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A new approach to the global asymptotic stability problem in a delay Lotka-Volterra differential equation

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1. Introduction

In the following we consider the nonautonomous Lotka-Volterra type delay differential equation

$$\dot{N}(t) = h(N(t))\left[r(t) - a_0(t)N(t) - \sum_{i=1}^{n} a_i(t)N(p_i(t)) + \sum_{j=1}^{m} b_j(t)N(q_j(t))\right] \tag{1.1}$$

where $h, r, a_i, b_j : R_+ \to R_+ \quad (0 \le i \le n,\ 1 \le j \le m)$ and $p_i : R_+ \to R$, $p_i(t) < t$ $(t \ge 0), \quad 1 \le i \le n, \quad q_j : R_+ \to R$, $q_j(t) < t \quad (t \ge 0), \quad 1 \le j \le m$, are some given functions.

Equations like (1.1) are important in the single species models and there is a considerable work done by several authors on the theory of global asymptotic stability of Lotka-Volterra type equations with delay. Especially, the books of Gopalsamy [3] and Kuang [12] are good sources for global attractivity results for Lotka-Volterra equations.

A large portion of the investigated models assume the so-called negative feedback effect which often appears as a nondelayed force, such as the term $-a_0(t)N(t)$ in Eq. (1.1). When such a term dominates the others, there are several kinds of conditions which guarantee that Eq. (1.1) is globally asymptotically stable. But it seems to us that even in this special case there are interesting open problems. One of these questions is how to handle equations without steady state solutions. In his recent work [13], Kung established sufficient conditions for global asymptotic stability of delay differential equations without steady state solutions. These results and the applied methods were essentially developed further in a recent paper [2] by Bereketoglu and the recent author. The method applied in these papers are based on Liapunov functionals, and it is effective if $-a_0(t)N(t)$ is a dominating term, which essentially means that $\sum_{i=1}^{n} a_i(t) + \sum_{j=1}^{m} b_j(t) \le q a_0(t), \quad t \ge T$, $q \in (0,1)$. In this condition the possible stabilization effects of the negative feedback terms

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with delays, such as the terms $-a_i(t)N(p_i(t))$ in Eq. (1.1), are not taking into account. On the other hand the Liapunov technique is much less effective if Eq. (1.1) is so called "pure-delay type", that is when $a_0(t) = 0 \quad (t \geq 0)$ - even if the equation has a saturated equilibria. Recently He [10] proved some global asymptotic stability results for "pure-delay type" equations under the a priori assumption that the equations are uniform persistent and the solutions are ultimately bounded on $[0, \infty)$. However, he also remarked that proving the uniform persistence and finding the ultimate upper bounds of the solutions - which are neccesary in the application of his theorem - are still open problems for "pure-delay type" equations.

Our focus in this paper is the global asymptotic stability analysis of the delay differential equation (1.1) when it does not have a steady state solution. We derive new sufficient and also necessary, in some special case necessary and sufficient conditions for Eq. (1.1) to be globally asymptotically stable. The methods to be used in this paper are different from those used in the above mentioned papers. To some extent, our approaches are related to some "simple" techniques, basically applying variation of constants formula and some basic comparison and oscillatory results from the theory of delay differential equations and inequalities.

This paper is organized as follows.

In Section 2 we obtain a priori upper bounds of the positive solutions of the nonautonomous logistic equation (1.1), based on the fact that the solutions can be classified with respect to the existence of the integral $\int_0^\infty a(t)h(N(t))dt$, where $a : R_+ \to R_+$ is a suitable function (see Theorem 2.1). In this section we also show that the conditions which are sufficient for the existence of an upper bound for the positive solutions are also necessary if the negative feedback terms with delays are not present, that is $a_i(t) = 0$ $(t \geq 0,\, 1 \leq i \leq n)$ in Eq. (1.1).

In Section 3 we give global asymptotic stability results when the negative feedback term without delay is dominating, but the nonautonomous equation does not have a steady state solution. Our sufficient conditions for global asymptotic stability of Eq. (1.1) are delay independent when the asymptotic equilibrium is constant. Otherwise the delays play role in our conditions but only trough the time dependent global attractor.

Section 4 contains an entirely new approach of the global asymptotic stability analysis of nonautonomous delay logistic equations. We show that the negative feedback terms with "small delay" help in the stabilization of Eq. (1.1). Roughly speeking we separate the negative feedback terms into two classes $a_1, ..., a_{n_0}$ and $a_{n_0+1}..., a_n$ such a way that the delays belonging to the coefficients a_i $(1 \leq i \leq n_0)$ are small enough and $\sum_{s=n_0+1}^{n} a_s(t) + \sum_{j=1}^{n} b_j(t) \leq q \sum_{i=0}^{n_0} a_i(t) \quad (t \geq 0)$ with a constant $q \in (0, 1)$. Our approach is related to the method based on the monotone semiflow theory and applied for autonomous differential equations. The reader is refereed to the recent book of Smith [15] for an extensive discussion of this theory and also for some applications to autonomous population model equations. Although these results are not directly applicable to our nonautonomous

case, they influenced us how we can apply our method introduced in 1989 in the paper [4] (see also [5], [7]).

In [4] the global attractivity results were proved for delay differential equations which are controlled by a time dependent negative delayed feedback term. In Section 4 of the present paper the main idea of [4] is combined with some comparison results from [8] and also some basic facts from the oscillation theory of delay differential equations to prove the global asymptotic stability results.

In Section 5 we discuss the sharpness of our results in autonomous cases and we compare them to some know results. Especially, we show that one of our result generalize Theorem 5.6 of Kuang in [12], p.35. We also construct examples which show the sharpness of our results and also that the delay in the negative feedback term can destroy the global asymptotic stability property if it is greater than the delay in a positive feedback term. Interesting to note that this fact is true even for small delays. We also raise up some open questions based on our investigations and examples.

2. A priori bounds

Consider the following Lotka-Volterra equation

$$\dot{N}(t) = h(N(t)) \left[r(t) - a_0(t)N(t) - \sum_{i=1}^{n} a_i(t)N(p_i(t)) + \sum_{j=1}^{m} b_j(t)N(q_j(t)) \right], \qquad (2.1)$$

where

(i) $h : R_+ \to R_+$ ($R_+ = [0, \infty)$)is a continuous function, $h(0) = 0$ and $h(u) > 0, \quad u > 0, \quad h(u) \to +\infty, \quad u \to +\infty$.

(ii) $r : R_+ \to R$ is locally Lebesgue integrable and locally bounded.

(iii) $a_i : R_+ \to R_+ \quad (0 \le i \le n)$ and $b_j : R_+ \to R_+ \quad (1 \le j \le m)$ are locally Lebesgue integrable and locally bounded on R_+.

(iv) $p_i : R_+ \to R \quad (1 \le i \le n)$ and $q_j : R_+ \to R \quad (1 \le j \le m)$ are locally Lebesgue integrable and locally bounded on R_+, moreover $p_i(t) < t \quad (t \ge 0), \quad \lim_{t \to +\infty} p_i(t) = +\infty$ $(1 \le i \le n)$ and $q_j(t) < t \quad (t \ge 0), \quad \lim_{t \to +\infty} q_j(t) = +\infty \quad (1 \le j \le m)$.

Let $t_{-1}^1(s) = \min_{1 \le i \le n} \left\{ \inf_{t \ge s} \{p_i(t)\} \right\}, \quad t_{-1}^2(s) = \min_{1 \le j \le m} \left\{ \inf_{t \ge s} \{q_j(t)\} \right\}$ and $t_{-1}(s) = \min \left\{ t_{-1}^1(s),\ t_{-1}^2(s) \right\}$ for any $s \ge 0$. We assume that $t_{-1}^1(s) > -\infty$ and $t_{-1}^2(s) > -\infty$.

Definition 2.1 *A function $N : [t_{-1}(0), \infty) \to R$ is called positive solution of Eq. (2.1) if it is positive on $[t_{-1}(0), \infty)$, Lebesgue integrable and bounded on $[t_{-1}(0), 0]$, absolutely continuous on $[0, \infty)$ and N satisfies Eq. (2.1) for almost every $t \ge 0$.*

In this paper we do not deal with the existence and uniqueness of the solutions. We assume that some additional conditions are satisfied for the right hand side of Eq. (2.1) such that the solutions of Eq. (2.1) exist on $[t_{-1}(0), \infty)$. It is worth noting that the uniqueness of the solutions are not needed in our results.

For any function $c : R_+ \to R$ the functions $c^+ : R_+ \to R_+$ and $c^- : R_+ \to R_+$ are defined by the relations $c^+(t) = \max\{0,\ c(t)\}$ and $c^-(t) = \max\{0,\ -c(t)\}$, $t \geq 0$.

The following result gives a possible separation of the solutions of Eq. (2.1) with respect to the existence of the integral $\int_0^\infty a(t)h(N(t))dt$, where $a : R_+ \to R_+$ is a given function.

