}\inf\int_{t-\tau(t)}^{t}\beta(s)ds < +\infty.$$

3. Main results

The following theorem is our first main result.

Theorem 3.1. *Consider the impulse system (1.1)-(1.2).*

(a) Suppose that the equation

$$\lambda = \sum_{j=1}^{N}|a_j|\,\underline{\gamma}(\tau_j)e^{\lambda\tau^j} \tag{3.1}$$

does not have any real root, where

$$\underline{\gamma}(\tau_j) = \lim_{t\to+\infty}\inf C(t-\tau_j,t), \quad 1\leq j\leq N. \tag{3.2}$$

Then the system (1.1)-(1.2) is oscillatory.

(b) Suppose that the equation

$$\lambda = \sum_{j=1}^{N} |a_j| \, \overline{\gamma}(\tau_j) e^{\lambda \tau^j} \tag{3.3}$$

has a positive real root, of multiplicity one, where

$$\overline{\gamma}(\tau_j) = \lim_{t \to +\infty} \sup C(t - \tau_j, t), \quad 1 \leq j \leq N. \tag{3.4}$$

Then the system (1.1)-(1.2) is nonoscillatory.

Proof. (a). For seek a contradiction, we suppose that system (1.1)-(1.2) has a positive solution x. In that case, we may (and do) assume that for some $T(\notin \{t_k\}_{k\geq 1}), x(t)$ is positive for $t \geq T$, $t \notin \{t_k\}_{k\geq 1}$, and x has positive limit from right and from left everywhere on $[T, \infty)$. Therefore by Lemma 1, we have the exponential representation (2.4) for x. Substituting α for x in (1.1), we arrive at the following

$$\alpha(t) = \sum_{j=1}^{N} |a_j| \exp \left(\int_{t-\tau_j}^{t} \alpha(s) ds \right) C(t - \tau_j, t) \tag{3.5}$$

using formula (2.4). This formula holds for $t \geq T_0$ such that t and the values $t - \tau_j$, $j = 1, ..., N$, do not belong to the set $\{t_k\}_{k\geq 1}$.

For each $j = 1, ..., N$, we can select a number $\underline{\gamma}_j^*$ such that

$$0 \leq \underline{\gamma}_j^* \leq \underline{\gamma}(\tau_j)$$

and $C(t - \tau_j, t) \geq \underline{\gamma}_j^*$ for all t large enough. Moreover, $\underline{\gamma}_j^*$ can be chosen such that the equation

$$\lambda = \sum_{j=1}^{N} |a_j| \, \underline{\gamma}_j^* e^{\lambda \tau^j} \tag{3.6}$$

does not have real root. This follows from the assumption that Eq. (3.1) does not have real root. Now, equation (3.4) can be replaced by an inequality in terms of the $\underline{\gamma}_j^{*\prime} s$:

$$\alpha(t) \geq \sum_{j=1}^{N} |a_j| \exp \left(\int_{t-\tau_j}^{t} \alpha(s) ds \right) \underline{\gamma}_j^*, \tag{3.7}$$

for almost every t large enough.

But Eq. (3.6) does not have any real root, and hence there exists an index $i_0 \in \{1, ..., N\}$, such that $\underline{\gamma}_{i_0}^* \cdot |a_{i_0}| \, \tau_{i_0} > 0$, moreover

$$\alpha(t) \geq |a_{i_0}| \, \underline{\gamma}_{i_0}^* \exp \left(\int_{t-\tau_{i_0}}^{t} \alpha(s) ds \right)$$

for almost every t large enough. Let T_1 be fixed such a way that α satisfies (3.7) for almost every $t \geq T_1 - 2\tau_{i_0}$. Set

$$\beta(t) = \int_{t-\tau_{i_0}}^{t} \alpha(s)ds, \qquad t \geq T_1 - \tau_{i_0}.$$

Then β is continuous on $[T_1 - \tau_{i_0}, \infty)$. Moreover

$$\beta(t) \geq |a_{i_0}| \underline{\gamma}^*_{i_0} \int_{t-\tau_{i_0}}^{t} \exp(\beta(u))du, \qquad t \geq T_1,$$

which by convexity can be reduced to

$$\beta(t) \geq |a_{i_0}| \underline{\gamma}^*_{i_0} \exp\left(\frac{1}{\tau_{i_0}} \int_{t-\tau_{i_0}}^{t} \beta(u)du\right), \qquad t \geq T_1.$$

