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Global attractivity and persistence in a discrete population model

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Abstract. We study the attractivity properties of the equilibrium points of the scalar delay difference equation $x_{n+1}-x_n = -\delta x_n + pf(x_{n-k})$ which arises in many contexts in ecology. New sufficient conditions for the global stability of a unique positive steady state are obtained. These conditions contain some earlier results [3], [5], [8], [11] as particular cases. Some persistence results for this equation are also proved.

1. Introduction

1.1. The problem and the main assumptions. Systems of difference equations are often used as mathematical deterministic models in population ecology. Such systems, even first-order ones, can possess rather rich (e.g. chaotic) dynamic behavior [1]. Here we shall be interested in the study of the asymptotic behavior of nonnegative solutions to the initial value problem for the delay difference equation

$$\begin{aligned} &x_{n+1} - x_n = -\delta x_n + pf(x_{n-k}),\ x_n \geq 0,\ \delta \in (0,1),\ p > 0, \\ &\text{with the initial conditions } x_i = a_i \geq 0,\ i = -k, -k+1, ..., 0, \end{aligned} \tag{1}$$

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where a_i are $(k+1)$ given constants. As usually, by a solution of this initial value problem we mean a sequence $\{x_n\}$ which is defined for $n \geq -k$ and which satisfies (1) for $n \geq 0$.

Equation (1) arises in many contexts in ecology and recently, have received a lot of interest (see, for example [3], [4], [5], [7], [8], [11] and the references therein). Here we shall investigate the attractivity properties of equilibrium points of this delay difference equation. Obviously, these points are determined from the equation

$$x = \frac{p}{\delta} f(x) \stackrel{def}{=} \varphi(x).$$

We suppose without further mentioning, that the continuous function $f : [0, +\infty) \to [0, +\infty)$ vanishes at ∞ : $f(+\infty) = 0$.

For convenience, we list some hypotheses on $f(x)$ which will be used only when it will be explicitly stated:

H1 $f : [0, +\infty) \to (0, +\infty)$ is strictly decreasing;

H2 $f : (0, +\infty) \to (0, +\infty)$ is a unimodal function (that is, it has at most one critical point) and $f(0) = 0$.

H3 $f : (0, +\infty) \to (0, +\infty)$ is a unimodal function and $f(x) = xf_1(x)$, $pf_1(0) > \delta$, where $f_1 : [0, +\infty) \to (0, +\infty)$ is continuous.

H4 $f : (0, +\infty) \to (0, +\infty)$ is unimodal function and $f(x) = xf_1(x)$, $pf_1(0) \leq \delta$, where $f_1 : [0, +\infty) \to (0, +\infty)$ is a continuous map.

If $f(x)$ has a unique critical point, it will denoted by x^*. Otherwise, we shall write $x^* = 0$. We also suppose that k is a fixed integer characterizing delay in the system.

Using the notations $\mathbf{x} = (x_{0,} \dots x_k)$, $S\mathbf{x} = S(x_0, x_{1,} \dots x_k) = (x_1, \dots, (1-\delta)x_k + pf(x_0))$, we obtain the nonlinear difference equation $\mathbf{x}_{n+1} = S\mathbf{x}_n$ associated with the semidynamical system $S : R_+^{k+1} \to R_+^{k+1}$.

If $k = 0$, (1) gives an example of a one-dimensional dynamic system and can be investigated by means of powerful tools of the corresponding theory (e.g., see [13]).

Thus we shall suppose that $k > 0$ and investigate the many-dimensional ($k+1$-dimensional) dynamic system (1). It could be expected that, in this case, the system will have much more complicated behavior. However, the information about this behavior, especially about its global characteristic, is not so complete as in dimension one.

Taking into account the unimodal character of $f(x)$ we can obtain at most two constant solutions of (1): $\{x_n^1\} \equiv \{0\}$ and $\{x_n^2\} \equiv \{c\}$ with $c > 0$. Below we will study the global asymptotic stability property of $\{x_n^1\}$ and, mainly, of $\{x_n^2\}$.

It should be noted that this problem has been investigated in the papers [3], [5], [7], [8], [11], [19], and the analogous continuous equation was studied in [6], [10], [12], [14], [18], [19].

Let us denote the inverse of $\varphi(x)$ on $[x^*, +\infty)$ by $\psi : (0, \varphi(x^*)] \to [x^*, +\infty)$ and set $\theta(x) = x - (1-\delta)^{k+1}\psi(x)$. We will prove that, under one of the conditions H1 or H3, the map

$$\xi(x) = \theta^{-1}((1-(1-\delta)^{k+1})\varphi(x)) : [\mathbb{A},\mathbb{B}] \to [\mathbb{A},\mathbb{B}],$$

is well defined and unimodal for some $\mathbb{A}, \mathbb{B} > 0 : \mathbb{A} \le c \le \mathbb{B}$, and $\xi(c) = c$. The main result of this paper asserts that, if c is globally attracting and locally stable point of ξ, then the equilibrium $\{x_n^2\}$ of the equation (1) is globally attracting.

Moreover, if $f(x)$ is three times continuously differentiable and satisfies H1 or H3, and

1. $2f'(x)f'''(x) < 3(f''(x))^2$ for all $x > 0$, $x \neq x^*$,
2. $0 > (1-(1-\delta)^{k+1})\frac{p}{\delta}f'(c) > -\frac{1}{2}\left(1+\sqrt{1+4(1-\delta)^{k+1}(1-(1-\delta)^{k+1})}\right)$,

then the equilibrium c of (1) is globally asymptotically stable.