Theorem 2.1 *Assume (i)-(iv) and suppose that there exists a locally Lebesgue integrable, locally bounded function $a : R_+ \to (0, \infty)$ such that*

$$\int_0^\infty a(t)dt = +\infty, \tag{2.2}$$

moreover

$$\widehat{r} = \sup_{t\geq 0}\left\{\frac{r^+(t)}{a(t)}\right\} < \infty \quad \textit{and} \quad \widehat{b} = \sup_{t\geq 0}\left\{\frac{\sum_{j=1}^m b_j(t)}{a(t)}\right\} < \infty. \tag{2.3}$$

Then

(A) *For any solution $N : [t_{-1}(0), \infty) \to R_+$ of Eq. (2.1) the relation*

$$\int_0^\infty a(t)h(N(t))dt < \infty, \tag{2.4}$$

yields $\lim_{t\to+\infty} N(t) = 0$.

(B) *If*

$$\liminf_{t\to+\infty} \frac{r(t)}{a(t)} > 0 \quad \textit{and} \quad \sup_{t\geq 0}\left\{\sum_{i=0}^n \frac{a_i(t)}{a(t)}\right\} < \infty, \tag{2.5}$$

then for any positive solution $N : [t_{-1}(0), \infty) \to R_+$ one has $\int_0^\infty a(t)h(N(t))dt = +\infty$.

Proof. (A) Assume that the solution $N : [t_{-1}(0), \infty) \to R_+$ satisfies (2.4). Then from Eq. (2.1) we obtain

$$\dot{N}(t) \leq a(t)h(N(t))\left[\frac{r^+(t)}{a(t)} + \sum_{j=1}^m \frac{b_j(t)}{a(t)}N(q_j(t))\right], \quad \text{a.e. } t \geq 0.$$

Let $\alpha(t) = a(t)h(N(t))$, $t \geq 0$. Then (2.3) yields

$$\dot{N}(t) \leq \widehat{r}\cdot\alpha(t) + \widehat{b}\cdot\alpha(t)\cdot\max_{1\leq j\leq m}\{N(q_j(t))\}, \quad \text{a.e. } t \geq 0.$$

Integrating both sides of the above inequality, we find

$$N(t) \le c + \widehat{b} \int_0^t \alpha(s) \cdot \max_{1\le j\le m} \{N(q_j(s))\}\, ds, \quad t \ge 0,$$

where $c = N(0) + \widehat{r} \cdot \int_0^\infty \alpha(t)dt < \infty$.

Let $u(t) = \max_{t_{-1}(0)\le s\le t} \{N(s)\}, \quad t \ge 0, \quad$ and $\quad c_1 = \max\left\{c, \max_{t_{-1}(0)\le s\le 0} \{N(s)\}\right\}$. Then

$$u(t) \le c_1 + q \int_0^t \alpha(s)u(s)ds, \quad t \ge 0,$$

and by using the Gronwall-inequality, we obtain

$$u(t) \le c_1 e^{q\int_0^t \alpha(s)ds} \le c_1 e^{q\int_0^\infty \alpha(s)ds} < \infty, \quad t \ge 0.$$

This yields that the solution N of Eq. (2.1) is bounded on $[t_{-1}(0), \infty)$, and hence under our conditions, we find

$$M_1 = \int_0^\infty a(t)h(N(t)) \left[\frac{r^+(t)}{a(t)} + \sum_{j=1}^m \frac{b_j(t)}{a(t)} N(q_j(t))\right] dt < \infty.$$

On the other hand, from Eq. (2.1), we obtain

$$N(t) + \int_0^t a(s)h(N(s)) \left[\frac{r^-(s)}{a(s)} + \frac{a_0(s)}{a(s)}N(s) + \sum_{i=1}^n \frac{a_i(s)}{a(s)} N(p_i(s))\right] ds =$$

$$N(0) + \int_0^t a(s)h(N(s)) \left[\frac{r^+(s)}{a(s)} + \sum_{j=1}^m \frac{b_j(s)}{a(s)} N(q_j(s))\right] ds \le N(0) + M_1 < \infty, \quad t \ge 0.$$

This yields

$$\int_0^\infty a(s)h(N(s)) \left[\frac{r^-(s)}{a(s)} + \frac{a_0(s)}{a(s)} + \sum_{i=1}^n \frac{a_i(s)}{a(s)} N(p_i(s))\right] ds < \infty,$$

and hence $N(\infty) = \lim_{t\to+\infty} N(t)$ exists and is finite. It remains to show that $N(\infty) = 0$. Otherwise, $N(\infty) > 0$, and there is a constant $T > 0$ such that $N(t) \ge \frac{1}{2}N(\infty) > 0$ for $t > T$. Thus $\beta = \inf_{t\ge T} \{h(N(t))\} \ge \inf\left\{h(u) : u \ge \frac{1}{2}N(\infty)\right\} > 0$, and hence $\int_T^\infty a(t)h(N(t))dt \ge \beta \int_T^\infty a(t)dt = +\infty$ contradicting (2.4). The proof of statement (A) is complete.

(B) For the seeking of contradiction, assume that under condition (2.5) there is a positive solution $N : [t_{-1}(0), \infty) \to R_+$ of Eq. (2.1) such that $\int_0^\infty a(t)h(N(t))dt < \infty$.

Then by statement (A) we have $\lim_{t\to+\infty} N(t) = 0$. Therefore (2.5) yields that there exists a $T \geq 0$ and $c > 0$ such that

$$h(N(t))\left[r(t) - a_0(t)N(t) - \sum_{i=1}^{n} a_i(t)N(p_i(t)) + \sum_{j=1}^{M} b_j(t)N(q_j(t))\right] \geq a(t)h(N(t)) \cdot c,$$

for $t \geq T$. Therefore Eq. (2.1) implies

$$\dot{N}(t) \geq a(t)h(N(t)) \cdot c, \quad t \geq T,$$

and hence

$$N(t) \geq N(T) + \int_T^t a(s)h(N(s))ds \cdot c, \quad t \geq T.$$

This yields that $\lim_{t\to+\infty} N(t) > 0$ which is a contradiction. The proof of statement (B) is complete. ■

In the next theorem we give an upper bound for the solutions of Eq. (2.1).

Theorem 2.2 *Assume that (i)-(iv) are satisfied, and*

(v) $\quad a_0(t) > 0, \quad t \geq 0, \quad \int_0^\infty a_0(t)dt = +\infty$, *moreover*

$$\widehat{\gamma} = \limsup_{t\to+\infty} \frac{r^+(t)}{a_0(t)} < \infty \quad \text{and} \quad \widehat{\beta} = \limsup_{t\to+\infty} \frac{\sum_{j=1}^{m} b_j(t)}{a_0(t)} < 1. \tag{2.6}$$

Then for any solution $N : [t_{-1}(0), \infty) \to R_+$ of Eq. (2.1), one has

$$\limsup_{t\to+\infty} N(t) \leq \frac{\widehat{\gamma}}{1-\widehat{\beta}}. \tag{2.7}$$

In the proof of the above theorem we need the following lemma which is a simple generalization of a similar result proved in [1] in that case when $a(t) = a_0 (> 0)$, $t \geq 0$, is a constant. The proof of the next lemma is omitted.

Lemma 2.1 *Let $a : R_+ \to R_+$ and $g : R_+ \to R$ be locally Lebesgue integrable functions such that*

$$\int_0^\infty a(t)dt = +\infty \quad \text{and} \quad \sup_{t\geq 0} |g(t)| < \infty.$$

Then for any $T \geq 0$,

$$\begin{aligned}
\liminf_{t\to+\infty} g(t) &\leq \liminf_{t\to+\infty} e^{-\int_T^t a(s)ds} \int_T^t e^{\int_T^u a(s)ds} a(u)g(u)du \leq \\
&\leq \limsup_{t\to+\infty} e^{-\int_T^t a(s)ds} \int_T^t e^{\int_T^u a(s)ds} a(u)g(u)du \leq \limsup_{t\to+\infty} g(t),
\end{aligned}$$

whenever $\lim_{t\to+\infty} g(t)$ *exists.*

The proof of Theorem 2.2. Let $N : [t_{-1}(0), \infty) \to R_+$ be any solution of Eq. (2.1). Let $\alpha(t) = a_0(t)h(N(t)), \quad t \geq 0$. Then either $\int_0^\infty \alpha(t)dt < \infty$ or $\int_0^\infty \alpha(t)dt = +\infty$. If $\int_0^\infty \alpha(t)dt < \infty$ then by the statement (A) of Theorem 2.1 we have $\lim_{t\to+\infty} N(t) = 0$ and hence (2.7) is satisfied. Now, assume that $\int_0^\infty \alpha(t)dt = +\infty$. From Eq. (2.1), we have

$$\dot{N}(t) \leq -\alpha(t)N(t) + \alpha(t)\frac{r^+(t)}{a_0(t)} + \alpha(t)\sum_{j=1}^{m}\frac{b_j(t)}{a_0(t)}N(q_j(t)), \quad \text{a.e.} \quad t \geq 0.$$

