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Stability Analysis of a Single Neuron Model with Delay

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Abstract

In this paper we study the asymptotic behavior and numerical approximation of the single neuron model equation $\dot{x}(t) = -dx(t) + af(x(t)) + bf(x(t-\tau)) + I$, $t \geq 0$ (1), where $d > 0$ and $f(x) = 0.5(|x+1| - |x-1|)$. We obtain new sufficient conditions for global asymptotic stability of constant equilibriums of (1), give several numerical examples to illustrate our results, and formulate conjectures on the asymptotic behavior of the solutions based on our numerical experiments.

keywords: delayed cellular neural networks, global asymptotic stability, numerical approximations

1 Introduction

Cellular neural networks (CNNs), introduced by Chua and Yang in 1988 ([5]), have been successfully applied in various engineering and scientific topics: in signal processing systems, especially in static image treatment [6], in solving nonlinear algebraic equations [1]. In these applications the existence and stability of the equilibrium solutions and the qualitative properties (oscillation, periodicity, asymptotic representation of the solutions) play very important role. Because of the importance of the qualitative properties of the solutions the model equations of CNNs models have been extensively studied in the past decade (see, e.g., [2], [9] [16]–[20], [26] and [28], and the references therein).

In a standard CNN model the model equations are ordinary differential equations (ODEs) assuming that the interactions in the system are instantaneous. On the other hand it is known that in the real models of electronic networks time delays are likely to be present, due to the finite switching speed of amplifiers. So in the so-called delayed CNNs (DCNNs) the model equations are delay differential equations, which have much more complicated dynamics than the ODEs. The time delay in the response of a neuron can influence stability (see, e.g., [13]) or it creates oscillation (see, e..g., [11]).

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Recently, DCNN models are applied in the artificial neural networks ([23],[24]). In the applications DCNNs are usually required to be globally asymptotically stable, completely stable, absolutely stable or stable independently of the delays. These different types of stability of DCNNs have been rigorously done and many criteria have been obtained so far (see, e.g., [4], [17], [19], [26]–[28]). Most of these methods and results are devoted to the case when a non-delayed, linear terms dominate the others.

In this paper our attention is focused on a single neuron or the averaged potential of a population of neurons coupled by mutual inhibitory synapses. In that case, based on the paper [14], the model equation is a scalar delay differential equation of the form

$$C\dot{x}(t) = -\frac{x(t)}{R} + \alpha f(x(t)) + \beta f(x(t-\tau)) + \tilde{I}, \qquad t \geq 0,$$

in which $C > 0$, $R > 0$ and $\tilde{I}$ is called capacitance, resistance and the external current input constants of the neuron, respectively; $x(t)$ is the voltage of the neuron and f is a feedback function. The feedback time delay τ may be caused by finite conduction velocities, synaptic transmission or other mechanisms. In retinal network, an extraordinary value of $\tau = 0.1$ sec has been measured (see, e.g., [7] and [21]).

In our study we focus on global stability results, oscillation properties of the solution of the equation

$$\dot{x}(t) = -dx(t) + af(x(t)) + bf(x(t-\tau)) + I, \qquad t \geq 0, \tag{1.1}$$

in the case when the feedback function f is a Hopfield activation function defined by

$$f(x) = \frac{1}{2}(|x+1| - |x-1|) = \begin{cases} 1, & x > 1, \\ x, & -1 \leq x \leq 1, \\ -1, & x < -1. \end{cases} \tag{1.2}$$

We assume throughout this paper that $d > 0$. Note that even in this single neuron model with this simple nonlinearity there is no complete knowledge on the asymptotic or global asymptotic stability of the equilibrium points of (1.1). The standard condition can be found in the literature for asymptotic stability of the trivial solution of (1.1) is

$$d > |a| + |b| + |I|$$

(see, e.g., [4] or [16]). We will show in Section 2 that this condition can be relaxed, the weaker condition

$$d > a + |b| + |I| \tag{1.3}$$

implies the global asymptotic stability of the unique equilibrium point of (1.1) (see Theorem 2.3). In the second part of Section 2 we will study the case when $d \leq a + |b| + |I|$. Then (1.1) may have more equilibrium points, and the dynamics of the equation can be more interesting. We will study in details the case when $b > 0$ using the technique of monotone semiflows. In the case when $a + b - |I| < d \leq a + b + |I|$ we have a complete understanding of the dynamics of (1.1) (see Theorems 2.8), but in the remaining cases we have only partial theoretical results (see Theorems 2.9 and 2.12). In the latter cases we made numerical studies, and based on those experiments we conjecture that if $b > 0$, then every solution of (1.1) tends to a constant equilibrium, i.e., (1.1) is completely stable.

In the case when $b < 0$ and $a + b + |I| < d \leq a + |b| + |I|$ we will present numerical studies and conjecture cases when the solutions of (1.1) are asymptotically periodic.

In Section 3 we will define two numerical approximation techniques we used in the simulations. Note that these methods were originally introduced in [10] for more general delay equations.

We note that if e is an equilibrium point of (1.1) such that $-1 < e < 1$ (see Lemma 2.4 below) then the linearization of (1.1) around this equilibrium gives equation

$$\dot{z}(t) = cz(t) + bz(t - \tau), \qquad t \geq 0, \tag{1.4}$$

where $c = a - d$. The asymptotic behavior of the solution of equation (1.4) is well-known.

Theorem 1.1 (see, e.g., [13]) *The trivial (zero) solution of equation* (1.4) *is asymptotically stable independently of the delay if and only if* $-c > |b|$*. Moreover, the exact stability region of the trivial solution of* (1.4) *is bounded by the line* $b = -c$ *and by the curve*

$$c = s\cot(\tau s), \quad b = -\frac{s}{\sin(\tau s)}, \qquad s \in \left[0, \frac{\pi}{\tau}\right].$$

(See Figure 1 for the region.)

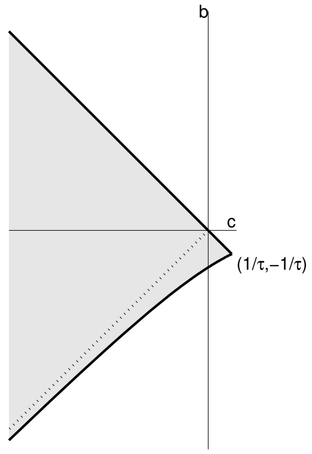

Figure 1: Stability region of the linearized equation.