The paper is organized as follows. Subsection 1.2 contains some preliminary facts, notations, definitions and results. In the next section we prove some persistence results necessary for the third section, where the main theorem is proved and some applications are given.

1.2. Some preliminary facts, notations, definitions, and results. In the paper we shall use the following notations and definitions:

$R_+ = [0, +\infty)$, $q = 1 - \delta \in (0, 1)$,

If $G, F : X \to X$, then $F^j(x) = \underbrace{F \circ \ldots \circ F}_{j \text{ times}}(x)$, where $F \circ G(x) = F(G(x))$,

Elements of R_+^{k+1} will be given in bold print, e.g. $\mathbf{a}, \mathbf{b} \in R_+^{k+1}$.

Let X be a metric space, $B, A \subset X, \varepsilon > 0$. An $\varepsilon-$neighborhood of A is denoted by $U_\varepsilon(A)$ and $F^j(A) \to B$ as $j \to \infty$ means that for every $\varepsilon > 0$ there exists j_0 such that for all $j \ge j_0$ we have: $F^j(A) \subset U_\varepsilon(B)$.

Definition 1. *The semidynamical system $T : X \to X$ has a global attractor A if $T^j x \to A$ as $j \to +\infty$ for every $x \in X$. Equation (1) is said to be globally attracting if there exists $a \ge 0$ such that $x_j \to a$, $j \to +\infty$ for each nonzero solution $\{x_j\}_{j\ge 0}$ of (1) (in this case $X = R_+^{k+1} \setminus \{\mathbf{0}\}$, $T = S$ and $A = \{(a, \ldots, a)\} = \{\mathbf{a}\}$).*

We note that if equation (1) is globally attracting, then $S\mathbf{a} = \mathbf{a}$ and therefore $\mathbf{a} = (c, ..., c)$ or $\mathbf{a} = (0, ..., 0)$.

We shall relate the global attractivity of delay difference equation (1) to the global attractivity properties of some one-dimensional maps. In order to do this we will use the following simple propositions.

Proposition 2. *Let the dynamical system $g : [a, b] \to [a, b]$, $g \in C[a, b]$, have a globally attracting set $[\alpha, \beta]$. Then*

$$\alpha \leq \liminf_{j\to\infty} g^j(x) \leq \limsup_{j\to\infty} g^j(x) \leq \beta, \text{ for every } x \in [a, b].$$

Lemma 3. *Let X be a compact metric space and $T : X \to X$ be a continuous map such that $T^j(x) \to K$, $j \to \infty$ for some nonempty, closed, invariant subset $K \subset X$ (in other words, K is a global attractor for this system). Suppose that the set K is stable. Then $T^j(X) \to K$ as $j \to \infty$.*

Proof. Since K is stable, for every $\varepsilon-$neighborhood $U_\varepsilon(K)$ of K we can find a $\delta-$neigbourhood $U_\delta(K)$, such that $T^j(z) \in U_\varepsilon(K)$ for every $j \geq 0$, $z \in U_\delta(K)$. Because of the global attractivity of K, for every $w \in X$ we can find a minimal integer $j(w)$ such that $T^{j(w)}(w) \in U_\delta(K)$. It is clear, that there exists a neighborhood $U_{\mu(w)}(w)$ with property $T^{j(w)}(U_{\mu(w)}(w)) \subset U_\delta(K)$. The family of open sets $\{(U_{\mu(w)}(w), w \in X\}$ covers the compact set X and therefore there exists a finite subcover $\{(U_{\mu(k)}(w_k), k = 1, \ldots, r\}$. Let $s = \max\limits_{k=1,\ldots,r} j(w_k)$. Obviously, $T^j(X) \subset U_\varepsilon(K)$ for every $j \geq s$. ■

Remark 1. *The stability assumption for K is essential in the formulation of this Lemma.*

Corollary 4. *Let the map $g : [a, b] \to [a, b]$ have a globally attracting stable fixed point c. Then $g^j[a, b] \to c$ as $j \to \infty$.*

In our work we shall use some facts from the theory of one-dimensional maps with negative Schwarzian derivative.

Definition 5. *Let f be a real function having at least three continuous derivatives. The Schwarzian derivative of f at point x, denoted by $(\mathcal{S}f)(x)$, is given by*

$$(\mathcal{S}f)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

The following formula can be checked by direct computation:

$$\mathcal{S}(f \circ g)(x) = (\mathcal{S}f)(g(x))(g'(x))^2 + (\mathcal{S}g)(x). \tag{2}$$

Using the identities $f \circ f^{-1}(x) = x$, $(\mathcal{S}x)(x) \equiv 0$ and (2), we obtain

Lemma 6. *1. If $(\mathcal{S}f)(x) > 0$ (resp., < 0) for all x, then $(\mathcal{S}f^{-1})(x) < 0$ (resp., > 0) for all x;*

2. If $(\mathcal{S}f)(x) < 0$, $(\mathcal{S}g)(x) \leq 0$, then $\mathcal{S}(g \circ f)(x) < 0$.

The next theorem, which can be easily deduced from Singer's results ([16], see also [13]), will play a crucial role in the sequel.

Proposition 7. *Let the function $g : [a, b] \to [a, b]$, $g(x) \in C^3[a, b]$, have at most one critical point in $[a, b]$. If a unique fixed point $c \in [a, b]$ of g is locally asymptotically stable and the Schwarzian derivative $(\mathcal{S}g)(x) < 0$, then c is a global attractor of g.*

2. Persistence.

First we introduce the concept of uniform persistence. In recent years this concept has attracted a lot of interest and has played an important role in mathematical ecology (see, for example, [9]). Since we intend to study the only those solutions $\{x_n\}_{n\geq 0}$ of (1), which are biologically meaningful, in the sequel, we will assume that $x_n \geq 0$ for all n.