Thus

$$N(t) \leq e^{-\int_0^t \alpha(s)ds}\left[N(0) + \int_0^t e^{\int_0^u \alpha(s)ds}\alpha(u)\left[\frac{r^+(u)}{a_0(u)} + \sum_{j=1}^{m}\frac{b_j(u)}{a_0(u)}N(q_j(u))\right]du\right], \quad t \geq 0. \tag{2.8}$$

Since $\int_0^\infty \alpha(t)dt = \infty$, by Lemma 2.1 and condition (2.6) we find

$$\limsup_{t\to+\infty} e^{-\int_0^t \alpha(s)ds}\int_0^t e^{\int_0^u \alpha(s)ds}\alpha(u)\sum_{j=1}^{m}\frac{b_j(u)}{a_0(u)}du \leq \widehat{\beta} < 1,$$

and

$$M_T = \sup_{t\geq T} e^{-\int_0^t \alpha(s)ds}\int_0^t e^{\int_0^u \alpha(s)ds}\alpha(u)\frac{r^+(u)}{a_0(u)}du < \infty.$$

So, for arbitrary $\beta_1 \in \left(\widehat{\beta}, 1\right)$ we may (and do) fix a constant $T > 0$ such that

$$e^{-\int_0^t \alpha(s)ds}\int_0^t e^{\int_0^u \alpha(s)ds}\alpha(u)\sum_{j=1}^{m}\frac{b_j(u)}{a_0(u)}du \leq \beta_1 < 1, \quad t \geq T. \tag{2.9}$$

Let M be a constant such that

$$M > \max\left\{\frac{N(0) + M_T}{1-\beta_1}, \quad \max_{t_{-1}(0)\leq s\leq T} N(s)\right\}. \tag{2.10}$$

Now, we show that $N(t) < M$ for all $t \geq 0$. Otherwise, there exists a $T_1 > T$ such that

$$0 \leq N(t) < M, \quad t_{-1}(0) \leq t < T_1, \quad \text{and} \quad N(T_1) = M.$$

But, in that case (2.8)-(2.10) yield

$$N(T_1) \leq N(0) + M_T + \beta_1 M < M,$$

which is a contradiction. So N is bounded on $[t_{-1}(0),\infty)$ and hence $\widehat{N} = \limsup_{t\to+\infty} N(t) < \infty$. By Lemma 2.1, from (2.8), it follows

$$\widehat{N} \leq \limsup_{t\to+\infty} \frac{r^+(t)}{a_0(t)} + \limsup_{t\to+\infty} \sum_{j=1}^{m} \frac{b_j(t)}{a_0(t)} \cdot \widehat{N} = \widehat{\gamma} + \beta_1 \cdot \widehat{N},$$

and hence (2.7) is satisfied. The proof of the theorem is complete.

The next result shows that the condition $\widehat{\beta} < 1$ in the above theorem is important.

Theorem 2.3 *Assume that conditions (i)-(v) are satisfied, moreover* $a_1(t) = ... = a_n(t) = 0, \quad r(t) \geq 0, \quad t \geq 0$.
If either

$$\widehat{\beta}_0 = \liminf_{t\to+\infty} \sum_{j=1}^{m} \frac{b_j(t)}{a_0(t)} > 1$$

or

$$\widehat{\beta}_0 = 1 \quad \textit{and} \quad \widehat{\gamma}_0 = \liminf_{t\to+\infty} \frac{r(t)}{a_0(t)} > 0,$$

then for any positive solution $N : [t_{-1}(0),\infty) \to R_+$ *of Eq. (2.1), one has* $\lim_{t\to+\infty} N(t) = +\infty$.

Proof. Let $N : [t_{-1}(0),\infty) \to R_+$ be a positive solution of Eq. (2.1). First we show that $\int_0^\infty a_0(t)h(N(t))dt = +\infty$. Otherwise $\int_0^\infty \alpha(t)dt < \infty$, where $\alpha(t) = a_0(t)h(N(t)), \quad t \geq 0$. Since $a_i(t) = 0$, $0 \leq i \leq n$, $t \geq 0$, from Eq. (2.1) we have

$$N(t) = e^{-\int_0^t \alpha(s)ds} N(0) + e^{-\int_0^t \alpha(s)ds} \int_0^t e^{\int_0^u \alpha(s)ds} \alpha(u) \left[\frac{r(u)}{a_0(u)} + \sum_{j=1}^{m} \frac{b_j(u)}{a_0(u)} N(q_j(u)) \right] du, \quad t \geq 0, \tag{2.11}$$

and hence, by using Lemma 2.1, we find

$$\liminf_{t\to+\infty} \; N(t) \geq \liminf_{t\to+\infty} \frac{r(t)}{a_0(t)} = \widehat{\gamma}_0 > 0. \tag{2.12}$$

On the other hand $\int_0^\infty a_0(t)dt = +\infty$ and hence (2.12) and Theorem 2.1 yield $\int_0^\infty a_0(t)h(N(t))dt = +\infty$. This is a contradiction and so $\int_0^\infty a_0(t)h(N(t))dt = +\infty$. It remains to show that $N(t) \to +\infty$, as $t \to +\infty$. Otherwise $\liminf_{t\to+\infty} N(t) < \infty$ and by Lemma 2.1 and (2.11), we obtain $\infty > \liminf_{t\to+\infty} N(t) \geq \widehat{\gamma}_0 + \widehat{\beta}_0 \liminf_{t\to+\infty} N(t)$, which can not be satisfied if either $\widehat{\beta}_0 > 1$ or $\widehat{\beta}_0 = 1$ and $\widehat{\gamma}_0 > 0$. Therefore $N(t) \to +\infty$ as $t \to +\infty$, and the proof of the theorem is complete. ■

3. Global asymptotic stability when $a_0(t)$ is dominating

In this section we give global asymptotic stability results for Eq. (2.1) when $a_0(t) > 0$, $t \geq 0$.

Definition 3.1 *We say that Eq. (2.1) is globally asymptotically stable if for any two positive solutions $N_1, N_2 : [t_{-1}(0), \infty) \to R_+$ of Eq. (2.1),*

$$\lim_{t\to+\infty} (N_1(t) - N_2(t)) = 0.$$

It is clear that Eq. (2.1) is globally asymptotically stable if and only if there exists a function K:$[t_{-1}(0), \infty) \to R_+$ such that for any positive solution $N : [t_{-1}(0), \infty) \to R_+$ of Eq. (2.1) one has

$$\lim_{t\to+\infty} (N(t) - K(t)) = 0,$$

that is K is a global attractor with respect to the positive solutions of Eq. (2.1).

Now, assume that the following hypothesis is satisfied

(vi) $a_0(t) > 0$, $t \geq 0$, $\int_0^\infty a_0(t)dt = +\infty$, and the limits

$$c_0 = \lim_{t\to+\infty} \frac{r(t)}{a_0(t)}, \quad \alpha_0 = \lim_{t\to+\infty} \frac{\sum_{i=1}^{n} a_i(t)}{a_0(t)}, \quad \beta_0 = \lim_{t\to+\infty} \frac{\sum_{j=1}^{m} b_j(t)}{a_0(t)}$$

exist and are finite, moreover

$$q_0 = \alpha_0 + \beta_0 < 1.$$

In the following result we show that $\frac{c_0}{1+\alpha_0-\beta_0}$ is the limit of all positive solutions of Eq. (2.1) whenever $c_0 > 0$, otherwise the solutions of Eq. (2.1) tend to zero at $+\infty$.

Theorem 3.1 *Assume (i)-(iv) and (vi). Then*

(A) If $c_0 \leq 0$ then for any solution $N : [t_{-1}(0), \infty) \to R$ of Eq. (2.1), one has $\lim_{t\to+\infty} N(t) = 0$, and hence the zero solution of Eq. (2.1) is a global attractor.

(B) If $c_0 > 0$ then for any positive solution $N : [t_{-1}(0), \infty) \to R_+$ of Eq. (2.1) one has

$$\lim_{t\to+\infty} N(t) = \frac{c_0}{1 + \alpha_0 - \beta_0}, \tag{3.1}$$

and hence Eq. (2.1) is globally asymptotically stable.

Proof. (A) Let $c_0 \leq 0$ and fix an arbitrary positive solution $N : [t_{-1}(0), \infty) \to R_+$ of Eq. (2.1). If $\int_0^\infty a_0(t)h(N(t))dt < \infty$ then by Theorem 2.1 we have $\lim_{t\to+\infty} N(t) = 0$.