2 Stability Results

We consider again the single neuron model equation

$$\dot{x}(t) = -dx(t) + af(x(t)) + bf(x(t - \tau)) + I, \qquad t \geq 0, \tag{2.1}$$

with the initial condition

$$x(t) = \varphi(t), \qquad t \in [-\tau, 0]. \tag{2.2}$$

We assume throughout this paper that

$$f(x) = \frac{1}{2}(|x+1| - |x-1|) \tag{2.3}$$

and

$$d > 0, \quad a, b \in \mathbb{R}, \quad b \neq 0. \tag{2.4}$$

For a given $c > 0$ and $\psi\colon [-r, 0] \to (0, \infty)$ consider the equation

$$\dot{y}(t) = -dy(t) + af(y(t)) + |b|f(y(t-\tau)) + c, \qquad t \geq 0 \tag{2.5}$$

associated to (2.1), and the initial condition

$$y(t) = \psi(t) \qquad t \in [-\tau, 0]. \tag{2.6}$$

Lemma 2.1 *Assume* (2.3)–(2.4). *Let* $\psi\colon [-\tau, 0] \to (0, \infty)$, $c > 0$, *and let* y *be the corresponding solution of* (2.5)-(2.6). *Then there exists* $M > 0$ *such that*

$$0 < y(t) < M, \qquad t \geq 0.$$

Proof Since $y(0) > 0$ and y is continuous on $[0, \infty)$, $y(t) > 0$ for small enough $t \geq 0$. Suppose there exists $T > 0$ such that

$$y(t) > 0 \quad \text{for } t \in [-\tau, T), \quad \text{and} \quad y(T) = 0.$$

Then $\dot{y}(T-) \leq 0$. On the other hand, (2.5) implies

$$\dot{y}(T) = -dy(T) + af(y(T)) + |b|f(y(T-\tau)) + c = |b|f(y(T-\tau)) + c > 0,$$

which is a contradiction. Therefore $y(t) > 0$ for all $t > 0$.

To prove that y is bounded from above, assume that $\limsup_{t\to\infty} y(t) = \infty$. Then there exists a monotone increasing sequence t_n such that

$$\lim_{n\to\infty} t_n = \infty, \qquad \lim_{n\to\infty} y(t_n) = \infty, \qquad \text{and} \qquad y(t_n) = \max\{y(t)\colon\, t \in [-\tau, t_n]\}.$$

Then $\dot{y}(t_n-) \geq 0$, which contradicts to the relations

$$\dot{y}(t_n) = -dy(t_n) + af(y(t_n)) + |b|f(y(t_n - \tau)) + c \leq -dy(t_n) + a + |b| + c < 0$$

for large enough n. $\square$

Lemma 2.2 *Assume* (2.3)–(2.4), $c > 0$, *and*

$$d > a + |b| + c. \tag{2.7}$$

Let $\psi\colon [-\tau, 0] \to (0, \infty)$, *and let* y *be the corresponding solution of* (2.5)-(2.6). *Then*

$$\lim_{t\to\infty} y(t) = \frac{c}{d - a - |b|}. \tag{2.8}$$

Proof It follows from Lemma 2.1 that

$$\limsup_{t\to\infty} y(t) = M \qquad \liminf_{t\to\infty} y(t) = m$$

are finite and $m \geq 0$. There are two cases: either $M = m$, or $M > m$. In the first case $M = \lim_{t\to\infty} y(t)$, and (2.5) yields

$$0 = -dM + af(M) + |b|f(M) + c. \tag{2.9}$$

In the second case there exists a sequence t_n such that

$$t_n \to \infty \quad \text{as } n \to \infty, \qquad \dot{y}(t_n) = 0,\ n = 1, 2\ldots, \qquad \text{and} \qquad \lim_{n\to\infty} y(t_n) = M.$$

We may also assume that

$$\lim_{n\to\infty} y(t_n - \tau) = m^*$$

for some $m \leq m^* \leq M$, since otherwise we can select a subsequence of t_n with this property. Then

$$\begin{aligned} 0 &= \lim_{n\to\infty} \dot{y}(t_n) \\ &= \lim_{n\to\infty} \Big(-dy(t_n) + af(y(t_n)) + |b|f(y(t_n - \tau)) + c \Big) \\ &= -dM + af(M) + |b|f(m^*) + c \\ &\leq -dM + af(M) + |b|f(M) + c. \end{aligned} \tag{2.10}$$

Therefore in both cases (2.10) holds. Suppose $M \geq 1$. Then (2.10), $f(M) = 1$ and (2.7) imply the contradiction

$$0 \leq -d + a + |b| + c < 0.$$

Therefore $0 \leq M < 1$. This means there exists $t_1 > 0$ such that for $t \geq t_1$ (2.5) is equivalent to

$$\dot{y}(t) = (-d + a)y(t) + |b|y(t - \tau) + c, \qquad t \geq t_1. \tag{2.11}$$

Define

$$K = \frac{c}{d - a - |b|}.$$

It follows from (2.7) that $K < 1$. Introducing $z(t) = y(t) - K$ we can rewrite (2.11) as

$$\dot{z}(t) = (-d + a)z(t) + |b|z(t - \tau), \qquad t \geq t_1. \tag{2.12}$$

Since $d - a > |b|$, Theorem 1.1 yields the trivial solution of (2.12) is asymptotically stable (independently of the size of the delay), therefore (2.8) holds. $\square$

Theorem 2.3 *Assume* (2.3)–(2.4), *and*

$$d > a + |b| + |I|. \tag{2.13}$$

Then any solution x *of* (2.1)-(2.2) *satisfies*

$$\lim_{t\to\infty} x(t) = \frac{I}{d-a-b}. \tag{2.14}$$

Proof Fix any $\psi\colon [-r,0] \to (0,\infty)$ such that

$$\psi(s) > |\varphi(s)|, \qquad s \in [-r,0],$$

and let $c > |I|$ be such that $d > a + |b| + c$. Let y denote the solution of the corresponding IVP (2.5)-(2.6). Since $y(0) > |x(0)|$, relation $|x(t)| < y(t)$ holds for sufficiently small $t > 0$. Suppose there exists $T > 0$ such that

$$|x(t)| < y(t), \qquad t \in [-\tau, T), \qquad \text{and} \quad |x(T)| = y(T). \tag{2.15}$$