Lemma 2 applies readily to the above inequality and yields the fact that

$$\lim_{t\to+\infty} \inf \beta(t) = \lim_{t\to+\infty} \inf \int_{t-\tau_{i_0}}^{t} \alpha(s)ds < \infty.$$

This implies that $m := \lim_{t\to+\infty} \inf \alpha(t) < \infty$, and thus there exists a sequence $\{\sigma_n\}_{n\geq 1}$ such that $\sigma_n \to \infty$ and $\alpha(\sigma_n) \to m$, as $n \to +\infty$. Using inequality (3.7), for $t = \sigma_n$, we obtain

$$m \geq \sum_{j=1}^{N} |a_j| \underline{\gamma}^*_j e^{m\tau_j}.$$

This yields that Eq. (3.6) has a real root, in contradiction with our assumption. The proof of (a) is complete

(b) By assumption, Eq. (3.3) has a real root of multiplicity one. So, for each $j = 1, ..., N$, we can select numbers $\overline{\gamma}^*_j$ such that $\overline{\gamma}^*_j < \overline{\gamma}(\tau_j)$ and equation

$$\lambda = \sum_{j=1}^{N} |a_j| \overline{\gamma}^*_j e^{\lambda\tau_j} \tag{3.8}$$

has a real root, moreover $C(t - \tau_j, t) < \overline{\gamma}^*_j$ for all t large enough. Let λ^* denote the largest real root of (3.8).

We will build a fixed point of (3.5), which will define a positive solution of impulsive system (1.1) and (1.2). The solution of Eq. (3.5) will be seeked on an interval $[\sigma - \tau_N, \infty)$, where σ is suitably large. We start from

$$\alpha_0 = 0$$

and look for α_n such that

$$\alpha_n(t) = 0, \quad \sigma - \tau_N \leq t \leq \sigma,$$

$$\alpha_n(t) = \sum_{j=1}^{N} |a_j| \exp\left(\int_{t-\tau_j}^{t} \alpha_{n-1}(s)ds \right) C(t-\tau_j, t) \tag{3.9}$$

for $t > \sigma$ and $n \geq 1$.
Obviously, $0 \leq \alpha_0(t) \leq \lambda^*, \quad t \geq \sigma - \tau_N$, and by induction it can be easily seen that

$$\alpha_0(t) \leq \alpha_1(t) \leq ... \leq \alpha_n(t) \leq ... \leq \lambda^*$$

for $t \geq \sigma - \tau_N$. So, the sequence α_n converges pointwise to a function we denote α^*. But the functions α_n are pointwise continuous on $[\sigma - \tau_N, \infty)$ and also they are uniformly bounded on $[\sigma - \tau_N, \infty)$. Thus we conclude that the integrals $\int_{t-\tau_j}^{t} \alpha_n(s)ds$ converge to $\int_{t-\tau_j}^{t} \alpha^*(s)ds$ for any fixed $t \geq \sigma$. Thus α^* is bounded on $[\sigma-\tau_N, \infty)$, locally integrable and it satisfies Eq. (3.5) for almost every $t \geq \sigma$. But for each $j = 1, ..., N$, the function $\int_{t-\tau_j}^{t} \alpha^*(s)ds$ is continuous on $[\sigma, \infty)$, therefore (3.5) shows that the function α^* have discontinuities of first kind contained in the union of the points of discontinuity of the functions $C(t-\tau_j, t)$.

Set

$$x(t) = \exp\left(- \int_{\sigma-\tau_N}^{t} \alpha^*(u)du \right) \frac{1}{C(\sigma - \tau_N, t)}.$$

Then $x(t)$ is positive, piecewise continuous, with jumps concentrated at points $t = t_k, t_k > \sigma$. x is absolutely continuous with respect to the Lebesgue measure in each interval (t_k, t_{k+1}) and it verifies Eq. (1.1) inside the interval. Finally, the jumps from $x(t_k-) \quad$ to $\quad x(t_k+)$ are given by $(1+\beta_k)x(t_k-)$, that is x verifies Eq. (1.2). The proof of the theorem is complete.