Now, assume that $\int_0^\infty \alpha(t)dt = +\infty$, $\alpha(t) = a_0(t)h(N(t))$, $t \geq 0$. Then

$$\dot{N}(t) \leq \alpha(t)\frac{r(t)}{a_0(t)} - \alpha(t)N(t) + \alpha(t)\sum_{j=1}^{m} \frac{b_i(t)}{a_0(t)} N(q_j(t)), \quad \text{a.e.} \quad t \geq 0,$$

and hence

$$N(t) \leq e^{-\int_0^t \alpha(u)du} N(0) + e^{-\int_0^t \alpha(u)du} \int_0^t e^{\int_0^s \alpha(u)du} \frac{r(s)}{a_0(s)} ds +$$

$$e^{-\int_0^t \alpha(u)du} \int_0^t e^{\int_0^s \alpha(u)du} \alpha(s) \sum_{j=1}^{m} \frac{b_j(s)}{a_0(s)} N(q_j(s)) ds, \quad t \geq 0.$$

We know, in virtue of Theorem 2.1, that $\widehat{N} = \limsup_{t \to +\infty} N(t) < \infty$, and by using Lemma 2.1 we find

$$\widehat{N} \leq c_0 + \beta_0 \widehat{N},$$

that is $\widehat{N} \leq \frac{c_0}{1-\beta_0} \leq 0$. On the other hand from the nonnegativity of N we have $0 \leq \widehat{N}$, so either $N(t) \to +\infty$, $t \to +\infty$, or $\int_0^\infty \alpha(t)dt < \infty$. The proof of statement (A) is complete.

(B) Assume that $c_0 > 0$ and let $N : [t_{-1}(0), \infty) \to R_+$ be any positive solution of Eq. (2.1). By Theorem 2.2, we have

$$0 \leq m_0 = \liminf_{t \to +\infty} N(t) \leq M_0 = \limsup_{t \to +\infty} N(t) < \infty,$$

and by statement (B) of Theorem 2.1 we know that $\int_0^\infty \alpha(t)dt = +\infty$,

$\alpha(t) = a_0(t)h(N(t)), \quad t \geq 0.$

But, by using the variation of constant formula, Eq. (2.1) can be written in the following equivalent form

$$N(t) = e^{-\int_0^t \alpha(s)ds} N(0) + e^{-\int_0^t \alpha(s)ds} \int_0^t e^{\int_0^u \alpha(s)ds} \alpha(u) \frac{r(u)}{a_0(u)} du -$$

$$\sum_{i=1}^{n} e^{-\int_0^t \alpha(s)ds} \int_0^t e^{\int_0^u \alpha(s)ds} \alpha(u) \frac{a_i(u)}{a_0(u)} N(p_i(u)) du + \sum_{j=1}^{m} e^{-\int_0^t \alpha(s)ds} \int_0^t e^{\int_0^u \alpha(s)ds} \alpha(u) \frac{b_j(u)}{a_0(u)} N(q_j(u)) du,$$

for $t \geq 0$.

By using Lemma 2.1, we find

$$M_0 \leq c_0 - \alpha_0 m_0 + \beta_0 M_0$$

and

$$m_0 \geq c_0 - \alpha_0 M_0 + \beta_0 m_0.$$

From that, it follows

$$0 \leq M_0 - m_0 \leq (\alpha_0 + \beta_0)(M_0 - m_0) \leq q_0(M_0 - m_0),$$

and since $q_0 < 1$, we obtain $M_0 = m_0$. This means that $N(\infty) = \lim_{t \to +\infty} N(t)$ exists and $N(t)$ satisfies the required relation (3.1). The proof of the theorem is complete. ■

Now, we investigate the case when $h(u) = u$, $u \geq 0$, that is when Eq. (2.1) reduces to

$$\dot{N}(t) = N(t)\left[r(t) - a_0(t)N(t) - \sum_{i=1}^{n} a_i(t)N(p_i(t)) + \sum_{j=1}^{m} b_j(t)N(q_j(t))\right], \quad \text{a.e.} \quad t \geq 0. \tag{3.2}$$

Definition 3.2 *For a locally Lebesgue integrable and locally bounded function $g : R_+ \to R$, the equation*

$$\dot{K}(t) = K(t)\left[r(t) - a_0(t)K(t) - \sum_{i=1}^{n} a_i(t)K(p_i(t)) + \sum_{j=1}^{m} b_j(t)K(q_j(t)) + g(t)\right], \quad \textit{a.e.} \quad t \geq 0, \tag{3.3}$$

is called the $g-$perturbed equation of Eq. (3.2).

It is clear that when $h(u) = u$, $u \in R$ and (vi) is satisfied then the function $K_{g_0}(t) = \frac{c}{1+\alpha-\beta}$, $t \geq t_{-1}(0)$, is a solution of the g_0 perturbed equation (3.3), where $g_0 : R_+ \to R$ is a suitable function such that $g_0(t) \to 0$, $t \to +\infty$.

In the next result we generalize Theorem 3.1 in the following sense: We prove that under certain conditions, for any positive solution $N : [t_{-1}(0), \infty) \to R_+$ of Eq. (3.2) one has

$$\lim_{t\to+\infty} \frac{N(t)}{K_g(t)} = 1,$$

where K_g is a solution of the related g-perturbed equation (3.3)

In fact, if K_g is a bounded function on R_+, then (3.5) yields

$$\lim_{t\to+\infty} (N(t) - K_g(t)) = 0, \tag{3.4}$$

which means that K_g is a global attractor with respect to the positive solutions of Eq. (3.2), or equivalently Eq. (3.2) is globally asymptotically stable with respect to the positive solutions.

Theorem 3.2 *Assume (ii)-(iv) and*

$$a_0(t) > 0, \quad t \geq 0, \quad \int_0^{\infty} a_0(t)dt = +\infty.$$

Let $g : R_+ \to R$ be a locally Lebesgue integrable and locally bounded function and $K_g : [t_{-1}(0), \infty) \to R_+$ is a solution of (3.3) such that

$$K_g(p_i(t)) \geq \delta \cdot K_g(t), \quad 1 \leq i \leq n, \quad K_g(q_j(t)) \geq \delta \cdot K_g(t), \quad 1 \leq j \leq m, \tag{3.5}$$

$\delta > 0$ is a constant and

$$\frac{g(t)}{a_0(t)K_g(t)} \to 0, \quad t \to +\infty, \tag{3.6}$$

moreover

$$\liminf_{t\to+\infty} \frac{r(t)}{a_0(t)} > 0 \text{ and } \widehat{q} = \limsup_{t\to+\infty} \frac{\sum_{i=1}^{n} a_i(t)K_g(p_i(t)) + \sum_{j=1}^{m} b_j(t)K_g(q_j(t))}{a_0(t)K_g(t)} < 1. \tag{3.7}$$

Then, for any positive solution $N : [t_{-1}(0), \infty) \to R_+$ of Eq. (3.2) relation (3.4) holds, and if K_g is bounded on R_+ then K_g is a global attractor with respect to the positive solutions of Eq. (3.2), and hence (3.2) is globally asymptotically stable.

Proof. Let

$$x(t) = \frac{N(t)}{K_g(t)} - 1, \quad t \geq t_{-1}(0),$$

for any arbitrarily fixed positive solution $N : [t_{-1}(0), \infty) \to R_+$ of Eq. (3.2).

Then

$$\dot{x}\,(t) = \frac{\dot{N}\,(t)}{K_g(t)} - \frac{N(t)}{K_g(t)}\,\frac{\dot{K_g}\,(t)}{K_g(t)} =$$

$$\frac{N(t)}{K_g(t)}\left[-a_0(t)(N(t) - K_g(t)) - \sum_{i=1}^{n} a_i(t)(N(p_i(t)) - K_g(p_i(t))) + \right.$$

$$\left. + \sum_{j=1}^{m} b_j(t)N(q_j(t)) - K_g(q_j(t))) - g(t)\right],$$

for a.e. $t \geq 0$.

Let $\alpha(t) = a_0(t)N(t)$ and $g_0(t) = \frac{g(t)}{a_0(t)K_g(t)}, \quad t \geq 0$. Then

$$\dot{x}\,(t) = -\alpha(t)x(t) - \alpha(t)\sum_{i=1}^{n} \frac{a_i(t)K_g(p_i(t))}{a_0(t)K_g(t)}x(p_i(t)) +$$

$$+\alpha(t)\sum_{j=1}^{m} \frac{b_j(t)K_g(q_j(t))}{a_0(t)K_g(t)}x(q_j(t)) - \alpha(t)g_0(t), \quad \text{a.e.} \quad t \geq 0.$$

Because of $\widehat{q} < 1$, there exist $T > 0$ and $q \in (0, 1)$ such that

$$\sum_{i=1}^{n} \frac{a_i(t)K_g(p_i(t))}{a_0(t)K_g(t)} + \sum_{j=1}^{m} \frac{b_j(t)K_g(q_j(t))}{a_0(t)K_g(t)} \leq q, \quad t \geq T. \tag{3.8}$$

In that case

$$|x(t)| \leq e^{-\int_T^t \alpha(s)ds}\,|x(T)| +$$

$$+e^{-\int_T^t \alpha(s)ds}\int_T^t e^{\int_0^u \alpha(s)ds} \cdot q \cdot \alpha(u) \max\left\{\max_{1\leq i\leq n}\,|x(p_i(u))|\,, \max_{1\leq j\leq m}\,|x(q_j(u))|\right\} du +$$

$$e^{-\int_T^t \alpha(s)ds}\int_T^t e^{\int_0^u \alpha(s)ds}\alpha(u)\,|g_0(u)|\,du, \quad t \geq T. \tag{3.9}$$

Let M be a constant for which

$$|x(t)| \leq M, \quad t_{-1}(0) \leq t \leq T, \quad \text{and} \quad M > \frac{c_1}{1-q}$$

are satisfied, where $c_1 = |x(T)| + \sup_{u\geq T}\,|g_0(u)|\,.$

Now, we show $|x(t)| \leq M$ for any $t \geq T$. Otherwise there exists a $T_1 > T$ such that

$$|x(t)| < M, \quad T \leq t < T_1, \quad \text{and} \quad |x(T_1)| = M.$$

But, from (3.9), it follows

$$|x(T_1)| \leq |x(T)| + \sup_{u \geq T} |g_0(u)| + q \cdot M < M,$$

which is a contradiction. So $|x(t)| \leq M$, $t \geq t_{-1}(0)$, and hence $\widehat{x} = \limsup_{t \to +\infty} |x(t)| < \infty$.