It follows from Lemma 2.1 that $|x(T)| = y(T) \neq 0$, therefore $\frac{d}{dt}|x(t)|$ exists at T, and $\frac{d}{dt}|x(T)| = \dot{x}(T)\,\mathrm{sign}\, x(T)$. Hence

$$\begin{aligned}
\frac{d}{dt}|x(T)| &= \Big(-dx(T) + af(x(T)) + bf(x(T-\tau)) + I\Big)\,\mathrm{sign}\, x(T) \\
&= -d|x(T)| + af(|x(T)|) + bf(x(T-\tau))\,\mathrm{sign}\, x(T) + I\,\mathrm{sign}\, x(T) \\
&\le -d|x(T)| + af(|x(T)|) + |b|f(|x(T-\tau)|) + c \\
&< -dy(T) + af(y(T)) + |b|f(y(T-\tau)) + c \\
&= \dot{y}(T).
\end{aligned}$$

This contradicts to assumption (2.15), therefore $|x(t)| < y(t)$ holds for all $t > 0$. Moreover, Lemma 2.2 yields

$$\lim_{t\to\infty} y(t) = \frac{c}{d-a-|b|} < 1$$

holds, therefore there exists $t_1 > 0$ such that $|x(t)| < 1$ for $t \ge t_1$. Then (2.1) is equivalent to

$$\dot{x}(t) = (-d+a)x(t) + bx(t-\tau) + I, \qquad t \ge t_1.$$

This implies (2.14) using an argument similar to that in the proof of Lemma 2.2. $\square$

L is an equilibrium of (2.1) if

$$-dL + af(L) + bf(L) + I = 0. \tag{2.16}$$

If $L \ge 1$, $L \le -1$ and $-1 \le L \le 1$, then $f(L) = 1$, $f(L) = -1$ and $f(L) = L$, respectively. Therefore in this three cases we get three possible solutions of (2.16):

$$e_1 = \frac{a+b+I}{d}, \qquad e_2 = \frac{-a-b+I}{d}, \qquad \text{and} \qquad e_3 = \frac{I}{d-a-b}, \tag{2.17}$$

assuming $d \neq a+b$ in the third case. Conversely, e_1, e_2 and e_3 defined by (2.17) are equilibrium points of (2.1), if $e_1 \geq 1$, $e_2 \leq -1$ and $-1 \leq e_3 \leq 1$. The next cases can be checked easily:

Lemma 2.4 *Assume* (2.3)–(2.4), *and let e_1, e_2 and e_3 be defined by* (2.17). *Then*

(i) if $d > \max(a+b+|I|, 0)$, then $-1 < e_3 < 1$ is the only equilibrium point of (2.1).

(ii) if $\max(0, a+b-|I|) < d \leq a+b+|I|$, then (2.1) *has only one equilibrium point:*

  (1) if $I > 0$, then $e_1 \geq 1$ is the equilibrium,

  (2) if $I < 0$, then $e_2 \leq -1$ is the equilibrium,

(iii) if $0 < d = a+b$ and $I = 0$, then any number $e \in [-1, 1]$ is an equilibrium of (2.1), *and it has no other equilibrium outside $[-1, 1]$,*

(iv) if $0 < d = a+b-|I|$ and $I \neq 0$, then (2.1) *has two equilibrium points:*

  (1) if $I > 0$, then $e_1 > 1$ and $e_2 (= e_3) = -1$ are equilibriums,

  (2) if $I < 0$, then $e_1 (= e_3) = 1$ and $e_2 < -1$ are equilibriums,

(v) if $0 < d < a+b-|I|$, then $e_1 > 1$, $e_2 < -1$ and $-1 < e_3 < 1$ are the equilibrium points of (2.1).

Next we assume that $b > 0$. First we recall some results from the theory of monotone dynamical systems formulated for (2.1).

Theorem 2.5 (see, e.g., [25]) *Assume* (2.3) *and $b > 0$.*

(i) Let $\varphi, \psi \colon [-\tau, 0] \to \mathbb{R}$ be such that

$$\varphi(s) \leq \psi(s), \qquad s \in [-\tau, 0],$$

and let $x(t; \varphi)$ and $x(t; \psi)$ denote the solution of (2.1) *corresponding to initial function φ and ψ, respectively. Then*

$$x(t; \varphi) \leq x(t; \psi), \qquad t \geq 0.$$

(ii) Let $x(t; c)$ be the solution of (2.1) *corresponding to a constant $\varphi(s) = c$ initial function. If $-dc + af(c) + bf(c) + I \geq 0$, then $x(t; c)$ is nondecreasing, and if $-dc + af(c) + bf(c) + I \leq 0$, then $x(t; c)$ is nonincreasing function.*

Next we study the asymptotic behavior of (2.1) starting from constant initial conditions. Consider a constant initial function $\varphi(s) = c$, then the corresponding solution will be denoted by $x(t; c)$.

Theorem 2.6 *Assume* (2.3) *and* $b > 0$. *Then every solution of* (2.1) *starting from a constant initial function tends to a constant equilibrium.*

Proof It follows from Theorem 2.5 (ii) that all solutions of (2.1) corresponding to a constant initial function are monotone functions. On the other hand Lemma 2.1 yields the solutions of (2.1) are bounded functions. Therefore $\lim_{t\to\infty} x(t;c)$ always exists, and hence it is an equilibrium point of Equation (2.1). $\square$

Theorems 2.3 and 2.6, and Lemma 2.4 have the following corollary, which gives a complete description of the asymptotic property of the solution of (2.1) starting from constant initial functions.

Theorem 2.7 *Assume* (2.3), $b > 0$, *and let* e_1, e_2 *and* e_3 *be defined by* (2.17). *Then*

(i) if $d > \max(a + b + |I|, 0)$, *then* e_3 *is a globally asymptotically stable equilibrium of* (2.1);

(ii) if $\max(0, a + b - |I|) < d \le a + b + |I|$, *then*

  (1) if $I > 0$, *then* $x(t;c) \to e_1$ *for any* $c \in \mathbb{R}$,

  (2) if $I < 0$, *then* $x(t;c) \to e_2$ *for any* $c \in \mathbb{R}$ *as* $t \to \infty$;

(iii) if $0 < d = a + b$ *and* $I = 0$, *then*

  (1) if $c > 1$, *then* $x(t;c) \to 1$ *monotone decreasingly;*

  (2) if $c \in [-1, 1]$, *then* $x(t;c)$ *is constant; and*

  (3) if $c < -1$, *then* $x(t;c) \to -1$ *monotone increasingly as* $t \to \infty$;

(iv) if $0 < d = a + b - |I|$ *and* $I \neq 0$, *then if* $c > e_1$, *then* $x(t;c) \to e_1$ *monotone decreasingly;*

  (1) if $I > 0$, *then if* $c \in (-1, e_1)$, *then* $x(t;c) \to e_1$ *monotone increasingly; if* $c < -1$, *then* $x(t;c) \to -1$ *monotone increasingly;*

  (2) if $I < 0$, *then if* $c \in (e_2, 1)$, *then* $x(t;c) \to e_2$ *monotone decreasingly; if* $c < e_2$, *then* $x(t;c) \to e_2$ *monotone increasingly as* $t \to \infty$;

(v) if $0 < d < a + b - |I|$, *then*

  (1) if $c > e_1$, *then* $x(t;c) \to e_1$ *monotone decreasingly;*

  (2) if $c \in (e_3, e_1)$, *then* $x(t;c) \to e_1$ *monotone increasingly;*

  (3) if $c \in (e_2, e_3)$, *then* $x(t;c) \to e_2$ *monotone decreasingly; and*

  (4) if $c < e_2$, *then* $x(t;c) \to e_2$ *monotone increasingly as* $t \to \infty$.