Definition 8. *System (1) is said to be uniformly persistent if there exists a positive number α such that*

$$\lim_{n\to+\infty} \inf x_n \geq \alpha$$

for every trajectory $\{x_n\}_{n\geq 0}$ with positive initial values.

For the equation (1) we indicate below the conditions of uniform persistence and the method to obtain the lower survival bounds α, while in the next section some global attractivity results for this equation will be established.

Let us consider the map $\varphi : [0, +\infty) \to [0, +\infty)$ when either the condition H3 or H1 holds. In both these cases there is an invariant attracting set [$\mathbb{A}$,$\mathbb{B}$] with $0 < \mathbb{A} \leq \mathbb{B} < +\infty$ for the map $\varphi(x) : (0, +\infty) \to (0, +\infty)$.

Indeed, under condition H3, it is sufficient to consider the following particular cases F1-F3 depending on the mutual disposition of $\varphi(x^*)$, $\varphi^2(x^*)$, x^*, c (see Figures F1-F3):

F1. $\varphi : [\mathbb{A},\mathbb{B}] \to [\mathbb{A},\mathbb{B}]$ with $\mathbb{B} = \varphi(x^*)$ and $0 < \mathbb{A} = \varphi^2(x^*) \le x^* \le c \le \mathbb{B}$;

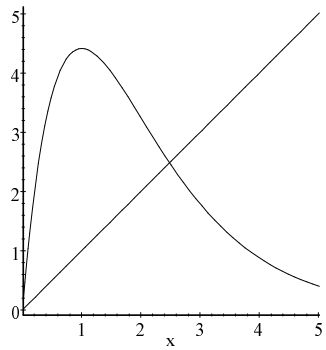

Fig. F1

F2. $\varphi : [\mathbb{A},\mathbb{B}] \to [\mathbb{A},\mathbb{B}]$ with $x^* \le \mathbb{A} \le c \le \mathbb{B} \le \varphi(x^*)$ and $\varphi(\mathbb{A}) = \mathbb{B}$, $\varphi(\mathbb{B}) = \mathbb{A}$:

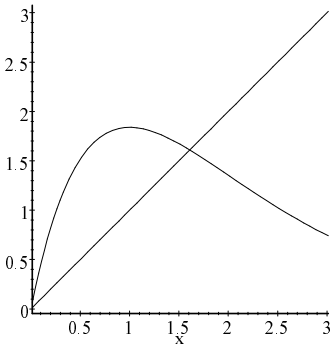

Fig. F2

F3. $\varphi : [\mathbb{A},\mathbb{B}] \to [\mathbb{A},\mathbb{B}]$ with $0 < \mathbb{A} \le c \le x^* = \mathbb{B}$ and $\varphi(x)$ increases on $[\mathbb{A},\mathbb{B}]$:

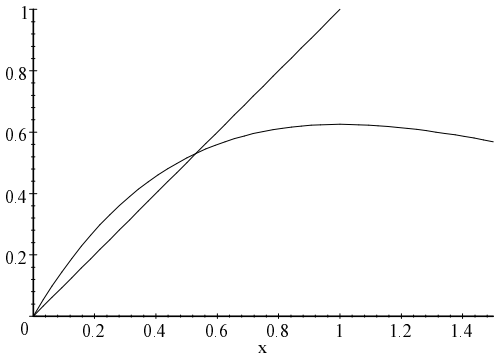

Fig. F3

It should be noted that case F3 is trivial (see proposition 9 below) and, mainly, we shall consider cases F1 and F2. Moreover, the diagram F2 is appropriate to describe φ when assumption H1 is fulfilled (supposing $x^* = 0$). The assertion below follows immediately from the graphical analysis of diagrams F1-F3.

Proposition 9. *Let us consider the map $\varphi : [\mathbb{A},\mathbb{B}] \to [\mathbb{A},\mathbb{B}] \ni c$, where c is the unique fixed point of $\varphi(x)$ in $[\mathbb{A},\mathbb{B}]$. Then the following statements hold:*

1. Assume $x^* < c$. If $m = \inf\limits_{n\geq 0} \varphi^n(z)$, $M = \sup\limits_{n\geq 0} \varphi^n(z)$ for some z and $c \leq m \leq M$ or $c \geq M \geq m$, then $c = m = M$.
2. If $c \leq x^*$, then c is global attractor.

Corollary 10. *If $\varphi(x) \in C^1[\mathbb{A},\mathbb{B}]$ and $\varphi'(c) \geq 0$, then c is global attractor.*

Proof. Under our assumptions, $\varphi'(c) \geq 0$ implies the estimate $\varphi'(c) < 1$. ■

Now we will analyze (1) under condition H2 in more detail.

Consider the above-mentioned map $S : R_+^n \to R_+^n$. Denote the solution starting from $\mathbf{a} = (a_{-k}, a_{-k+1}, ...a_0) \in R_+^{k+1}$ by $\gamma_+(\mathbf{a}) = \{S^n\mathbf{a}, n \geq 0\}$. If x_j is the first component of the vector $S^j\mathbf{a}$ (we denote it by $x_j = (S^j\mathbf{a})_1$), then, obviously, $\{x_n \geq 0\}_{n\geq 0}$ is a solution of the initial value problem (1). Since, by variation of constants formula, we have

$$x_{n+k+1} = q^{n+1}x_k + p\sum_{s=0}^{n} q^{n-s}f(x_s)$$

and $f(x)$ is bounded, $\delta \in (0,1)$, it follows that the set $\gamma_+(\mathbf{a}) \subset R_+^{k+1}$ is also bounded. Thus, the $\omega-$ limit set $\omega(\mathbf{a})$ of $\gamma_+(\mathbf{a})$ is compact, invariant and $S^i\mathbf{a} \to \omega(\mathbf{a})$ as $i \to +\infty$ (see, for example, lemma 4.2 in [15]).