Remark 1. *What we really proved in part (b) of Theorem 3.1 is that if we solve Eq. (3.5) from an initial time σ large enough with initial function $\alpha = 0$ on $[\sigma - \tau_N, \infty)$,then the solution of system (1.1)-(1.2) generated by the solution of Eq. (3.5) is not oscillating. The same result holds if we start from any initial function α less than or equal to the largest real root of Eq. (3.3). This means that the positivity holds for a fairly large set of the solutions of system (1.1) and (1.2).*

Remark 2. *Note that no assumption was made regarding the distribution of t_k 's, the except for the fact that $t_k \to +\infty$, as $k \to +\infty$. In particular, it might happen that $t_{k+1} - t_k$ has zero as a* $\liminf$ *and/or $+\infty$ as a* $\limsup$.

Remark 3. *The assumption in part (a) of Theorem 1 could include notably the case when one of the* $\underline{\gamma}(\tau_j)$ *'s is equal to* $+\infty$*. By extension, we consider it as a situation where Eq. (3.1) does not have any real root. The proof works the same.*

The following corollary gives a sharp result when Eq. (1.1) is not critical, that is, the admissible parameters $(\overline{a}, \overline{\tau})$ defining (1.1) are in $\mathcal{N}_- \cup \mathcal{N}_+$.

Corollary 1. *Suppose that Eq. (1.1) is not critical and* $\underline{\gamma}(\tau_j) = \overline{\gamma}(\tau_j) = 1$ *for* $j = 1, ..., N$*. Then system (1.1) and (1.2) is oscillatory if and only if Eq. (1.1) is oscillatory.*

Proof. It is immediate from Theorem 1.

Remark 4. *For each* $1 \leq j \leq N$, $\underline{\gamma}(\tau_j) = \overline{\gamma}(\tau_j)$ *holds, if, in particular,* $t_{k+1} - t_k \geq \tau_* > 0, \quad k \geq 1$*, and the* β_k *'s tend to zero as* $k \to +\infty$*. Indeed, in that case, there are at most* $[\frac{\tau}{\tau_*}] + 1$ *points* t_k *in each interval* $[t - \tau, t]$*. With* $\beta_k \to 0$ *as* $k \to +\infty$*, we conclude that* $C(t - \tau, t)$ *tends to 1, as* $t \to +\infty$*. Therefore by formula (2.3), we conclude that* $\underline{\gamma}(\tau) = \overline{\gamma}(\tau) = 1$ *for each* $\tau \geq 0$*.*

Our oscillatory results, both in Theorem 3.1 and its corollary, are obtained under more restrictive conditions than in the nonimpulsive case. For example, in Theorem 3.1, Equation (3.3) is supposed to have a positive real root of multiplicity one; in Corollary 1, it is assumed that Eq. (1.1) is not critical. Such restrictions can be eliminated if additional restrictions are imposed on the impulse equation. Thus, the proof of Theorem 3.1 can be readily adapted to yield the following

Corollary 2. *Suppose that for each* $j = 1, ..., N, \quad \overline{\gamma}(\tau_j) \geq C(t - \tau_j, t) \quad$ *for* t *large enough, and Eq. (3.3) has a positive root. Then system (1.1) and (1.2) is nonoscillatory. If, in particular,* $t_{k+1} - t_k \geq \tau_* > 0$ *for* k *large enough and the sequence* β_k *tends to zero, with* $\beta_k \geq 0$ *for* k *large enough, then system (1.1) and (1.2) is nonoscillatory if Eq. (1.1) is nonoscillatory.*

Now, we turn our attention to the case when Eq. (1.1) is critical, that is, its characteristic equation has precisely one (double) real root. We also assume that β_k 's are eventually negative. Let us first draw a few basic conditions to be verified for system (1.1) and (1.2) to have positive solution in this case. If x is a positive solution of system (1.1) and (1.2) on an interval $[\sigma, \infty)$ then from $u_k = x(t_k+) - x(t_k-) = \beta_k x(t_k-)$, it follows that $u_k < 0 \quad$ for $\quad k \geq k(\sigma)$. But x satisfies (1.3), where y is a solution of Eq. (1.1). Since Eq. (1.1) is nonoscillatory and $a_i < 0$, we have that v is positive on $[0, \infty)$ and it behaves asymptotically as $te^{\omega t}$ at infinity, that is, $\lim_{t \to +\infty} x(t) \frac{1}{t} e^{-\omega t} = C_0$ exists, $C_0 > 0$, where ω is the unique

real root of Eq. (1.7) (see, e.g. [5]). Moreover, any eventually positive/negative solution of Eq. (1.1) behaves as $e^{\omega t}$ or either as $te^{\omega t}$ at $+\infty$.