To illustrate Theorem 2.7 we numerically computed solutions of (2.1) corresponding to several constant initial functions to different parameter values. The corresponding solutions can be seen in Figures 2–7.

Figure 2 illustrates case (i) of Theorem 2.7, here $d = 4$, $a = 1$, $b = 1$, $I = -1$ and $\tau = 1$. We see that all solutions tend to $e_3 = -0.5$.

In Figure 3 case (ii) (1) of Theorem 2.7 is illustrated. The solutions of (2.1) correspond to $d = 2$, $a = 1$, $b = 1$, $I = 1$ and $\tau = 1$.

Figure 4 corresponds to parameter values $d = 2$, $a = -1$, $b = 3$, $I = 0$ and $\tau = 1$. We see that solutions starting from constant value greater than 1 tend to 1, and similarly, solutions starting from a constant less than -1 tend to -1, and solutions starting from constants between -1 and 1 remain constant.

In Figure 5 solutions of (2.1) with $d = 2$, $a = 1$, $b = 3$, $I = 2$ and $\tau = 1$ can be seen. In this case (2.1) has only two equilibriums: $e_1 = 3$ and $e_2 = -1$. This corresponds to case (iv) (1) of Theorem 2.7. Case (iv) (2) is illustrated in Figure 6, where $d = 1$, $a = -1$, $b = 3$, $I = -1$, $\tau = 1$, and the equilibriums are $e_1 = 1$, $e_2 = -3$.

In Figure 7 an example for case (v) of Theorem 2.7 is studied. Here $d = 2$, $a = 1$, $b = 3$, $I = 1$ and $\tau = 1$, and the corresponding equation has three equilibriums: $e_1 = 2.5$, $e_2 = -1.5$ and $e_3 = -0.5$. We can see from the graph that e_1 and e_2 are attractive with respect to solutions starting from constant initial functions.

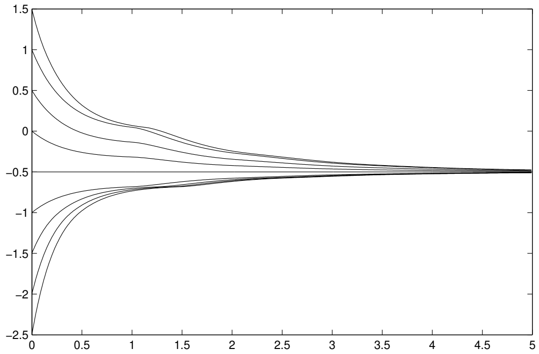

Figure 2: case (i), $d = 4$, $a = 1$, $b = 1$, $I = -1$, $\tau = 1$, and $\varphi =$ constant.

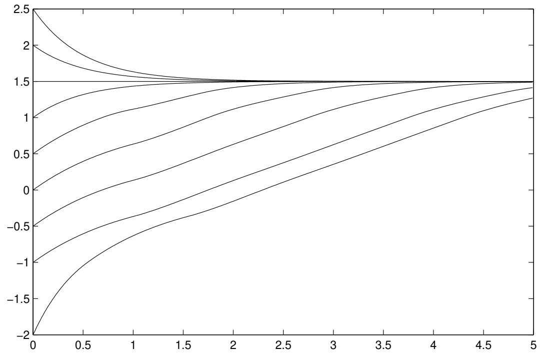

Figure 3: case (ii) (1), $d = 2$, $a = 1$, $b = 1$, $I = 1$, $\tau = 1$, and $\varphi =$ constant.

Newt we show that in case (ii) of Theorem 2.7 the single equilibrium point of (2.1) is globally asymptotically stable for nonconstant initial functions, as well.

Theorem 2.8 *Assume* (2.3), $b > 0$, *and* $\max(0, a + b - |I|) < d \le a + b + |I|$. *Let* $x(t;\varphi)$ *be any solution of* (2.1)-(2.2), *and* e_1, e_2 *and* e_3 *be defined by* (2.17). *Then*

(1) if $I > 0$, *then* $x(t;\varphi) \to e_1$, *as* $t \to \infty$,

(2) if $I < 0$, *then* $x(t;\varphi) \to e_2$, *as* $t \to \infty$.

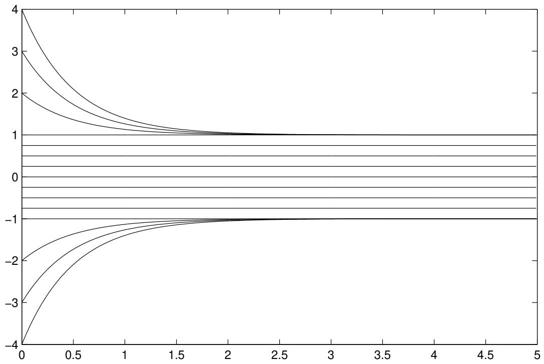

Figure 4: case (iii), $d = 2$, $a = -1$, $b = 3$, $I = 0$, $\tau = 1$, and $\varphi =$ constant.

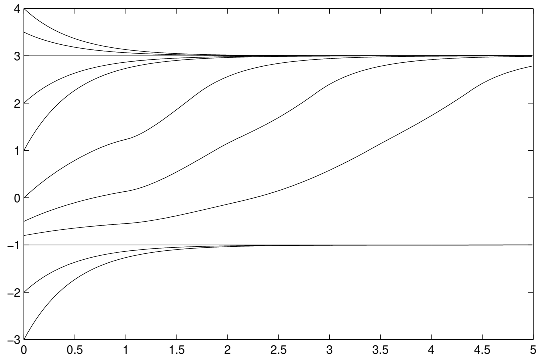

Figure 5: case (iv) (1), $d = 2$, $a = 1$, $b = 3$, $I = 2$, $\tau = 1$, and $\varphi =$ constant.