We leave for the reader to verify the following proposition:

Lemma 11. *Let $\{x_n \geq 0\}_{n\in Z}$ be a solution of (1). Then $\{x_n\}$ satisfies the equation*

$$x_{n+k+1} = p\sum_{s=n}^{-\infty} q^{n-s}f(x_s),$$

where $q = 1 - \delta \in (0,1)$.

Corollary 12.

1. *If* $\{x_n \geq 0\}_{n\in Z}$ *is a nonnegative solution of (1) and* $x_j = 0$ *for some* $j \in Z$, *then* $x_n \equiv 0$ *for all* $n \in Z$.

2. *Let H4 hold and* $f_1(x) < \max_{x\geq 0} f_1(x) = f_1(0) = M < \infty$ *for all* $x > 0$. *Then* $\omega(\mathbf{a}) = \mathbf{0}$ *for every* $\mathbf{a}$ *(i.e. the steady state* $\{x_n^1\} = \{0\}$ *is a global attractor of (1) independently of* k*).*

3. *If H3 holds, then the system (1) is uniformly persistent: namely, if* $\mathbf{a} \neq \mathbf{0}$, *then* $\mathbf{0} \notin \omega(\mathbf{a})$ *and there exists* $m > 0$, *which is independent of* k, *and such that*

$$\liminf_{j\to\infty} x_j > m \text{ for all } \{x_j\} \neq \{0\}$$

for any solution $\{x_n \geq 0\}_{n\geq 0}$ *of initial value problem (1).*

Proof. We prove the last statement, the two others are left to the reader. By contrary, suppose that $\{x_i\} \neq \{0\}$ is a solution of (2) such that $\liminf_{n\to\infty} x_n = 0$ and $M_1 = \sup_i\{x_i\} > 0$. It is clear that we can find $n_1, n_2 \geq k$ such that $m_1 = \min_{n_1-k\leq s\leq n_1+n_2+k-1} x_j > 0$ satisfies the relations

$$f_1(m_1)p(1-q^{k+n_2}) > \delta \text{ and } \min_{t\in[m_1,M_1]} (tf_1(t)) = m_1 f_1(m_1).$$

Then we obtain

$$x_{n_1+k+n_2} = q^{k+n_2}x_{n_1} + p\sum_{\mu=0}^{k+n_2-1} q^\mu f(x_{n_1+n_2-\mu}) \geq p\sum_{\mu=0}^{k+n_2-1} q^\mu m_1 f_1(m_1) > m_1.$$

Analogously, we can prove that $x_{n_1+k+n_2+1} > m_1$,etc., in contradicting to $\liminf_{i\to+\infty} x_i = 0$. Finally, we note that m_1 can be chosen independently on the solution $\{x_i\}$. ■.

Remark 2. *The part 2 of Corollary 12 can be deduced also from the general theorem 11.5.1 of [19].*

It is also clear that assumption H1 implies the uniform persistence of (1), and from now on we suppose that either H1 or H3 is fulfilled.

Now we can state how the lower bound α in the definition of uniform persistence can be estimated:

Lemma 13. *Let $K = [\alpha_1, \beta_1]$ be the minimal invariant globally attracting set of the map $\varphi(x) : [\mathbb{A},\mathbb{B}] \to [\mathbb{A},\mathbb{B}]$. Then (1) is uniformly persistent and in Definition 8 we can choose $\alpha = \alpha_1$.*

Proof. Let$\{x_j\}_{j\geq 0}$, $x_j = (S^j\mathbf{a})_1$ be a nonzero trajectory of the uniformly persistent system (1). Then $\omega(\mathbf{a})$ is a closed, invariant and compact set. We use the notations $m = \min\limits_{\mathbf{q}\in\omega(\mathbf{a})} \inf\limits_{j\geq 0}\{x_j(\mathbf{q})\}$ and $M = \max\limits_{\mathbf{q}\in\omega(\mathbf{a})} \sup\limits_{j\geq 0}\{x_j(\mathbf{q})\}$, where $x_j(\mathbf{q}) = (S^j\mathbf{q})_1$. Since $\omega(\mathbf{x})$ is a compact invariant set, there exist a solution $\{y_j\}_{j\geq 0}$ and an integer i such that $y_{i+k+1} = M$. Then $y_{i+k+1} - y_{i+k} \geq 0$ and therefore $-\delta y_{i+k} + pf(y_i) \geq 0$, $y_{i+k} \leq \frac{p}{\delta} f(y_i)$. As a consequence, we obtain

$$M = y_{i+k+1} = qy_{i+k} + pf(y_i) \leq \frac{p}{\delta} f(y_i) = \varphi(y_i) \quad \textit{and} \tag{3}$$

$$M \leq \sup_{y\in[m,M]} \varphi(y). \tag{4}$$

Analogously, since $\omega(\mathbf{a})$ is a compact invariant set, there exist a solution $\{z_j\}_{j\geq 0}$ and an integer r such that $z_{r+k+1} = m$. Then $z_{r+k+1} - z_{r+k} \leq 0$ and therefore $-\delta z_{r+k} + pf(z_r) \leq 0$, $z_{r+k} \geq \frac{p}{\delta} f(z_r)$. Hence

$$m = z_{r+k+1} = qz_{r+k} + pf(z_r) \geq \frac{p}{\delta} f(z_r) = \varphi(z_r) \tag{5}$$

$$m \geq \min_{x\in[m,M]} \varphi(x). \tag{6}$$

From (4), (6) we conclude that

$$[m, M] \subset \varphi([m, M]) \subset \varphi^2([m, M]) \subset ... \subset \varphi^j([m, M]) \subset ... \tag{7}$$