On the other hand, in that case, (1.3) implies

$$x(t) \leq y(t-\sigma), \quad t \geq \sigma,$$

which means that $y(t)$ is positive. Therefore it behaves asymptotically either as $e^{\omega t}$ or as $te^{\omega t}$ at $+\infty$. But, from (1.3), we can deduce that

$$x(t) \leq y(t-\sigma) + \sum_{\sigma \leq t_j \leq t_\ell} y(t-t_j)u_j \tag{3.10}$$

where ℓ is any fixed integer such that $\ell > k(\sigma)$. The above inequality shows that assuming that $y(t)$ behaves asymptotically as $e^{\omega t}$ at $+\infty$ implies that x is eventually negative (since we know that v behaves asymptotically as $te^{\omega t}$ at $+\infty$). Therefore, if y is a "continuous" part of a nonoscillatory solution of system (1.1) and (1.2), we have that y is eventually positive, decreasing and

$$\lim_{t \to +\infty} y(t) \frac{1}{t} e^{-\omega t} = c, \quad c > 0,$$

that is y behaves asymptotically as $te^{\omega t}$ at $+\infty$. Let us now turn our attention to the jumps u_k, in this case. Under the above assumptions we have that the sequence $\{u_k\}_{k \geq k(\sigma)}$ is a sequence of negative numbers verifying equation (1.5). Thus, we obtain from it that

$$|u_k| \leq |\beta_k|\, y(t_k - \sigma), \quad k \geq k(\sigma). \tag{3.11}$$

Note, the sequence $\{u_k\}$ is uniquely determined in terms of the $y(t_k - \sigma)$'s and once we know it, a unique solution x is determined in terms of y and u_j 's, according to Eq. (1.3). So, assuming that y yields a negative sequence $\{u_k\}$, the question reduces to see whether x, as given by (1.3), is positive.

Now we are in a situation to state and prove our second main result in this paper.

Theorem 3.2. *Assume that Eq. (1.1) is critical, i.e. its characteristic equation has exactly one (double) real root and the impulse equation (1.2) is such that*

$$\beta_k \leq 0 \quad \textit{for} \quad k \geq 0, \quad \textit{and} \quad \sum_{k=1}^{\infty} |\beta_k|\, t_k < +\infty. \tag{3.12}$$

Then the system (1.1) and (1.2) is nonoscillatory.

Proof. To prove our theorem, we should show that there exists a σ such that system (1.1) and (1.2) with initial time σ has at least one nonoscillatory solution on $[\sigma, \infty)$.

We first solve Eq. (1.5) for a negative sequence, for a suitable solution y of Eq. (1.1). From the previous discussion, y should be taken so that it behaves asymptotically as $te^{\omega t}$, as $t \to +\infty$, where ω is the unique real root of (1.7). Let us choose a solution y of Eq. (1.1) verifying the following estimates

$$0 < C_1(1+t)e^{\omega t} \leq y(t) \leq C_2(1+t)e^{\omega t}, \quad t \geq 0, \tag{3.13}$$

where $0 < C_1 \leq C_2 < \infty$. From the asymptotic representation of the fundamental solution v of Eq. (1.1), we also have $0 < v(t) \leq K(1+t)e^{\omega t}, \quad t \geq 0$, for some constant $K > 0$.
Let σ be fixed such that $\sigma \notin \{t_k\}_{k\geq 1}$. Then we have

$$u_{k(\sigma)} = \beta_{k(\sigma)} u(t_{k(\sigma)} - \sigma) < 0.$$

Suppose that all the u_j 's up to $j = k-1 \quad (k > k(\sigma))$ are negative. In that case, formula (1.5) implies that

$$|u_j| \leq \left|\beta_j\right| y(t_j - \sigma) \quad k(\sigma) \leq j \leq k-1.$$

Substituting the right hand side of the above expression for u_j in (1.5) leads to the following inequality

$$\frac{u_k}{\beta_k} \geq y(t_k - \sigma) - \sum_{\sigma \leq t_j \leq t_k} v(t_k - t_j) y(t_j - \sigma) \left|\beta_j\right|$$