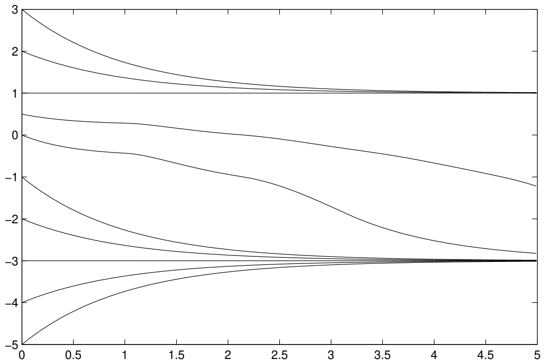

Figure 6: case (iv) (2), $d = 1$, $a = -1$, $b = 3$, $I = -1$, $\tau = 1$, and $\varphi =$ constant.

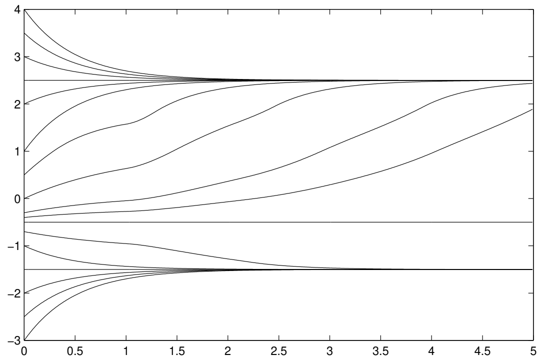

Figure 7: case (v), $d = 2$, $a = 1$, $b = 3$, $I = 1$, $\tau = 1$, and $\varphi =$ constant.

Proof Consider case (1). Pick constants h and k such that

$$h < e_3, \qquad k > e_1, \qquad \text{and} \qquad h < \varphi(t) < k, \quad t \in [-\tau, 0].$$

Then by Theorem 2.5

$$x(t; h) \leq x(t; \varphi) \leq x(t; k), \qquad t \geq 0.$$

Since by Theorem 2.7

$$\lim_{t\to\infty} x(t; h) = e_1 = \lim_{t\to\infty} x(t; k),$$

the theorem is proved. Case (2) can be proved similarly. □

Finally we consider case (v) of Theorem 2.7, i.e., assume $0 < d < a + b - |I|$. Then the linearized equation (1.4) has an unstable trivial solution (see Figure 1). We show that in this case the solutions of (2.1) either tend to e_1 or e_2, or oscillate around e_3.

Theorem 2.9 *Assume* (2.3), $b > 0$, *and* $0 < d < a + b - |I|$. *Let* e_1, e_2 *and* e_3 *be defined by* (2.17), *and let* $x(t;\varphi)$ *be any solution of* (2.1)-(2.2). *Then either*

(i) $\lim_{t\to\infty} x(t;\varphi) = e_1$,

(ii) $\lim_{t\to\infty} x(t;\varphi) = e_2$, *or*

(iii) there exists a sequence $t_n \geq 0$ *such that*

$$\lim_{n\to\infty} t_n = \infty, \quad |t_{n+1} - t_n| \leq \tau, \qquad \text{and} \qquad x(t_n;\varphi) = e_3,$$

i.e., x *oscillates around* e_3.

Proof We distinguish three cases. If there exists $\varepsilon > 0$ and $t_0 \geq 0$ such that $x(t;\varphi) > e_3 + \varepsilon$ for $t \in [t_0 - \tau, t_0]$, then by Theorem 2.7, $x(t; e_3 + \varepsilon) \to e_1$. Theorem 2.5 implies $x(t;\varphi) > x(t + t_0; e_3 + \varepsilon)$, therefore there exists $T > 0$ such that $x(t;\varphi) > 1$ for $t > T$. But then

$$\dot{x}(t;\varphi) = -dx(t;\varphi) + a + b + I,$$

and therefore $x(t;\varphi) \to e_1$.

If there exists $\varepsilon > 0$ and $t_0 \geq 0$ such that $x(t;\varphi) < e_3 - \varepsilon$ for $t \in [t_0 - \tau, t_0]$, then we get by a similar argument that $x(t;\varphi) \to e_2$.

In the remaining case statement (iii) holds. □

Corollary 2.10 *Assume* (2.3), $b > 0$, *and* $0 < d < a + b - |I|$. *Let* e_1, e_2 *and* e_3 *be defined by* (2.17), *and let* $x(t;\varphi)$ *be any solution of* (2.1)-(2.2). *Then*

(i) if $\varphi(t) > e_3$, $t \in [-r, 0]$, *then* $\lim_{t\to\infty} x(t;\varphi) = e_1$,

(ii) if $\varphi(t) < e_3$, $t \in [-r, 0]$, *then* $\lim_{t\to\infty} x(t;\varphi) = e_2$.

The next result shows that there are solutions of (2.1) (different from the constant function e_3) satisfying case (iii) of Theorem 2.9.

Proposition 2.11 *Assume* (2.3), $b > 0$, *and* $0 < d < a + b - |I|$. *Let* e_3 *be defined by* (2.17). *Then there exist initial functions* φ *such that the corresponding solutions* $x(t;\varphi)$ *of* (2.1)-(2.2) *satisfy case (iii) of Theorem 2.9, moreover* $x(t;\varphi) \to e_3$ *as* $t \to \infty$.

Proof Consider the linear equation

$$\dot{z}(t) = (-d + a)z(t) + bz(t - \tau) \tag{2.18}$$

associated to (2.1). The characteristic equation $\lambda = -d + a + be^{-\lambda\tau}$ of (2.18) has a complex root $\lambda = \alpha + i\beta$ with $\alpha < 0$ and $\beta > \pi/\tau$ (see, e.g., [13]). Then $z(t) = ce^{\alpha t}\cos\beta t$ is a solution of (2.18) for any $c \in \mathbb{R}$. Pick any c satisfying $|c| < \min(1 - e_3, 1 + e_3)$, and let $x(t) = z(t) + e_3$. Then $|x(t)| < 1$, and it is a solution of (2.1) satisfying $x(t) \to e_3$. $\square$