Now, a direct application of Lemma 3 completes the proof of Lemma 13. ■

3. Global Attractivity.

3.1. Main result and its corollaries. We recall that, under hypothesis H1 or H3, the map $\varphi : (0, +\infty) \to (0, +\infty)$ has an invariant attracting set $[\mathbb{A},\mathbb{B}]$ with $\mathbb{A}>0$. The following theorem is the main result of this paper.

Theorem 14. *Assume one of the conditions H1 or H3 is fulfilled. Let $\psi : [\mathbb{A}, \mathbb{B}] \to [x^*, \mathbb{B}]$ be the inverse of $\varphi(x)$ on $[x^*, \mathbb{B}]$, and set $\theta(x) = x - q^{k+1}\psi(x)$ and*

$$\xi(x) = \theta^{-1}((1 - q^{k+1})\varphi(x)) : [\mathbb{A},\mathbb{B}] \to [\mathbb{A},\mathbb{B}], \qquad q = 1 - \delta.$$

If $c > 0$ is a globally attracting and locally stable fixed point of the unimodal map $\xi(x)$ then c is a globally attracting steady state solution of equation (1).

Corollary 15. *Let the map $\xi(x)$ have a negative Schwarzian derivative, $\mathcal{S}\xi$, on $[0,+\infty)$. If c is a locally asymptotically stable (LAS) fixed point of $\xi(x)$, then the equilibrium c of (1) is globally attracting.*

Corollary 16. *Assume that one of the conditions H1 or H3 is fulfilled. If*

1. $(\mathcal{S}\varphi)(x) = \frac{\varphi'''(x)}{\varphi'(x)} - \frac{3}{2}\left(\frac{\varphi''(x)}{\varphi'(x)}\right)^2 < 0$, $x \geq 0$,
2. $0 > (1-q^{k+1})\varphi'(c) > -\frac{1}{2}\left(1+\sqrt{1+4q^{k+1}(1-q^{k+1})}\right)$,

then the equilibrium point c of (1) is globally asymptotically stable.

The following propositions are weaker than the above stated results, but their assumptions can easily be verified.

Corollary 17. *Let c be a unique globally attracting and locally stable point of the map $\zeta(x) = q^{k+1}c + (1-q^{k+1})\varphi(x) : [0,+\infty) \to (0,+\infty)$. Then the steady state c of (1) is globally stable.*

Corollary 18. *Let the map $\varphi(x)$ have a negative Schwarzian derivative $\mathcal{S}\varphi$. If c is a locally asymptotically stable fixed point of $\zeta(x)$, then the equilibrium c of (1) is globally attracting.*

Remark. It should be noted that in these four assertions the conditions of global stability depend on the delay k. We also mention that the condition $0 \geq (1-q^{k+1})\varphi'(c) > -1$ is sufficient for the point c to be LAS.

Corollary 19. *[5]. Let c be a unique globally attracting and locally stable point of the unimodal map $\varphi(x) : (0,+\infty) \to (0,+\infty)$. Then the steady state c of (1) is globally attracting.*

Corollary 20. *Assume that the Schwarzian derivative $\mathcal{S}\varphi$ of the map $\varphi(x)$ is negative. If c is a locally asymptotically stable fixed point, then the steady state c of (1) is globally attracting.*

(We mention that the condition $|\varphi'(c)| < 1$ is sufficient for the point c to be LAS).

Proof of Theorem 14. Under the assumptions of the theorem, the map $\varphi : (0,+\infty) \to (0,+\infty)$ has a globally attracting set $[\mathbb{A},\mathbb{B}] \ni c$ with $\mathbb{B} = \max\varphi(x) = \varphi(x^*)$ and $0 < \mathbb{A} \leq \min\{\varphi^2(x^*),\ x^*\}$. Hence $\varphi : [\mathbb{A},\mathbb{B}] \to [\mathbb{A},\mathbb{B}]$, and $\varphi(c) = c$.

Let $\mathbf{x}_j = S^j\mathbf{a}, \quad j \geq 0$ with $x_j = (S^j\mathbf{a})_1$ be a nonzero solution of the uniformly persistent system (1). We know that according to our assumptions the ω-limit set $\omega(\mathbf{a})$ is an invariant and compact set. Thus, using the notations $m = m(\mathbf{q}) = \min_{\mathbf{q}\in\omega(\mathbf{a})} \inf_{j\geq 0}\{x_j(\mathbf{q})\}$ and $M = M(\mathbf{q}) = \max_{\mathbf{q}\in\omega(\mathbf{a})} \sup_{j\geq 0}\{x_j(\mathbf{q})\}$, $x_j(\mathbf{q}) = (S^j\mathbf{q})_1$, we have from Proposition 9 that $m, M \in [\mathbb{A},\mathbb{B}]$. If either $c \leq x^*$, or $c \leq m \leq M$, or $m \leq M \leq c$ is satisfied, then the global asymptotic stability of equation (1) follows from Proposition 9 and relation (7). So, we will investigate the case $c > x^*$ and $m \leq c \leq M$ (therefore $\varphi'(c) < 0$).