Therefore, if we show that

$$y(t_k - \sigma) < \sum_{\sigma \leq t_j < t_k} v(t_k - t_j) y(t_j - \sigma) \left|\beta_j\right| \tag{3.14}$$

then $u_k < 0$. But, using the estimates in (3.13), a sufficient condition for (3.14) to hold is that

$$C_1(1 + t_k - \sigma) > \sum_{\sigma \leq t_j \leq t_k} KC_2(1 + t_k - t_j)(1 + t_j - \sigma) \left|\beta_j\right|,$$

which in turn is implied by the next one

$$C_1 > \sum_{\sigma \leq t_j \leq t_k} KC_2(1 + t_j - \sigma) \left|\beta_j\right|. \tag{3.15}$$

But condition (3.12) holds, and hence $\sum_{j=1}^{\infty} \left|\beta_j\right| < \infty$. Thus the right hand side of (3.15) is the tail of a convergent series, therefore, it can be made as small as desired proved that σ is taken large enough. With such a choice the u_k 's are negative for all $k \geq k(\sigma)$. Now, let us consider the solution x of system (1.1)-(1.2) defined by the above chosen y and σ through on formula (1.3). Since $u_k < 0, \quad k \geq k(\sigma)$, this formula yields

$$x(t) \geq C_1(1+t-\sigma)e^{\omega(t-\sigma)} - \sum_{\sigma \leq t_j \leq t} K(1+t-t_j)e^{\omega(t-t_j)} \left|u_j\right| \geq$$

$$C_1(1+t-\sigma)e^{\omega(t-\sigma)} - \sum_{\sigma \leq t_j \leq t} K(1+t-t_j)e^{\omega(t-t_j)} \left|\beta_j\right| y(t_j-\sigma) \geq$$

$$C_1(1+t-\sigma)e^{\omega(t-\sigma)} - \sum_{\sigma \leq t_j \leq t} K(1+t-t_j) \left|\beta_j\right| (1+t_j-\sigma)e^{\omega(t-\sigma)}C_2 \geq$$

$$(1+t-\sigma)e^{\omega(t-\sigma)}[C_1 - \sum_{\sigma \leq t_j < t} KC_2 \left|\beta_j\right| (1+t_j-\sigma)] > 0, \quad t \geq \sigma.$$

because of (3.15).
Therefore, condition (3.15) yields also the positivity of the solution x. The proof of the theorem is complete.

We will now consider sufficient conditions for oscillation in critical case, when β_k 's are eventually negative, tend to zero, but

$$\sum_{k=1}^{\infty} \left|\beta_k\right| t_k = +\infty.$$

We need the following lemma.

Lemma 3. *Suppose Eq. (1.1) is critical, β_k 's are eventually negative and tend to zero as $k \to +\infty$, and $t_{k+1} - t_k \geq \tau_* > 0, \quad k \geq 1$. Suppose that system (1.1)-(1.2) has a positive solution x on an interval $[\sigma, \infty)$, with initial time σ as large as necessary. Let α be defined by x in (2.1). Then, for each t large enough*

$$\alpha(t\pm) \geq m(t) := \inf_{t-\tau_N \leq s < t} \alpha(s) \tag{3.16}$$

and equality holds only at the points t_0 for which,

$$\alpha(t) = m_0, \quad t_0 - \tau_N \leq t < t_0, \quad C(t_0 - \tau_j, t_0) = 1, \quad 1 \leq j \leq N, \tag{3.17}$$

where m_0 denotes the unique real root of Eq. (1.7). Moreover, m is a nondecreasing function, and either converges to m_0, or is above m_0 at some point and then tends to$+\infty$.

Proof. Since β_k 's are eventually negative, by definition (2.3), we obtain that $C(t-s,t) \geq 1$ holds for each $0 \leq s \leq \tau_N$ and for t large enough. Let t be so large that this is satisfied. Then, in view of (2.4), α satisfies (3.4), and hence

$$\alpha(t\pm) \geq \sum_{j=1}^{N} |a_j| \exp\left(\int_{t-\tau_j}^{t} \alpha(s)ds \right) \geq \sum_{j=1}^{N} |a_j| \, e^{\tau_j m(t)}.$$

For convenience, we introduce the function

$$\psi(u) = \sum_{j=1}^{N} |a_j| \, e^{u\tau_j}, \quad u \geq 0.$$

But, since Eq. (1.1) is critical, Eq. (1.7) has exactly one real root m_0, and hence it is easy to see that

$$\psi(u) > u, \quad u \in R_+, \quad u \neq m_0, \quad \text{and} \quad \psi(m_0) = m_0 > 0.$$

Therefore, we conclude that

$$\alpha(t\pm) \geq m(t)$$

and, in fact,

$$\alpha(t\pm) > m(t)$$

except possibly at points t for which $m(t) = m_0$.