Let $\bar{x}(t)$ be a solution of (2.1) given in the proof of the last proposition, and let $\bar{\varphi}$ be its restriction to $[-r, 0]$. Suppose $\bar{x}(t)$ is stable. Then the solutions $x(t; \varphi)$ of (2.1) starting from initial functions φ close to $\bar{\varphi}$ remains in the neighborhood of $\bar{x}$, where $-1 < x(t; \varphi) < 1$ holds. But then define $z(t) = x(t) - e_3$ and $\bar{z}(t) = \bar{x}(t) - e_3$. Then both $z(t)$ and $\bar{z}(t)$ are solutions of (2.18), moreover the difference function $w(t) = z(t) - \bar{z}(t) = x(t) - \bar{x}(t)$ is also a solution of (2.18). But this is a contradiction, since in this case the trivial solution of (2.18) is unstable, and so $w(t)$ can not be bounded. Therefore solution $\bar{x}(t)$ of (2.1) is unstable, and hence it is difficult to observe it numerically. In Figure 8 we plotted such a solution starting from the initial function $\varphi(t) = 0.5e^{-0.43177t}\cos(2.3706t)$ (together with some other solutions). We can see that this solution first approaches 0, but after some time, due to numerical error, it gets off the unstable equilibrium, and one of the stable equilibrium attracts the solution. We made several numerical runnings to test the stability of the equilibrium points in this case for nonconstant initial functions, and we found that every numerical solution tends to e_1 or e_2.

Similarly to Theorem 2.9 and Corollary 2.10 one can prove the following result for cases (iii) and (iv) of Theorem 2.7.

Theorem 2.12 *Assume* (2.3), $b > 0$, *and let* e_1, e_2 *and* e_3 *be defined by* (2.17), *and let* $x(t; \varphi)$ *be any solution of* (2.1)-(2.2).

(i) Suppose $0 < d = a + b$ *and* $I = 0$. *Then*

(1) if $\varphi(t) > 1$ *for* $t \in [-r, 0]$, *then* $\lim_{t\to\infty} x(t; \varphi) = 1$,

(2) if $\varphi(t) < -1$ *for* $t \in [-r, 0]$, *then* $\lim_{t\to\infty} x(t; \varphi) = -1$;

(ii) Suppose $0 < d = a + b - |I|$ *and* $I > 0$. *Then*

(1) if $\varphi(t) > e_2$ *for* $t \in [-r, 0]$, *then* $\lim_{t\to\infty} x(t; \varphi) = e_1$,

(2) if $\varphi(t) < e_2$ *for* $t \in [-r, 0]$, *then* $\lim_{t\to\infty} x(t; \varphi) = e_2$;

(iii) Suppose $0 < d = a + b - |I|$ *and* $I < 0$. *Then*

(1) if $\varphi(t) > e_1$ *for* $t \in [-r, 0]$, *then* $\lim_{t\to\infty} x(t; \varphi) = e_1$,

(2) if $\varphi(t) < e_1$ *for* $t \in [-r, 0]$, *then* $\lim_{t\to\infty} x(t; \varphi) = e_2$;

Figure 9 studies case (iii) of Theorem 2.7. Here we can observe that solutions starting from different initial functions tend to a constant equilibrium (depending on the initial function). In Figure 10 we study case (iv) of Theorem 2.7. In this case, as well, the solutions tend to one of the two equilibrium points.

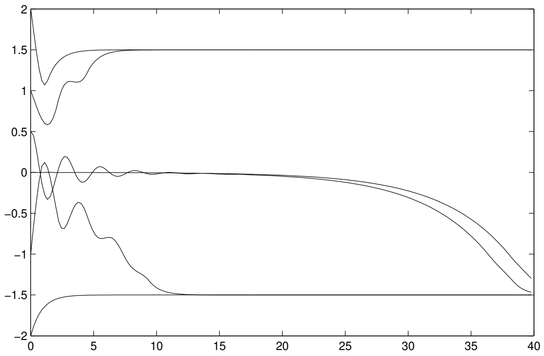

Figure 8: $d = 1$, $a = 0.5$, $b = 1$, $\tau = 2$, $I = 0$, $\varphi(t) = \cos(2t) + 1$, $\sin t + 1$, $0.5e^{-0.43177t}\cos(2.3706t)$, $0.01\sin 5t$, $t^2 - 1$, and $t - 2$, respectively.

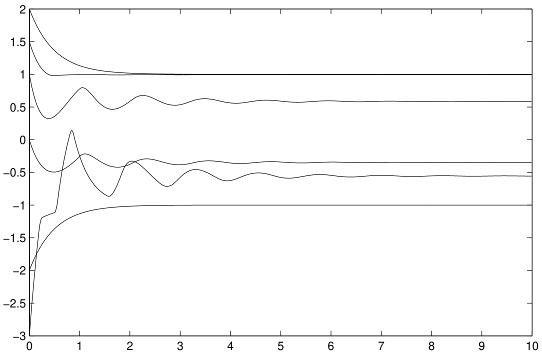

Figure 9: $d = 2$, $a = -1$, $b = 3$, $\tau = 1$, $I = 0$, $\varphi(t) = 2\cos t$, $t + 1.5$, $\cos 2t$, $\sin t$, $t^2 - 2$, and $-3\cos 10t$, respectively.

Theorem 2.9, 2.12 and our numerical studies suggest that not only in case (i) and (ii) of Theorem 2.7, but also in cases (iii)–(v) all solutions of (2.1) tend to a constant equilibrium.

Conjecture 2.13 *Assume* (2.3), $b > 0$, *and* $0 < d \leq a + b - |I|$. *Then every solution of* (2.1)-(2.2) *tends to a constant equilibrium.*

Finally, consider the case when $b < 0$. In this case the method of monotone semiflows (Theorem 2.5) does not work. Theorem 2.3 implies that e_3 is globally asymptotically stable if $d > \max(a + |b| + |I|, 0)$. We now study the case when

$$\max(a + b + |I|, 0) < d \leq a + |b| + |I|. \tag{2.19}$$

In this case Lemma 2.4 yields that e_3 is the only equilibrium of (2.1), but it is an open question whether this equilibrium point is globally asymptotically stable. Introduce $z(t) = x(t) - e_3$. As we have seen in the proof of Proposition 2.11, $z(t)$ satisfies equation (2.18) until $x(t)$ remains close to the equilibrium (more precisely, if $|x(t) - e_3| \leq 1$). It follows from Theorem 1.1 that the trivial solution of (2.18) is not asymptotically stable independently of the delay, as it was in the case of Theorem 2.3.