On the other hand from (3.6) and (3.8) it follows

$$\limsup_{t \to +\infty} \left(\frac{\sum_{i=1}^{n} a_i(t)}{a_0(t)} + \frac{\sum_{j=1}^{m} b_j(t)}{a_0(t)} \right) < \infty,$$

and hence by using statement B of Theorem 2.1, we have $\int_T^{\infty} \alpha(t)dt = \int_T^{\infty} a_0(t)h(N(t))dt = +\infty$. So, in virtue of Lemma 2.1, (3.9) yields $\widehat{x} \leq q\widehat{x}$ and hence $\widehat{x} = 0$. This means that

$$x(t) = \frac{N(t)}{K_g(t)} - 1 \to 0, \quad t \to +\infty,$$

and if K_g is bounded on R_+ then $\lim_{t \to +\infty} (N(t) - K_g(t)) = 0$ is also satisfied. The proof of the theorem is complete. ■

4. Global asymptotic stability when $a_0(t)$ is not necessarily dominating

The positivity of $a_0(t)$ is important in the global asymptotic stability results proved in the previous section, where Eq. (2.1) is controlled by an undelayed negative feedback term. Those theorems do not take into account the possible stabilization effect of the negative feedback terms with delay. In this section we prove some new results to show that the negative feedback terms with small delay may help to stabilize Eq. (2.1). The technique of the proof applied in this section is original in the literature of delay population model equations. This technique is borrowed from some earlier papers of the author [4, 5, 7], where the stabilization effect of negative feedback terms with small delays was initiated and analyzed.

We need the following definitions:

Definition 4.1 *We say that the solutions of Eq. (2.1) are eventually bounded by the constant $K_0 > 0$ if for any solution $N : [t_{-1}(0), \infty) \to R_+$ of Eq. (2.1)*

$$\limsup_{t \to +\infty} N(t) \leq K_0 \tag{4.1}$$

Let $L > 0$, $n_0 \in \{0, ..., n\}$ and consider the delay differential equation

$$\dot{u}_{\varepsilon}(t) = -\omega_{L,\varepsilon} \sum_{i=0}^{n_0} a_i(t) u_{\varepsilon}(p_i(t)), \quad \text{a.e.} \quad t \geq 0, \tag{4.2}$$

where $p_0(t) = t, \quad t \geq 0, \quad \omega_{L,\varepsilon} = \max\limits_{0 \leq s \leq L+\varepsilon} \{h(u)\}$.

The function $v_\varepsilon : \{(t,s) : 0 \leq s \leq t\} \to R$ is the fundamental solution of Eq. (4.2) if

$$\frac{\partial v_\varepsilon}{\partial t}(t,s) = -\omega_{L,\varepsilon} \sum_{i=0}^{n_0} a_i(t) v_\varepsilon(p_i(t),s), \quad \text{a.e.} \quad t \geq s \geq 0$$

and

$$v_\varepsilon(t,s) = \begin{cases} 1, & t = s, \\ 0, & t < s. \end{cases}$$

From the general theory of the delay differential equations it is known that the fundamental solution v_ε exists (see, e.g., [1], [9]).

Definition 4.2 *We say that Eq. (2.1) has property $P(L, n_0)$ if there exist $\varepsilon > 0$ and $T_\varepsilon \geq 0$ such that the fundamental solution v_ε of Eq. (4.2) is positive on $\{(t,s) : T_\varepsilon \leq s \leq t\}$.*

In the oscillation theory of delay differential equations, there are several explicit conditions which guarantee that the fundamental solution of a linear delay differential equation is positive. In this direction we refer to the book [6] and to the papers [4, 5, 7].

We need the following lemmas.

Lemma 4.1 *Assume that the solutions of Eq. (2.1) are eventually bounded by the constant $K_0 > 0$ and Eq. (2.1) has property $P(K_0, n_o)$. Then for any positive solution $N : [t_{-1}(0), \infty) \to R_+$ of Eq. (2.1), the fundamental solution $v_N : \{(t,s) : 0 \leq s \leq t\} \to R$ of the delay differential equation*

$$\dot{u}_N(t) = -h(N(t)) \sum_{i=0}^{n_0} a_i(t) u_N(p_i(t)), \quad \text{a.e.} \quad t \geq 0, \tag{4.3}$$

is positive on $T_N \leq s \leq t < \infty$, where $T_N \geq 0$ is a suitable constant depending on the solution $N(t)$.

Proof. Let $N : [t_{-1}(0), \infty) \to R_+$ be a positive solution of Eq. (2.1). Since Eq. (2.1) has property $P(K_0, n_0)$ there exists an $\varepsilon > 0$ such that the fundamental solution $v_\varepsilon : \{(t,s) : 0 \leq s \leq t\} \to R$ of the equation

$$\dot{u}(t) = -\omega_{K_0,\varepsilon} \sum_{i=0}^{n_0} a_i(t) u_\varepsilon(p_i(t)), \quad \text{a.e.} \quad t \geq 0, \tag{4.4}$$

is positive on $T_\varepsilon \leq s \leq t < \infty$, where T_ε is a suitable constant. On the other hand the solutions of Eq. (2.1) are eventually bounded with K_0 and hence there exists T_N such that $T_N \geq T_\varepsilon$ and

$$N(t) \leq K_0 + \varepsilon, \quad t \geq T_N.$$

In that case $h(N(t)) \leq \omega_{K_0,\varepsilon} = \max\limits_{0 \leq u \leq K_0+\varepsilon} \{h(u)\}, \quad t \geq T_N$, and for the fundamental solution v_ε of Eq. (4.4) we have that $v_\varepsilon(t,s) > 0, \quad t \geq s \geq T_\varepsilon$, and hence

$$\frac{\partial v_\varepsilon}{\partial t}(t,s) = -\omega_{K_0,\varepsilon} \sum_{i=0}^{n_0} a_i(t) v_\varepsilon(p_i(t),s) \leq -h(N(t)) \sum_{i=0}^{n_0} a_i(t) v_\varepsilon(p_i(t),s), \quad \text{a.e.} \quad t \geq s \geq T_N.$$

By using a comparison theorem for the positive solutions of linear delay differential equations from [8], we find

$$0 < v_\varepsilon(t,s) \le v_N(t,s), \quad t \ge s \ge T_N,$$

where v_N is the fundamental solution of Eq. (4.3). The proof of the lemma is complete. ■

Lemma 4.2 *Assume that the conditions of Lemma 4.1 are satisfied and $g : R_+ \to R$ is a locally Lebesgue integrable function such that $\sup_{t\ge 0} |g(t)| < \infty$. Then for any positive solution $N : [t_{-1}(0), \infty) \to R_+$ of Eq. (2.1) and for any $T \ge T_N$,*

$$\liminf_{t\to+\infty} g(t) \le \liminf_{t\to+\infty} \int_T^t v_N(t,u)a(u)h(N(u))g(u)du \le$$

$$\limsup_{t\to+\infty} \int_T^t v_N(t,u)a(u)h(N(u))g(u)du \le \limsup_{t\to+\infty} g(t),$$

assuming that $\int_0^\infty a(s)h(N(s))ds = +\infty$, where $a(t) = \sum_{i=0}^{n_0} a_i(t), \quad t \ge t_0$.