However, even in that case, one can see from (3.4) that the inequality is strict except if $\alpha(s) = m_0$ for $t - \tau_N \leq s < t$ and $C(t-\tau_j, t) = 1$ for each $1 \leq j \leq N$. This proves the first part of our statement and as a consequence, we have that m is nondecreasing. Now, assume that $m(t_0) = m_0$ for some t_0. Then there exists $t_0' \geq t_0$ such that $\alpha(t_0'\pm) > m_0$. In fact, we have $\alpha(t) \geq m_0$ for $t \geq t_0$ and $\alpha(t)$ will be greater than m_0 at any point t such that $C(t-\tau_j, t) > 1$ for at least one index $j \in \{1, ..., N\}$.

Once $\alpha(t_0'\pm)$ takes values larger than m_0, it cannot take value m_0 any more since, in view of the above observation, assuming $\alpha(t'\pm) = m_0$ for some $t' \geq t_0'$ implies $\alpha = m(t')$, therefore $\alpha(s)=m_0, \quad t' - \tau_N \leq s < t'$ and $C(t'-\tau_j, t') = 1, \quad 1 \leq j \leq N$. This property makes it possible to find a first finite time $t_0" > t_0'$ where $\alpha(t_0"\pm) = m_0$.

But using (2.3), (3.4) and the monotonicity property of m, we can see that

$$m(t+\tau_N) = \inf_{t \leq s < t+\tau_N} \alpha(s) \geq \sum_{j=1}^{N} |a_j| \, e^{\tau_j m(t)} \inf_{t \leq s < t+\tau_N} C(s-\tau_j, s) \geq$$

$$\sum_{j=1}^{N} |a_j| \, e^{\tau_j m(t)} \inf_{t \leq s < t+\tau_N} C(s-\tau_1, s) = \psi(m(t)) \inf_{t \leq s < t+\tau_N} C(s-\tau_1, s) \qquad (3.18)$$

and hence

$$m(t+\tau_N) \geq \psi(m(t)).$$

From this, it follows that either $m(t) < m_0$ for each $t \geq 0$, therefore, $m(t)$ tends to m_0 as t tends to $+\infty$, or $m(t) \geq m_0$ and then $m(t)$ becomes larger than m_0 for some $t' > t$, and tends to $+\infty$ as t tends to $+\infty$. The proof of the lemma is complete.

Now, based on estimation (3.16) we derive an oscillatory result which takes the impulse equation into account. We provide a proof of this result based on the Laplace transform and also on the fact that under our conditions the non-oscillatory solutions of system (1.1)-(1.2) are small solutions, that is, they converge to zero at $+\infty$ faster then any exponential function. We note that the Laplace transform method was first used in [7] in the oscillation theory of delay differential equations.

Define the function

$$c(t) = \inf_{t \leq s < t+\tau_N} C(s-\tau_1, s).$$

Theorem 3.3. *Suppose that Eq. (1.1) is critical and $\beta_k's$ and $t_k's$ satisfy the conditions in Lemma 3.1, moreover for each σ,*

$$\prod_{j=1}^{k} c(\sigma + j\tau_N) \to +\infty, \quad as \quad k \to +\infty. \tag{3.19}$$

Then system (1.1) and (1.2) is oscillatory.

Proof. Suppose on the contrary that system (1.1) and (1.2) is eventually nonoscillatory, that is, for some σ, it has positive solution on $[\sigma, \infty)$. In exponential representation, $x(t)$ reads as

$$x(t) = \exp\left(-\int_{\sigma}^{t} \alpha(s)ds\right) \frac{1}{C(\sigma, t)},$$

where α verifies (3.4) and C is defined in (2.3). In view of the assumption (3.19) and inequality (3.18) we have $\alpha(t) \to +\infty$ as $t \to +\infty$. This implies that x tends to zero faster than any exponential at $+\infty$. The same holds for the function

$$x^*(t) = \exp\left(-\int_{\sigma}^{t} \alpha(s)ds\right),$$

which satisfies

$$x^*(t) = C(\sigma, t)x(t), \quad t \geq \sigma. \tag{3.20}$$

But x^* is continuous, absolutely continuous with respect to the Lebesgue measure on each finite interval and verifies the equation

$$\frac{dx^*}{dt} = C(\sigma, t)\left(\sum_{j=1}^{N} a_j x(t-\tau_j)\right) =$$

$$\sum_{j=1}^{N} a_j C(t-\tau_j, t) x^*(t-\tau_j), \quad t > \sigma. \tag{3.21}$$