We now establish the conditions guaranteeing the equality $m(\mathbf{q}) = M(\mathbf{q})$ for each $\mathbf{q} \in \omega(\mathbf{a})$. It is clear that these conditions will imply the global asymptotic stability of the steady state solution $\{x_n^2\} \equiv \{c\}$.

We shall consider only the case F1 (with $x^* = x^\bullet$) from the previous subsection, case F2 is left to the reader.

Let $\varphi_1^{-1} : [\varphi^3(x^*), \mathbb{B}] \to [\mathbb{A}, x^*]$ be the inverse of $\varphi(x)$ on $[\mathbb{A}, x^*]$ and $\psi : [\mathbb{A}, \mathbb{B}] \to [x^*, \mathbb{B}] \ni c$ be the inverse of $\varphi(x)$ on $[x^*, \mathbb{B}]$.

Since $\omega(\mathbf{a})$ is a compact set , there exist a solution $\{y_j\}_{j\geq 0}$, $y_j = (S^j\mathbf{q})_1$, $\mathbf{q} \in \omega(\mathbf{a})$ and an integer i such that $y_{i+k+1} = M$. Since $\varphi(x) = \frac{p}{\delta}f(x)$ is a unimodal map and $c \leq M \leq \mathbb{B} = \varphi(x^*)$, we obtain from (3) that

$$y_i \in [\kappa, \psi(M)] \subset [\kappa, c],$$

where $\kappa = \min\{\varphi_1^{-1}(M), \mathbb{A}\}$ if $\varphi_1^{-1}(M) \neq \emptyset$ and $\kappa = \mathbb{A}$ in the opposite case. The upper estimate

$$y_i \leq \psi(M) \tag{8}$$

plays very important role in the proof. Since

$$M = y_{i+k+1} = qy_{i+k} + pf(y_i) = q^2y_{i+k-1} + pqf(y_{i-1}) + pf(y_i) =$$

$$= q^{k+1}y_i + p(q^kf(y_{i-k}) + \ldots + f(y_i)),$$

we obtain the inequalities:

$$M \leq q^{k+1}\psi(M) + (1 - q^{1+k}) \max_{x\in[m,M]} \varphi(x),$$

$$M \leq q^{k+1}x_1 + (1 - q^{1+k}) \max_{x\in[m,M]} \varphi(x). \tag{9}$$

Finally, if we set $\theta(x) = x - q^{k+1}\psi(x)$, we get

$$\theta(M) \leq (1 - q^{1+k}) \max_{x\in[m,M]} \varphi(x). \tag{10}$$

Analogously, since $\omega(\mathbf{a})$ is a compact invariant set, there exist a solution $\{z_j\}_{j\geq 0}$, $z_j = (S^j\mathbf{p})_1$, $\mathbf{p} \in \omega(\mathbf{a})$ and an integer r such that $z_{r+k+1} = m$. Next, since $\varphi(x) = \frac{p}{\delta}f(x)$ is a unimodal map and $z_r \geq m \in [\mathbb{A}, c]$, we have from (5) that

$$z_r \in R_+ \setminus (\varphi_1^{-1}(m), \psi(m)).$$

Moreover, since $z_r \geq m > \varphi_1^{-1}(m)$, we have

$$z_r \geq \psi(m).$$

Since

$$m = z_{r+k+1} = qz_{r+k} + pf(z_r) = q^2 z_{r+k-1} + pqz_{r-1} + pf(z_r) =$$

$$= q^{k+1}z_r + p(q^k f(z_{r-k}) + \ldots + f(z_r)),$$

we have:

$$m \geq q^{k+1}x_1 + (1 - q^{1+k}) \min_{x\in[m,M]} \varphi(x), \tag{11}$$

$$m \geq q^{k+1}\psi(m) + (1 - q^{1+k}) \min_{x\in[m,M]} \varphi(x). \tag{12}$$

Finally, (12) can be rewritten as

$$\theta(m) \geq (1 - q^{1+k}) \min_{x\in[m,M]} \varphi(x). \tag{13}$$

It should be noted that

$$\begin{aligned} \theta(x) &= x - q^{k+1}\psi(x) : [\mathbb{A}, \mathbb{B}] \to [\mathbb{A} - q^{k+1}\mathbb{B}, \mathbb{B} - q^{k+1}x^*] \supseteq \\ &\supseteq [(1 - q^{k+1})\mathbb{A}, (1 - q^{k+1})\mathbb{B}] = (1 - q^{k+1})\varphi([\mathbb{A}, \mathbb{B}]) \end{aligned}$$

is an increasing function, since $\psi(x)$ is decreasing. Therefore, $\theta^{-1}((1 - q^{k+1})\varphi(x)) : [\mathbb{A}, \mathbb{B}] \to [\mathbb{A}, \mathbb{B}]$ is a well defined unimodal map with a unique positive fixed point c.

Finally, from estimates (10), (13) we obtain

$$[m, M] \subset \xi([m, M]) \subset \xi^2([m, M]) \subset \ldots \subset \xi^j([m, M]) \subset \ldots$$

The statement of Theorem 14 is now an immediate consequence of these relations and Lemma 3. ■

We note that the proof of Corollaries 19 and 20 immediately follow from relation (7), Lemma 3 and Proposition 7. From estimates (9) and (11) we obtain that

$$[m, M] \subset \zeta([m, M]) \subset \zeta^2([m, M]) \subset ... \subset \zeta^j([m, M]) \subset ...$$

Since $\mathcal{S}\zeta = \mathcal{S}\varphi$, these relations, Lemma 3 and Proposition 7 yield Corollary 18.