Proof. Let $T \ge T_N$ be a fixed number and $y : [t_{-1}(T), \infty) \to R_+$ be a positive solution of Eq. (4.3), where $N : [t_{-1}(T), \infty) \to R_+$ of Eq. (2.1) such that $\int_0^\infty a(s)h(N(s))ds = +\infty$. We show that $y(t) \to 0$, as $t \to +\infty$. In fact, y satisfies

$$\dot{y}(t) = -h(N(t)) \sum_{i=0}^{n_0} a_i(t)y(p_i(t)), \quad t \ge T,$$

and hence $\dot{y}(t) \le 0, \quad t \ge T$. Thus $y(\infty) = \lim_{t\to+\infty} y(t)$ exists. Assume that $y(\infty) > 0$. Then there exists a constant $t_1 \ge T$ such that $y(p_i(t)) \ge c > 0, \quad t \ge t_1$. But in that case

$$\dot{y}(t) \le -ch(N(t)) \sum_{i=0}^{n_0} a_i(t), \quad t \ge t_1,$$

and hence

$$y(t) \le y(t_1) - \int_{t_1}^t ch(N(s)) \sum_{i=0}^{n_0} a_i(s)ds \to -\infty, \quad \text{as} \quad t \to +\infty.$$

This is a contradiction and hence $y(t) \to 0, \quad t \to +\infty$. Now, let $\eta(t) = 1, \quad t \ge t_1(T)(T \ge T_N)$. Then

$$\dot{\eta}(t) = -h(N(t)) \sum_{i=0}^{n_0} a_i(t)\eta(p_i(t)) + h(N(t)) \sum_{i=0}^{n_0} a_i(t), \quad t \ge T.$$

Thus, by using the variation of constants formula this yields

$$\eta(t) = y_1(t;T) + \int_T^t v_n(t,u)a(u)h(N(u))du, t \geq T,$$

where $a(t) = \sum_{i=0}^{n_0} a_i(t), \quad t \geq 0, \quad$ and $\quad y_1(\cdot;T) : [t_{-1}(T), \infty) \to R$ is the solution of Eq. (4.3) with initial condition

$$y_1(t;T) = 1, \quad t_{-1}(T) \leq t \leq T,$$

and v_N is the fundamental solution of Eq. (4.3)

Under our conditions $y_1(t;T) > 0, \quad t \geq T, \quad$ and $\quad v_N(t,s) \geq 0, \quad t \geq s \geq T_N$, and hence $\lim_{t\to+\infty} y_1(t;T) = 0$.

On the other hand

$$\int_T^t v_n(t,u)a(u)h(N(u)) = 1 - y_1(t;T) \to 1, \quad \text{as} \quad t \to +\infty,$$

for any fixed $T \geq T_N$.

Now, we show that for any fixed T_1 such that $T_1 > T \quad (\geq T_N)$

$$\lim_{t\to+\infty} \int_T^{T_1} v_N(t,u)a(u)h(N(u))du = 0.$$

In fact, as $t \to +\infty$,

$$\int_T^{T_1} v_N(t,u)a(u)h(N(u))du = \int_T^t v_N(t,u)a(u)h(N(u))du - \int_{T_1}^t v_N(t,u)a(u)h(N(u))du \to 0.$$

Now, we turn to the proof of the required inequality. Let $\overline{g} = \limsup_{t\to+\infty} g(t)$ and $g_M = \sup_{t\geq T_N} g(t)$. Then $\overline{g} < \infty$ and $g_M < \infty$. Moreover for an arbitrarily fixed $\varepsilon > 0$ there exists a $T_\varepsilon > T \ (\geq T_N)$ such that $g(t) \leq \overline{g} + \varepsilon, \quad t \geq T_\varepsilon$. Thus

$$\limsup_{t\to+\infty} \int_T^t v_N(t,u)a(u)h(N(u))g(u)du \leq \limsup_{t\to+\infty} \int_T^{T_\varepsilon} v_N(t,u)a(u)h(N(u))g(u)du +$$

$$\limsup_{t\to+\infty} \int_{T_\varepsilon}^t v_N(t,u)a(u)h(N(u))g(u)du \leq g_M \limsup_{t\to+\infty} \int_T^{T_\varepsilon} v_N(t,u)a(u)h(N(u))du +$$

$$(\overline{g} + \varepsilon) \cdot \lim_{t\to+\infty} \int_{T_\varepsilon}^t v_N(t,u)a(u)h(N(u))du = \overline{g} + \varepsilon.$$

But $\varepsilon > 0$ was arbitrary, and hence the upper estimation part of the required inequality is proved. The proof of the lower part is similar and hence it is omitted. ■

Now, we state and prove the main result of this section which is analogous to Theorem 3.1.

Theorem 4.1 *Assume that conditions (i)-(iv) are satisfied and Eq. (2.1) has property* $P(K_0, n_0)$, *where* $K_0 > 0$, $n_0 \in \{0, ..., n\}$. *Let* $a(t) = \sum_{i=0}^{n_0} a_i(t)$, $t \geq 0$. *Suppose*

$$a(t) > 0, \quad t \geq 0, \qquad \int_0^\infty a(t)dt = +\infty,$$

and the limits

$$c = \lim_{t\to+\infty} \frac{r(t)}{a(t)} > 0, \quad \alpha = \lim_{t\to+\infty} \frac{\sum_{i=n_0+1}^{n} a_i(t)}{a(t)}, \quad \beta = \lim_{t\to+\infty} \frac{\sum_{j=1}^{m} b_j(t)}{a(t)}$$

exist and are finite, moreover

$$q = \alpha + \beta < 1.$$

If the solutions of Eq. (2.1) are eventually K_0 *bounded, then for any solution* $N : [t_{-1}(0), \infty) \to R_+$ *of Eq. (2.1), one has*

$$\lim_{t\to+\infty} N(t) = \frac{c}{1+\alpha-\beta} = \lim_{t\to+\infty} \frac{r(t)}{\sum_{i=0}^{n} a_i(t) - \sum_{j=1}^{m} b_j(t)} \tag{4.5}$$

and hence Eq. (2.1) is globally asymptotically stable.

Proof. Let $N : [t_{-1}(0), \infty) \to R_+$ be a positive solution of Eq. (2.1). Then by Theorem 2.1 we have that $\int_0^\infty a(t)h(N(t))dt = +\infty$, since $c > 0$. On the other hand, N is eventually bounded by K_0, moreover Eq. (2.1) has property $P(K_0, n_0)$. So, by Lemma 4.1, the fundamental solution v_N of Eq. (4.3) is positive on $T_N \leq s \leq t < \infty$.

Eq. (2.1) can be written in the form

$$\dot{N}(t) = -h(N(t)) \sum_{i=0}^{n_0} a_i(t)N(p_i(t)) + f(t), \quad \text{a.e.} \quad t \geq 0 \tag{4.6}$$

where

$$f(t) = -h(N(t)) \sum_{i=n_0+1}^{n_0} a_i(t)N(p_i(t)) + h(N(t)) \sum_{j=1}^{m} b_j(t)N(q_j(t)) + h(N(t))r(t), \quad t \geq 0,$$

By using the variation of constants-formula for Eq. (4.6), we have

$$N(t) = y(t) + \int_{T_N}^{t} v_N(t, s)f(s)ds, \quad t \geq T_N, \tag{4.7}$$

where $y : [t_{-1}(T_N), \infty) \to R$ is a function satisfying

$$\dot{y}\,(t) = -h(N(t)) \sum_{i=0}^{n_0} a_i(t) y(p_i(t)), \quad \text{a.e.} \quad t \geq T_N, \tag{4.8}$$

$$y(t) = N(t), \quad t \leq T_N.$$

Since Eq. (4.8) is not oscillatory and $\int_0^\infty h(N(t))a(t)dt = +\infty$, from the proof of Lemma 4.2 we know that $y(t) \to 0$, $t \to +\infty$.

On the other hand N is bounded on $[t_{-1}(0), \infty)$, and hence $0 \leq \underline{N} = \liminf_{t\to+\infty} N(t) \leq \overline{N} = \limsup_{t\to+\infty} N(t) < \infty$. By applying Lemma 4.2, under our conditions, we find

$$\liminf_{t\to+\infty} \int_{T_N}^{t} v_N(t,s) f(s) ds =$$

$$\liminf_{t\to+\infty} \int_{T_N}^{t} v_N(t,s) h(N(s)) a(s) \left(\frac{-\sum_{i=n_0+1}^{n} a_i(s) N(p_i(s))}{a(s)} + \frac{\sum_{j=1}^{m} b_j(s) N(q_j(s))}{a(s)} + \frac{r(s)}{a(s)} \right) ds \geq$$

$$-\alpha\overline{N} + \beta\underline{N} + c,$$

and a similar way

$$\limsup_{t\to+\infty} \int_{T_N}^{t} v_N(t,s) f(s) ds \leq -\alpha\underline{N} + \beta\overline{N} + c.$$

Since $y(t) \to 0, \quad t \to +\infty, \quad$ (4.7) and the above estimations yield

$$\underline{N} \geq -\alpha\overline{N} + \beta\underline{N} + c$$

and

$$\overline{N} \leq -\alpha\underline{N} + \beta\overline{N} + c.$$

So

$$0 \leq \overline{N} - \underline{N} \leq (\alpha + \beta)(\overline{N} - \underline{N}),$$

which implies that $\underline{N} = \overline{N}$, since $\alpha + \beta < 1$. This means that (4.7) is satisfied and the proof of the theorem is complete. ■

In the above theorem all of the conditions are explicit except the assumption on the eventually boundedness of the solutions of Eq. (2.1). In the following theorems we give some applications of Theorem 4.1 for some cases when the eventually boundedness of the solutions of Eq. (2.1) is guaranteed by explicit conditions.