Since x^* tends to zero at $+\infty$, faster than any exponential function, it is possible to compute the Laplace transform of x^* and the abscissa of convergence is -∞. Denote

$$z(s) = \int_{\sigma}^{+\infty} e^{-s(t-\sigma)} x^*(t)dt,$$

for any complex s. Taking this Laplace transform of (3.21) on both sides, we get

$$z(s)\cdot s - x^*(\sigma) = \sum_{j=1}^{N} a_j \int_{\sigma}^{\infty} e^{-s(t-\sigma)} x^*(t-\tau_j)dt +$$

$$\sum_{j=1}^{N} a_j \int_{\sigma}^{\infty} e^{-s(t-\sigma)}[C(t-\tau_j, t)-1]x^*(t-\tau_j)dt.$$

Each integral

$$\int_{\sigma}^{\infty} e^{-s(t-\sigma)} x^*(t-\tau_j)dt$$

can be written in the form

$$\int_{\sigma-\tau_j}^{\sigma} e^{-s(t+\tau_j-\sigma)} x^*(t)dt + e^{-s\tau_j} z(s).$$

After obvious rearrangements, we arrive at $\Phi(s)z(s) = F(s)$ for all s, where Φ defined by (1.7) and function F defined by

$$F(s) = x^*(\sigma) + \sum_{j=1}^{N} a_j \int_{\sigma-\tau_j}^{\sigma} e^{-s(t+\tau_j-\sigma)} x^*(t)dt +$$

$$\sum_{j=1}^{N} a_j \int_{\sigma}^{\infty} (t-\sigma) e^{-s(t-\sigma)}[C(t-\tau_j, t)-1]x^*(t-\tau_j)dt.$$

are entire functions. Since Eq. (1.1) is critical, that is, Eq. (1.7) has exactly one (double) real root m_0, we have

$$\Phi(-m_0) = \Phi'(-m_0) = 0.$$

Thus, the derivative of the product $\Phi(s)z(s)$ is equal to zero at $s = -m_0$. But

$$F'(s) = \sum_{j=1}^{N} |a_j| \int_{\sigma-\tau_j}^{\sigma} (t + \tau_j - \sigma) e^{-s(t+\tau_j-\sigma)} x^*(t) dt +$$

$$\sum_{j=1}^{N} |a_j| \int_{\sigma}^{\infty} (t - \sigma) e^{-s(t-\sigma)} [C(t - \tau_j, t) - 1] x^*(t - \tau_j) dt > 0,$$

for any real s, especially $F'(-m_0) > 0$. So, assuming that system (1.1) and (1.2) has a positive solution leads to contradiction. This contradiction completes the proof of the theorem.
The condition started in Theorem 3.3 is far from being necessary. One can easily find situations when it does not apply. For example, if we assume that $t_{k+1} - t_k > \tau_1$, and $\tau_N > \tau_1$, the computation of $C(t, s)$ according to formula (2.3) gives

$$c(t) = \inf_{t \le s < t+\tau_N} C(s - \tau_1, s) = 1, \quad t \ge \sigma.$$

On the other hand, we have the following applications.

Corollary 3. *Suppose that $N = 1$, $t_{k+1} - t_k < \tau_1$, the sequence β_k is eventually negative and $\sum_{k=1}^{\infty} |\beta_k| = +\infty$. Moreover suppose that Eq. (1.1) is critical (which in this case is equivalent to condition $|a_1| \tau_1 = \frac{1}{e}$). Then system (1.1) and (1.2) is oscillatory.*

Corollary 4. *Assume that $t_{k+1} - t_k < \tau_1$ holds for any k large enough and β_k's are eventually negative, moreover for each subsequence $\{\beta_{k_l}\}_{l \ge 1}$ such that $0 < k_{l+1} - k_l \le L$, for some finite L, $\sum_{l=1}^{\infty} \left|\beta_{k_l}\right| = +\infty$. Suppose finally that Eq. (1.1) is critical. Then system (1.1) and (1.2) is oscillatory.*

As an example of latter hypotheses on the β_k's, we can consider any sequence β_k such that $\beta_k \le -\frac{M}{k}$, for k large and some constant $M > 0$.

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