First consider an example where $d = 2$, $a = 2\cot 2 + 2 \approx 1.54234$, $b = -2/\sin 2 \approx -1.09975$, $I = 1$ and $\tau = 2$. Note that these parameters lie on the lower boundary of the stability region of the linearized equation (2.18) (see Theorem 1.1 and Figure 1). In this case the trivial solution of (2.18) is stable but not asymptotically stable. Then it is known that the corresponding linear equation (2.18) has a periodic solution. (It is easy to check that $z(t) = \alpha\cos t$ solves (2.18) for any $\alpha \in [-1, 1]$.) First in Figure 11 we have the graph of a few solutions of the corresponding nonlinear equation (2.1). We found that all

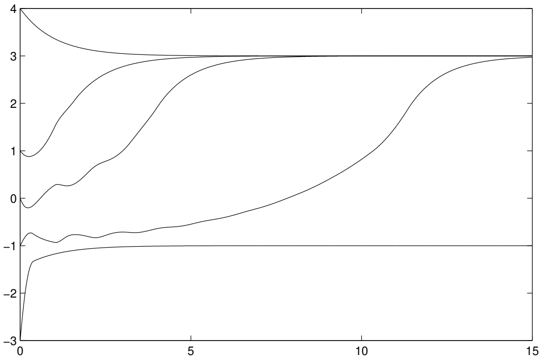

Figure 10: $d = 1$, $a = -1$, $b = 3$, $\tau = 1$, $I = 1$, $\varphi(t) = 4 - t^2$, $t + 1$, t^3, $0.5 \sin 5t - 1$, and $-3 \cos 2t$, respectively.

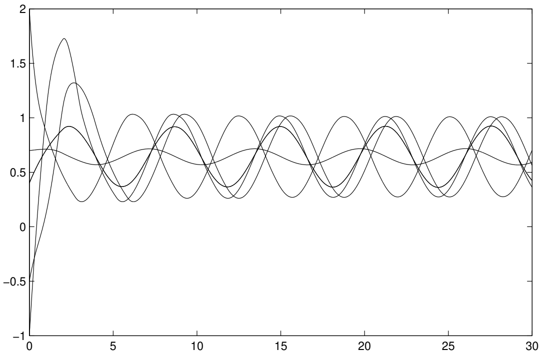

Figure 11: $d = 2$, $a = 1.54234$, $b = -1.09975$, $I = 1$, $\tau = 2$, and $\varphi(t) = t^2 + 2$, $0.7 + 0.1 \sin t$, 0.4, $-0.5 \cos 2t$ and $t - 1$, respectively.

the numerically observable solutions (except the constant equilibrium) are asymptotically periodic.

In the next example we use parameter values $d = 2$, $a = 1.54234$, $b = -0.8$, $I = 1$, and $\tau = 2$. Then it is easy to check that the linear equation (2.18) has an asymptotically stable trivial solution, therefore equilibrium e_3 of the nonlinear equation (2.1) is locally asymptotically stable, as well. Based on our numerical studies we conjecture that in this case e_3 is also globally asymptotically stable. We plotted some solutions of the corresponding equation (2.1) in Figure 12.

Finally, consider parameter values $d = 1$, $a = 0.5$, $b = -2$, $I = 0$, and $\tau = 2$. Then the zero solution of the linear equation (2.18) is unstable (see Figure 1). We found that the solutions of the nonlinear equations are asymptotically periodic. We can see some solutions of (2.1) in Figure 13. Of course, as in Proposition 2.11, we can find solutions of (2.18) which tend to 0. E.g., $z(t) = 0.3e^{-0.8146t} \cos(10.19475t)$ is a solution of (2.18), therefore $x(t) = z(t)$ is a solution of (2.1). In Figure 13 we plotted a numerical solution starting from this initial function. We can see that the numerical solution first follows the analytical solution $x(t)$, but after some time, due to numerical errors, a periodic solution attracts it.

Based on numerical studies we made the following conjecture on the asymptotic behavior of the solution.

Conjecture 2.14 *Assume* (2.3) *and* (2.19)*. If the trivial solution of the corresponding linear equation* (2.18) *is asymptotically stable, then* e_3 *is a globally stable equilibrium of* (2.1)*. Otherwise, "most of the solutions" of* (2.1) *are asymptotically periodic.*

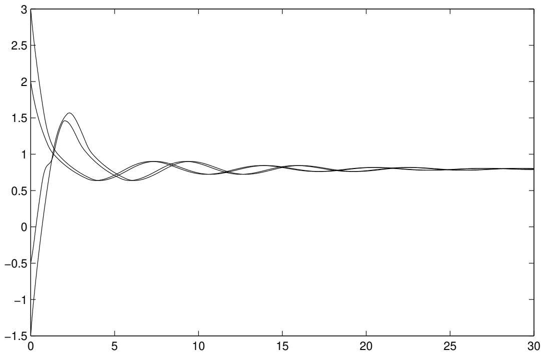

Figure 12: $d = 2$, $a = 1.54234$, $b = -0.8$, $I = 1$, $\tau = 2$, and $\varphi(t) = 3\cos t$, $t + 2$, $\sin 5t - 0.5$ and -1.5, respectively.

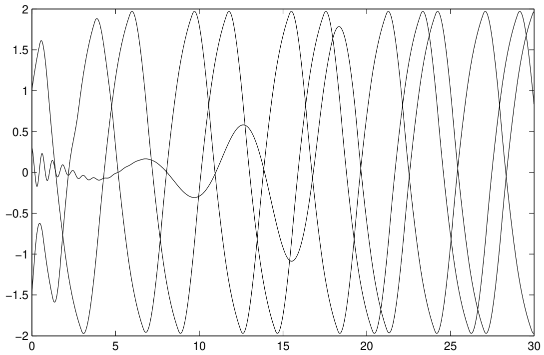

Figure 13: $d = 1$, $a = 0.5$, $b = -2$, $\tau = 2$, $I = 0$, and $\varphi(t) = t^3 - t + 1$, $0.3e^{-0.8146t}\cos(10.19475t)$, and $-1.5\cos 3t$, respectively.