Theorem 4.2 *Assume that the conditions of Theorem 2.2 are satisfied and the conditions of Theorem 4.1 also hold with the constant* $K_0 = \frac{\widehat{r}}{1-\widehat{\beta}}$, *where* $\widehat{\gamma}$ *and* $\widehat{\beta}$ *are defined in (2.6).*

Then the solutions of Eq. (2.1) are eventually bounded by K_0 *and for any positive solution* $N : [t_{-1}(0), \infty) \to R_+$ *of Eq. (2.1) relation (4.7) holds.*

Proof. The proof is an easy consequence of Theorem 2.2 and Theorem 4.1. ■

Theorem 4.3 *Assume that $h(u) = u$, $u \geq 0$, and there is no positive feedback term in Eq. (2.1), that is $b_1(t) = ... = b_m(t) = 0$, $t \geq 0$. Assume further that the conditions of Theorem 4.1 are satisfied whenever $K_0 = c$.*

Then for any positive solution $N : [t_{-1}(0), \infty) \to R_+$ of Eq. (2.1) one has

$$\limsup_{t \to +\infty} N(t) = \frac{c}{1+\alpha} = \lim_{t \to +\infty} \frac{r(t)}{\sum_{i=0}^{n} a_i(t)}$$

Proof. In virtue of Theorem 4.1 it is enough to show that the solutions of Eq. (2.1) are eventually bounded by the constant c.

To show this, let $N : [t_{-1}(0), \infty) \to R_+$ be a positive solution of Eq. (2.1) and define

$$x(t) = \ln \frac{N(t)}{c+\varepsilon}, \quad t \geq t_{-1}(0),$$

where $\varepsilon > 0$ is a fixed constant.

Then from Eq. (2.1), we have

$$\dot{x}(t) = r(t) - \sum_{i=0}^{n} a_i(t)(c+\varepsilon)e^{x(p_i(t))}, \quad \text{a.e.} \quad t \geq 0.$$

But, from the definition of c it follows the existence of a T_ε such that $r(t) \leq (c+\varepsilon)\sum_{i=0}^{n_0} a_i(t)$, $t \geq T_\varepsilon$, and hence

$$\dot{x}(t) \leq (c+\varepsilon)\sum_{i=0}^{n_0} a_i(t)(1 - e^{x(p_i(t))}), \quad \text{a.e.} \quad t \geq T_\varepsilon.$$

For any $u \in R$, $1 - e^u \leq -u$, and thus

$$\dot{x}(t) \leq -(c+\varepsilon)\sum_{i=0}^{n_0} a_i(t)x(p_i(t)), \quad \text{a.e.} \quad t \geq T_\varepsilon \tag{4.9}$$

Under the conditions of Theorem 4.1, if $\varepsilon > 0$ is small enough and T_ε is large enough, then the fundamental solution $v_\varepsilon : \{(t,s) : 0 \leq s \leq t\} \to R$ of the equation

$$\dot{u}(t) = -(c+\varepsilon)\sum_{i=0}^{n_0} a_i(t)u(p_i(t)) \tag{4.10}$$

is positive for $t \geq s \geq T_\varepsilon$. On the other hand, $\int_0^\infty \sum_{i=0}^{n_0} a_i(t)dt = +\infty$, and hence (see, e.g., the proof of Lemma 4.2) any solution of Eq. (4.12) tends to zero as $t \to +\infty$.

Inequality (4.9) yields that x is a solution of the equation

$$\dot{x}(t) = -(c+\varepsilon)\sum_{i=0}^{n_0} a_i(t)x(p_i(t)) - f(t), \quad \text{a.e.} \quad t \geq T_\varepsilon,$$

where $f(t) = -(c+\varepsilon)\sum_{i=0}^{n_0} a_i(t)x(p_i(t)) - \dot{x}(t) \geq 0$, a.e. $t \geq T_\varepsilon$. So, by using the variation of constant-formula, x can be written in the form

$$x(t) = y(t) - \int_{T_\varepsilon}^{t} v_\varepsilon(t,s)f(s)ds, \quad t \geq T_\varepsilon,$$

where y is a solution of the homogeneous equation (4.10). Thus $x(t) \leq y(t), \quad t \geq T_\varepsilon$, and hence $\limsup_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} y(t) = 0$, which yields that

$$\limsup_{t\to+\infty} N(t) = (c+\varepsilon) \limsup_{t\to+\infty} e^{x(t)} \leq (c+\varepsilon).$$

But $\varepsilon > 0$ can be arbitrarily small, so $\limsup_{t\to+\infty} N(t) \leq c$. The proof of the theorem is complete. ■

Corollary 4.1 *Let $r > 0$, $a_0 \geq 0$, $a_i \geq 0$, $\tau_i > 0$ $(1 \leq i \leq n)$ and $b_j \geq 0$, $\sigma_j > 0$ $(1 \leq j \leq m)$ be given constants and assume*

(i) $\sum_{j=1}^{m} b_j < a_0$ *and there exists an index $n_0 \in \{1, ..., n\}$ such that*

$$\sum_{i=n_0+1}^{n} a_i + \sum_{j=1}^{m} b_j < \sum_{i=0}^{n_0} a_i,$$

where $\sum_{i=n_0+1}^{n} a_i = 0$ whenever $n_0 = n$, by definition;

(ii) *The equation*

$$\lambda = -(K_0 + \varepsilon)\sum_{i=0}^{n_0} a_i e^{-\lambda\tau_i} \quad (\tau_0 = 0), \tag{4.11}$$

has a real root for any $\varepsilon > 0$ small enough, where

$$K_0 = \frac{r}{a_0 - \sum_{j=1}^{m} b_j} \quad \text{if} \quad 0 < \sum_{j=1}^{m} b_j < a_0 \quad \text{and} \quad K_0 = \frac{r}{\sum_{i=0}^{n_0} a_i} \quad \text{if} \quad b_1 = ... = b_m = 0. \tag{4.12}$$

Then for any positive solution $N : [-\gamma, \infty) \to R_+$ $\left(\gamma = \max\left\{\max_{1\leq i\leq n}\{\tau_i\}, \max_{1\leq j\leq m}\{\sigma_j\}\right\}\right)$ of the equation

$$\dot{N}(t) = N(t)\left[r - a_0 N(t) - \sum_{i=1}^{n} a_i N(t-\tau_i) + \sum_{j=1}^{m} b_j N(t-\sigma_j)\right], \quad t \geq 0, \tag{4.13}$$

one has

$$\lim_{t\to+\infty} N(t) = \frac{r}{\sum_{i=0}^{n} a_i - \sum_{j=1}^{m} b_j}.$$

Proof. The proof of the corollary is an easy consequence of Theorems 4.2 and 4.3 if we show that Eq. (4.2) has property $P(K_0, n_0)$, where K_0 is defined in (4.2). But in our case Eq. (4.2) is equivalent to the equation

$$\dot{u}_\varepsilon(t) = -(K_0 + \varepsilon) \sum_{i=0}^{n_0} a_i u_\varepsilon(t - \tau_i), \quad t \geq 0, \tag{4.14}$$

and by assumption its characteristic equation (4.11) has a real root. By using the oscillation theory of delay differential equations (see, e.g., [5], [6]) this implies that the fundamental solution of Eq. (4.14) is not oscillatory. So Eq. (4.14) has property $P(K_0, n_0)$ and by applying Theorem 4.2 and Theorem 4.3 the corollary is proved. ■

5. Some remarks and open problems

Let us consider the equation

$$\dot{N}(t) = N(t) \left[r - a_0 N(t) - \sum_{i=1}^{n} a_i N(t - \tau_i) + \sum_{j=1}^{m} b_j N(t - \sigma_j) \right] \tag{5.1}$$

where

$$r > 0, \quad a_0 \geq 0, \quad a_i \geq 0, \quad \tau_i \geq 0 \ (1 \leq i \leq n), \quad b_j \geq 0 \ (1 \leq j \leq m)$$

are given constants.

From Theorem 2.3 and Corollary 4.1 it follows that if $a_i = 0$ $(1 \leq i \leq n)$ then Eq. (5.1) is globally asymptotically stable if and only if

$$\sum_{j=1}^{m} b_j < a_0 \tag{5.2}$$

It is worth to note that there are only few results giving necessary (necessary and sufficient) conditions for a population model equation to guarantee its global asymptotic stability.