Together with Proposition 7, Theorem 14 implies Corollary 15, and we have to prove only Corollary 16.

Proof of Corollary 16: First we show that $|\xi'(c)| < 1$ under condition 2. of Corollary 16. Indeed,

$$\xi'(c) = \frac{d}{dx}\left[\theta^{-1}\left((1-q^{k+1})\varphi(c)\right)\right] = \frac{d}{dx}\left[\theta^{-1}(y)\right]_{y=\theta(c)}(1-q^{k+1})\frac{d}{dx}\varphi(c)$$

$$= (\theta'(c))^{-1}(1-q^{k+1})\varphi'(c) = \frac{(1-q^{k+1})\varphi'(c)}{1-q^{k+1}(\varphi'(c))^{-1}}$$

$$= \frac{(1-q^{k+1})(\varphi'(c))^2}{\varphi'(c)-q^{k+1}}.$$

Since $\varphi'(c) < 0$, the condition $|\xi'(c)| < 1$ is equivalent to

$$\frac{(1-q^{k+1})(\varphi'(c))^2}{\varphi'(c)-q^{k+1}} > -1, \tag{14}$$

which follows easily from condition 2 of Corollary 16.

Now, we evaluate the sign of $(\mathcal{S}\xi)(x)$. We note that the direct computation of the sign of $(\mathcal{S}\xi)(x)$ is rather difficult. However by using Lemma 6, it can be easily seen that $(\mathcal{S}\xi)(x) < 0$ if $(\mathcal{S}\theta)(x) > 0$ and $(\mathcal{S}\varphi)(x) < 0$. But $(\mathcal{S}\varphi)(x) < 0$ is assumed therefore our goal is to show $(\mathcal{S}\theta)(x) > 0$. Since $\psi(x)$ is the inverse of $\varphi(x)$, by Lemma 6, we have

$$\frac{2\psi'''\psi' - 3(\psi'')^2}{2(\psi')^2} = (\mathcal{S}\psi)(x) = (\mathcal{S}\varphi^{-1})(x) > 0 \implies 2\psi'''\psi' - 3(\psi'')^2 > 0.$$

Since $\theta(x) = x - q^{k+1}\psi(x)$, we obtain $\theta'(x) = 1 - q^{k+1}\psi'(x)$, $\theta''(x) = -q^{k+1}\psi''(x)$, $\theta'''(x) = -q^{k+1}\psi'''(x)$ and

$$(\mathcal{S}\theta)(x) = \frac{2\theta'''\theta' - 3(\theta'')^2}{2(\theta')^2} = \frac{q^{k+1}}{2(\theta')^2}\left[\left\{2\psi'''\psi' - 3(\psi'')^2\right\}q^{k+1} - 2\psi'''\right].$$

Hence $(\mathcal{S}\theta)(x) > 0$, if $(\mathcal{S}\varphi)(x) < 0$ and $\psi''' \le 0$. Let us now compute ψ'''. We have:

$$\begin{aligned}\psi'(\varphi(x)) &= \frac{1}{\varphi'(x)};\ \psi''(\varphi(x)) = -\frac{\varphi''(x)}{[\varphi'(x)]^3};\\ \psi'''(\varphi(x)) &= \frac{3(\varphi''(x))^2 - \varphi'''(x)\varphi'(x)}{[\varphi'(x)]^5} < 0,\end{aligned}$$

since $\varphi'(x) < 0$ for $x > x^*$ and $3(\varphi''(x))^2 - \varphi'''(x)\varphi'(x) > 0$ because of

$$\frac{2\varphi'''\varphi' - 3(\varphi'')^2}{2(\varphi')^2} = (\mathcal{S}\varphi)(x) < 0.$$

So, $(S\xi)(x) < 0$, and $|\xi'(c)| < 1$ implies that the fixed point c of $\xi(x)$ is locally asymptotically stable. Thus Proposition 7 yield that c is a locally asymptotically stable and globally attracting fixed point of ξ, and hence the statement of Corollary 16 follows from Theorem 14. ■

3.2. Discrete version of Nicholson's blowflies. Let us consider the equation

$$x_{n+1} - x_n = -\delta x_n + p x_{n-k}\exp(-a x_{n-k}),\ a > 0, \tag{15}$$

which is a discrete version of the Nicholson's blowflies delay-differential equation ([2], [3], [11], [18], [19]). In this case $f(x) = px\exp(-ax)$, $\varphi(x) = \frac{p}{\delta}xe^{-ax}$. If $p \le \delta$, we have only one equilibrium $\{x_n\} \equiv \{0\}$, and according to Corollary 12.1, it is a global attractor independently of k. Let now $p > \delta$. In this case $\{x_n\} \equiv \{0\}$ is an unstable equilibrium (Corollary 12) and the second rest point $\{x_n^2\} = \{\frac{1}{a}\ln\frac{p}{\delta}\}$ appears. Since $\varphi(x) = \frac{p}{\delta}xe^{-ax}$ (and therefore $\zeta(x)$) has negative Schwarzian derivative (see [16]), we can apply Corollaries 18 and 19. The conditions for the local exponential stability of the rest point c of the maps $\varphi(x)$ and $\zeta(x)$ have the forms $|\varphi'(c)| < 1$ and $|\zeta'(c)| < 1$, respectively. Since $\varphi'(c) = 1 - \ln\frac{p}{\delta}$ and $\zeta'(c) = -(1-q^{k+1})\ln\frac{p}{\delta e}$, we obtain some new conditions for the global stability (placed in Table 1).

Table 1. Comparison of different global attractivity conditions for Eq. (15).