3 Numerical Approximation

In this section we define two numerical schemes to approximate the solutions of (2.1). Our first method is the **chain method**, which was first introduced by Repin [22] and Janushevski [15], and later was also used in [10] and [12]. We can rewrite (2.1) in the form

$$\frac{d}{dt}\Big(x(t) + b\int_{t-\tau}^{t} f(x(s))\,ds\Big) = -dx(t) + (a+b)f(x(t)) + I, \qquad t \geq 0.$$

Fix a positive integer N, introduce the stepsize $h = \tau/N$, and to this equation we associate the system of ODEs

$$\dot{y}^{(N,0)}(t) = -dy^{(N,0)}(t) + af(y^{(N,0)}(t)) + I + \frac{1}{h}y^{(N,N)}(t) \tag{3.1}$$

$$\dot{y}^{(N,1)}(t) = -\frac{1}{h}y^{(N,1)}(t) + bf(y^{(N,0)}(t)) \tag{3.2}$$

$$\dot{y}^{(N,i)}(t) = -\frac{1}{h}y^{(N,i)}(t) + \frac{1}{h}y^{(N,i-1)}(t), \qquad i = 2,\ldots,N, \tag{3.3}$$

$$y^{(N,0)}(0) = \varphi(0), \tag{3.4}$$

$$y^{(N,i)}(0) = \int_{-ih}^{-(i-1)h} bf(\varphi(s))\,ds, \qquad i = 1,\ldots,N, \tag{3.5}$$

It can be shown (see the details in [12]) that

$$\lim_{N\to\infty} |y^{(N,0)}(t) - x(t)| = 0, \qquad \lim_{N\to\infty}\left|y^{(N,i)}(t) - \int_{t-ih}^{t-(i-1)h} bf(x(s))\,ds\right| = 0, \quad i = 1,\ldots,N.$$

Example 3.1 Consider the IVP

$$\begin{aligned}\dot{x}(t) &= -x(t) + f(x(t)) - f(x(t-1)), \quad t \geq 0, &(3.6)\\ x(t) &= 2, \quad t \in [-1, 0] &(3.7)\end{aligned}$$

Its solution can be computed using the method of steps:

$$x(t) = \begin{cases} 2 & t \in [-1, 0], \\ 2e^{-t} & t \in (0, \log 2] \\ -t + 1.69314718 & t \in (\log 2, 1 + \log 2] \\ 0.5t^2 - 2.69314718t + 3.12652087 & t \in (1 + \log 2, 2 + \log 2] \\ -0.166666667t^3 + 1.84657359t^2 - 6.31966805t & \\ \quad 6.38210568, & t \in (2 + \log 2, 3 + \log 2] \\ 0.0416666667t^4 - 0.782191197t^3 + 5.25640762t^2 & \\ \quad -14.5483473t + 14.1334177, & t \in (3 + \log 2, 4 + \log 2] \\ -0.00833333334t^5 + 2372144659t^4 - 2.61766040t^3 & \\ \quad +13.8705347t^2 - 34.9286972t + 33.1065757 & t \in (4 + \log 2, 5 + \log 2] \\ 0.0013888889t^6 - 0.0557762265t^5 + 0.9124628997t^4 & \\ \quad -7.743378692t^3 + 35.75663620t^2 - 84.76901587t & \\ \quad +80.39795420 & t \in (5 + \log 2, 6 + \log 2] \end{cases}$$

In this example we used scheme (3.1)–(3.5) to get approximate solution of IVP (3.6)-(3.7). In Table 1 we compared the numerical results to the true solution. We can observe linear convergence to the true solution.

Table 1: Chain method.

h	N	$\lvert y^{(N,0)}(2) - x(2)\rvert$	$\lvert y^{(N,0)}(4) - x(4)\rvert$	$\lvert y^{(N,0)}(6) - x(6)\rvert$
0.250000	5	0.090714	0.119166	0.033756
0.111111	10	0.047653	0.066927	0.029830
0.020408	50	0.009849	0.014230	0.008160
0.010101	100	0.004756	0.007093	0.004173
0.005025	200	0.002219	0.003469	0.002072

Our next scheme is based on the **method of lines**, which is used frequently to approximate PDEs (see, e.g., [29] and the references therein), and was used in [10] to approximate FDEs. Let $u(t,s) = x(t-s)$, then (2.1) is equivalent to

$$\begin{aligned}\frac{\partial u}{\partial t}(t,s) + \frac{\partial u}{\partial s}(t,s) &= 0, \qquad 0 \leq s \leq \tau, \quad t \geq 0 &(3.8)\\ \frac{\partial u}{\partial t}(t,0) &= -du(t,0) + af(u(t,0)) + bf(u(t,\tau)) + I, \quad t \geq 0 &(3.9)\end{aligned}$$

Let N be fixed, and $h = \tau/N$. Consider the system of ODEs

$$\dot{v}^{(N,0)}(t) = -dv^{(N,0)}(t) + af(v^{(N,0)}(t)) + bf(v^{(N,N)}(t)) \tag{3.10}$$

$$\dot{v}^{(N,i)}(t) = -\frac{1}{h}v^{(N,i)}(t) + \frac{1}{h}v^{(N,i-1)}(t), \qquad i = 1,\dots,N, \tag{3.11}$$

$$v^{(N,i)}(0) = \varphi(-ih), \qquad i = 0,\dots,N. \tag{3.12}$$

Then one can show (see details in [10]) that $\lim_{N\to\infty} |v^{(N,i)}(t) - u(t,ih)| = 0$, $i = 0,\dots,N$.

The schematic picture of the chain method and the method of lines can be seen in Figures 14 and 15, respectively. It can bee seen that the difference between the two methods is the computation of the first and second components, and the definition of the initial values of the variables.

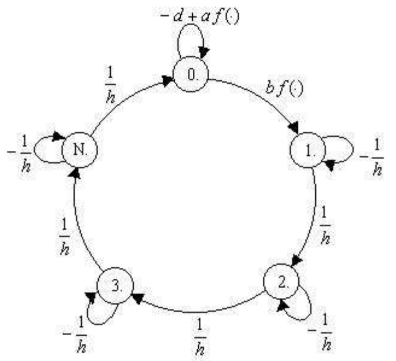

Figure 14: chain method

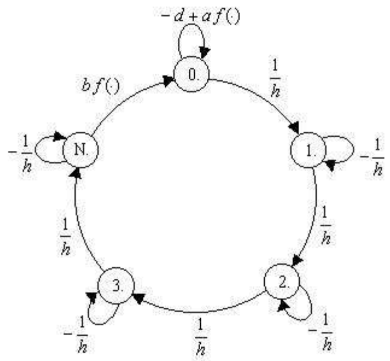

Figure 15: method of lines

Example 3.2 Consider again IVP (3.6)-(3.7), and now we we use scheme (3.1)–(3.5) to get its approximate solution. In Table 2 we compared the numerical results to the true solution. We can observe linear convergence to the true solution.

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Table 2: Method of lines.

h	N	$\lvert v^{(N,0)}(2) - x(2)\rvert$	$\lvert v^{(N,0)}(4) - x(4)\rvert$	$\lvert v^{(N,0)}(6) - x(6)\rvert$
0.250000	5	0.000444	0.123370	0.057385
0.111111	10	0.003646	0.061677	0.037875
0.020408	50	0.000445	0.012041	0.008720
0.010101	100	0.000153	0.006029	0.004431
0.005025	200	0.000044	0.003021	0.002234

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