Lenhart and Travis studied Eq. (5.1) in [14] and their main theorem states that Eq. (5.1) is globally asymptotically stable with respect to the positive solutions for all $\tau_i \geq 0$ $(1 \leq i \leq n)$ and $\sigma_j \geq 0$ $(1 \leq j \leq m)$ if and only if

$$\sum_{i=1}^{n} a_i + \sum_{j=1}^{m} b_j \leq a_0 \quad \text{and} \quad \sum_{j=1}^{m} b_j < a_0 + \sum_{i=1}^{n} a_i. \tag{5.3}$$

It is clear that if $a_i = 0$ $(1 \leq i \leq n)$, then condition (5.3) reduces to (5.2), so our result is a special case of the statement Lenhart and Travis. However, the proof of "only if" part of Lenhart and Travis theorem is not correct, as it was mentioned by Kuang in [12]. The next result shows that the second condition in (5.3) is, in fact, necessary for global asymptotic stability of Eq. (5.1).

Theorem 5.1 *If*

$$\sum_{j=1}^{m} b_j \geq a_0 + \sum_{i=1}^{n} a_i, \tag{5.4}$$

then for any positive solution $N : [t_0 - r, \infty) \to R_+$ $(t_0 \geq 0)$ *of Eq. (5.1) one has* $\limsup_{t\to+\infty} N(t) = +\infty$.

Proof. Let us assume (5.4) and that Eq. (5.1) has a positive solution $N : [t_0 - r, \infty) \to R_+$ such that $\widehat{N} = \sup_{t\geq t_0-r} N(t) < \infty$.

From Eq. (5.1), we have

$$N(t) = N(t_0)e^{r(t-t_0)-a_0\int_{t_0}^{t} N(s)ds-\sum_{i=1}^{n} a_i\int_{t_0}^{t} N(s-\tau_i)ds+\sum_{j=1}^{m} b_j\int_{t_0}^{t} N(s-\sigma_j)ds}, \quad t \geq t_0,$$

that is

$$N(t) = N(t_0)e^{r(t-t_0)+\left(\sum_{j=1}^{m} b_j-\sum_{i=1}^{n} a_i\right)\int_{t_0}^{t} N(s)ds} \cdot e^{c_1} \cdot e^{R_N(t)}, \quad t \geq t_0,$$

where

$$c_1 = -\sum_{i=1}^{n} a_i \int_{t_0-\tau_i}^{t_0} N(s)ds + \sum_{j=1}^{m} b_j \int_{t_0-\sigma_j}^{t_0} N(s)ds,$$

and

$$R_N(t) = \sum_{i=1}^{n} a_i \int_{t-\tau_i}^{t} N(s)ds - \sum_{i=1}^{n} b_j \int_{t-\sigma_j}^{t} N(s)ds, \quad t \geq t_0.$$

But

$$|R_N(t)| \leq \left(\sum_{i=1}^{n} a_i\tau_i + \sum_{j=1}^{m} b_j\sigma_j\right)\widehat{N} < \infty, \quad t \geq t_0,$$

and from condition (5.4) it follows

$$N(t) \geq c_2 \cdot e^{r(t-t_0)}, \quad t \geq t_0,$$

with a suitable positive constant $c_2 > 0$. This yields that $N(t) \to +\infty$ as $t \to +\infty$, which is a contradiction. The proof of the theorem is complete. ■

Condition (5.3) is not surprising, because Eq. (5.1) has a positive steady state if and only if

$$\sum_{j=1}^{m} b_j < a_0 + \sum_{i=1}^{n} a_i. \tag{5.5}$$

In Section 4, our main goal was (see Corollary 4.1) to find additional conditions to (5.4) to guarantee the global asymptotic stability property of the positive steady state solution of Eq. (4.1). But one can see from Corollary 4.1 that either $a_0 > 0$ or $b_1 = ... = b_m = 0$ additional restriction should be satisfied to get the required result. So the following open question arises:

Problem 5.1 *What are the most general delay dependent or independent additional conditions to (5.4) for Eq. (4.1) to be globally asymptotically stable?*

From Theorem 4.1, it can be seen that to answer Problem 5.2 one should answer first the following:

Problem 5.2 *What are the most general additional conditions to (5.4) which guarantee that the solutions of Eq. (4.1) are eventually bounded by a constant independent of the solutions?*

Some partial answers can be given for Problem 5.2 (see the results in the earlier sections of this paper) if either $a_0 > 0$ or $b_1 = ... = b_m = 0$. According to our best knowledge, no result is know for the case when $a_0 > 0$ and some a_i, b_j are positive, at the same time.

Consider the simple "pure delay-type" equation

$$\dot{N}(t) = N(t)\left[r - a_1 N(t-\tau_1) + b_1 N(t-\sigma_1)\right], \quad t \geq 0 \tag{5.6}$$

with both negative and positive delayed terms:

$$r > 0, \quad 0 < b_1 < a_1, \quad \tau_1 > 0, \quad \sigma_1 > 0. \tag{5.7}$$

From Theorem 5.1, it follows that (5.7) is a necessary condition for Eq. (5.6) to be globally asymptotically stable. But it turns out that, in general, (5.7) is not sufficient for global asymptotic stability if $\tau_1 > \sigma_1$. More precisely, for any constants $r > 0, \quad a_1 > 0, \quad \tau_1 > \sigma_1 \geq 0$ there exists a constant $b_1 \in (0, a_1)$ such that Eq. (5.6) has an unbounded solution on $[0, \infty)$. In fact, let $b_1 = a_1 e^{r(\sigma_1 - \tau_1)}$. Then $0 < b_1 < a_1$ and it is easy to verify that the function $N(t) = e^{rt}$ is an unbounded solution of Eq. (5.6) on $[0, \infty)$.

Based on our results we state the following conjecture as an open problem.

Conjecture 5.1 *If (5.7) is satisfied and the positive delay τ_1 is small enough, say $\tau_1 < \sigma_1$, then the positive steady state solution of Eq. (5.6) is a global attractor with respect to the positive solutions of Eq. (5.6).*

One more remark on the case $a_0 > 0$. Let us consider the equation

$$\dot{N}(t) = N(t)\left[r - a_0 N(t) - a_1 N(t-\tau_1) + b_1 N(t-\sigma_1)\right] \tag{5.8}$$

where the coefficients and the delays satisfy the following conditions

$$r > 0, \quad a_0 > 0, \quad 0 < b_1 < a_0 + a_1, \quad \tau_1 > 0, \quad \sigma_1 > 0. \tag{5.9}$$

From the Theorem 5.6 of Kuang (see [12], Section 2.5) we have that Eq. (5.8) is globally asymptotically stable if (5.9) holds and $b_1 \leq a_0 - a_1$. On the other hand, our Corollary 4.1 implies the following.

Proposition 5.1 *Let $\tau_1 > 0$, $\sigma_1 > 0$, $r > 0$. If*

$$0 < b_1 < a_0 \quad \text{and} \quad \frac{r}{a_0 - b_1} a_1 \tau_1 \, e^{\frac{r a_0}{a_0 - b} \tau_1} < \frac{1}{e} \tag{5.10}$$

then Eq. (5.8) is globally asymptotically stable.

The above proposition is an easy consequence of Corollary 4.1, since the equation

$$\lambda = -\left(\frac{r}{a_0 - b_1} + \varepsilon\right)\left(a_0 + a_1 e^{-\lambda\tau_1}\right)$$

has a real root for any $\varepsilon > 0$ small enough if (5.10) is satisfied as it can be shown easily (see also [6]).

Thus Corollary 4.1 generalizes Theorem 5.6 of Kuang given in [12] (p. 35) when the delay τ_1 is small enough and the other delay σ_1 is arbitrary.

It also worth to note that the above results and examples are strongly related to one of the open problems in [13] and the results in [11].

Unfortunately, the question of global asymptotic stability seems to be extremely difficult if $b_1 > a_0$ even assuming (5.9). To illustrate this, assume that $\tau_1 > 0, \quad \sigma_1 > 0, \quad a_1 > 0$ are given constants, such that $\tau_1 > \sigma_1$. Let r and b_1 be defined by

$$r = \frac{1}{\tau_1}\ln\frac{a_1(\tau_1 - \sigma_1)}{\sigma_1 a_0} \quad \text{and} \quad b_1 = \left(\frac{a_1(\tau_1 - \sigma_1)}{\sigma_1} + a_1\right)e^{-r(\tau_1 - \sigma_1)},$$

for a given constant $a_0 > 0$. Then it can be easily shown that $b_1 = a_0 e^{r\sigma_1} + a_1 e^{-r(\tau_1 - \sigma_1)}$, and for any small enough constant $a_0 > 0$, condition (5.9) is satisfied and $b_1 > a_0$. At the same time the function $N(t) = e^{rt}, \quad t \geq 0$, is an unbounded positive solution of Eq. (5.8), and hence Eq. (5.8) is not globally asymptotically stable. In this sense our condition $b_1 < a_0$ in Proposition 5.1 seems to be the best possible if $a_1 > 0$, $\tau_1 > \sigma_1 > 0$ and τ_1 is small enough.

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