Source	condition	case
present paper	$(q^{-k-1}-1)\ln\frac{p}{\delta} <$ $< q^{-k-1}(\frac{3}{2}+\frac{1}{2}\sqrt{1+4q^{k+1}(1-q^{k+1})}) - 1 \quad (\lvert\xi'(c)\rvert < 1)$	E_1
[5]	$\ln\frac{p}{\delta} < 2 \quad (\lvert\varphi'(c)\rvert < 1)$	E_2
[3]	$(q^{-k-1}-1)\ln\frac{p}{\delta} < 1$	E_3
[11]	$(q^{-k-1}-1)(\frac{p}{\delta}-1) < 1$	E_4
[7]	$(q^{-k-1}-1)(\frac{p}{\delta}-1) \le 1$	E_4

It is evident that condition E_1 is weaker than the others. It is interesting to note that condition E_1 implies that, if $\ln \frac{p}{\delta} = 2$ then equation (15) is globally stable independently of the delay k. We note that this result can also be deduced from Corollary 19, since in the case $\ln \frac{p}{\delta} = 2$ the fixed point $c = \frac{2}{a}$ of the map $\varphi(x) = 2x \exp(-ax)$ is still locally asymptotically (but not exponentially) stable. To prove this fact, it is sufficient to check that

$$(\varphi(x) - \frac{2}{a})^2 = (x - \frac{2}{a})^2 \left[1 - \frac{a^2}{6}(x - \frac{2}{a})^2 + o((x - \frac{2}{a})^2)\right] < (x - \frac{2}{a})^2$$

for all $x \neq \frac{2}{a}$ sufficiently close to $\frac{2}{a}$. Considering the conditions for the global attractivity of (15), which are independent of k, we remark that the result $\ln \frac{p}{\delta} \leq 2$ is the best possible. Indeed, for every $\ln \frac{p}{\delta} > 2$ and at least for even k and δ close to 1, equation (15) has a 2-periodic trajectory. This fact can be deduced by the investigation of period-doubling bifurcations (see, e.g. [17]) of the scalar equation

$$x_{n+1} = x_n(q + p \exp(-ax_n)), a > 0.$$

Applying Corollary 16 to equation (15), we get the following condition (the best to our knowledge) for the global attractivity in the discrete version of the Nicholson's blowflies delay-differential equation:

$$(1 - q^{k+1}) \ln \frac{p}{\delta} < 1 - q^{k+1} + \frac{1}{2}\left(1 + \sqrt{1 + 4q^{k+1}(1 - q^{k+1})}\right). \tag{E1}$$

The criterion for the local asymptotic stability of the equilibrium c of (15) can be obtained by direct computation of the characteristic equation, which has the form

$$\lambda^{k+1} = q\lambda^k + \delta \ln \frac{\delta e}{p}.$$

Obviously, c will be locally exponentially stable, if all the roots of this equation lie inside the unite ball $\{|z| < 1\}$. It is interesting to study the following conjecture which is strongly related to a remark of H. Smith (see [18], p.116) concerning the continuous case:

Conjecture 21. *The local exponential stability of the positive equilibrium of (15) is sufficient for its global attractivity.*

3.3. Discrete model of an erythropoietic system. Let us consider the difference equation

$$x_{n+1} - x_n = -\delta x_n + p\exp(-ax_{n-k}),\ a > 0, \tag{16}$$

see also [5], and [7], where the following sufficient condition for the global attractivity was established:

$$\frac{pa}{\delta} \le e. \tag{17}$$

This system has only one equilibrium $c > 0$ satisfying the transcendental equation $c = \frac{p}{\delta}\exp(-ac)$. To show the dependence c on $\frac{p}{\delta}$, we write $c = c(\frac{p}{\delta})$. Obviously, $c(z)$ is an increasing function for $z > 0$. Since the function $f(x) = \exp(-ax)$ has a negative Schwarzian derivative $(\mathcal{S}f)(x) = -\frac{a^2}{2} < 0$, we can apply Corollary 18. Therefore, if $(1-(1-\delta)^{k+1})\,|\varphi'(c)| < 1$ then equation (16) has the property of global attractivity. Since $\varphi'(c) = -ac$ and $c(z)$ is an increasing function, the last inequality will be fulfilled for all $\frac{p}{\delta} < z_0$ with $(1-(1-\delta)^{k+1})ac(z_0) = 1$. Let $p_0 > 0, \quad \delta_0 \in (0,1)$ be such that $z_0 = \frac{p_0}{\delta_0}$. Then $c(z_0) = \frac{p_0}{\delta_0}\exp(-ac(z_0))$ and hence

$$z_0 = \frac{p_0}{\delta_0} = c(z_0)\exp(ac(z_0)) = (1-(1-\delta)^{k+1})^{-1}a^{-1}\exp(1-(1-\delta)^{k+1})^{-1}$$

and a global attractivity condition for the equation (16) has the form:

$$\frac{pa(1-(1-\delta)^{k+1})}{\delta} < \exp(1-(1-\delta)^{k+1})^{-1}. \tag{18}$$

A similar analysis based on Corollary 16 gives the following globaly attracting condition for equation (16):

$$\frac{pa(1-(1-\delta)^{k+1}}{\delta} < \frac{\left(1+\sqrt{1+4q^{k+1}(1-q^{k+1})}\right)}{2}\exp\left(\frac{\left(1+\sqrt{1+4q^{k+1}(1-q^{k+1})}\right)}{2(1-q^{k+1})}\right). \tag{19}$$

Evidently, (17) implies (18), and (18) implies (19), so the estimate (19) is the sharpest possible here.

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