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# ON EQUI-STABILITY WITH RESPECT TO PARAMETERS IN FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study stability of certain delay differential equations through equi-stability of a corresponding family of more simple delay differential equations. We apply our method to formulate stability theorems for explicit and implicit (threshold-type) state-dependent delay differential equations.

## 1. Introduction

In this paper we show that stability (including asymptotic and exponential stability) properties of certain classes of delay equations can be obtained by investigating the same type of equi-stability of some more simple associated differential equations. We present this general comparison principle in Section 3. Our main interest in this paper is to apply this method for state-dependent delay equations. In this case we can reduce the stability investigation of such equations to studying stability properties of equations with delays which are state-independent, but which depend on a parameter (a function in our case). We note that this approach was motivated by papers [3] and [8].

One of our major tool in this direction will be the results of Section 2, where we study preservation of exponential stability of linear delay differential systems under perturbing the coefficient function and the delay function. We will show that if a trivial solution of a linear delay equation is exponentially stable, then there always exists a certain "neighborhood" of the parameters such that any equation corresponding to parameters from this "neighborhood"

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has an exponentially equi-stable trivial solution. Stability of delay perturbed differential equations has been studied by several papers (see, e.g., [2], [5], [19]), but not in the context of equi-stability. The results of Section 2 will extend some of our earlier works in this direction [9], [11].

Section 4 will contain applications of the results of the comparison principle of Section 3 and the perturbation results of Section 2. Inspired by some earlier results given for linear equations in the papers [14] and [20] and for state-dependent equations in [8] we prove some similar, sometimes more general stability results for nonlinear equations. It is worth to note that our method works for threshold-type differential equations, as well. To the best of our knowledge our approach is original in the stability investigation of such equations.

## 2. Perturbation Results

First we introduce some basic notations used throughout this paper: a positive integer $n$ and $r > 0$ are fixed. Let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices of the same dimension. By the notation $A \le B$ we mean that relation $a_{ij} \le b_{ij}$ holds for all $i$ and $j$, and by $\max(A, B)$ we denote a matrix with the $ij$-th component $\max(a_{ij}, b_{ij})$. Let $|\cdot|$ denote a fixed vector norm on $\mathbb{R}^n$ such that the corresponding induced matrix norm on $\mathbb{R}^{n\times n}$ (which is denoted by $|\cdot|$, as well) is monotone, i.e., it satisfies $|A| \le |B|$ for matrices $0 \le A \le B$, and $|A| = |\max(A, -A)|$. For example the $|\cdot|_1$ or $|\cdot|_\infty$ norms satisfy these properties. We denote the space of continuous functions $\psi: [-r, 0] \to \mathbb{R}^n$ equipped with the supremum norm $\|\psi\| \equiv \max\{|\psi(t)|\colon t \in [-r, 0]\}$ by $C$, and the identically zero function of $C$ by 0. For a function $x: [-r, \infty) \to \mathbb{R}$ we define $x_t\colon [-r, 0] \to \mathbb{R}^n$, $x_t(s) \equiv x(t + s)$ for $t \ge 0$ and $-r \le s \le 0$.

Consider the linear delay systems

$$\dot{x}(t) = A(t)x(t - \sigma(t)), \qquad t \ge 0 \tag{2.1}$$

and

$$\dot{y}(t) = B(t)y(t - \eta(t)), \qquad t \ge 0, \tag{2.2}$$

with the respective initial conditions

$$x(t) = \varphi(t), \qquad t \in [-r, 0] \tag{2.3}$$

and

$$y(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{2.4}$$

Throughout this paper $\varphi \in C$, and we assume

(H1) $A, B\colon [0, \infty) \to \mathbb{R}^{n\times n}$ are continuous functions;

(H2) the delay functions $\sigma, \eta\colon [0,\infty) \to \mathbb{R}$ are continuous, and

$$0 \le \sigma(t) \le \gamma(t) \qquad \text{and} \qquad 0 \le \eta(t) \le \gamma(t), \qquad t \ge 0$$

for some continuous $\gamma\colon [0,\infty) \to \mathbb{R}$ satisfying $0 \le \gamma(t) \le t + r$ and $\underline{\lim}_{t\to\infty} t - \gamma(t) > 0$.

The solution of (2.1) corresponding to the initial time 0 and the initial function $\varphi$ is denoted by $x(t;\varphi)$. If we want to emphasize that the solution corresponds to the coefficient $A$ and the delay $\sigma$ we use the more detailed notation $x(t;\varphi, A, \sigma)$.

The trivial solution (i.e., $x = 0$) of the linear equation (2.1) is exponentially stable with order $\alpha > 0$, if there exists a constant $K_\alpha \ge 1$ such that the solution of (2.1) corresponding to initial function $\varphi$ satisfies

$$|x(t;\varphi)| \le K_\alpha e^{-\alpha t}\|\varphi\|, \qquad t \ge 0. \tag{2.5}$$

We will consider $B$ and $\eta$ to be fixed such that the trivial solution of (2.2) be exponentially stable. Equation (2.1) is considered as a perturbed equation of (2.2), i.e., we assume that $A$ and $\sigma$ are "close" to $B$ and $\eta$, respectively. We will show in Theorem 2.2 that if the perturbations are "small enough", then the exponential stability of (2.2) is preserved for (2.1).

Preservation of stability under delay perturbation has been studied, e.g., in [2], [5], [11] and [19]. In these papers it was assumed that the delays and the coefficients are bounded. We relax this condition in this section. Our Theorem 2.2 extends the results of [11] using the approach of [8]. Note that it was shown in [2] that there always exists a "neighborhood" of $B$ and $\eta$ inside which the exponential stability is preserved, but the proof gives only the existence of such a "neighborhood", not the size of it. We will define the "neighborhood" explicitly. Moreover, in Theorem 2.3 we define such a "neighborhood", inside which the exponential stability of the corresponding equation is uniform with respect to the parameters, i.e., the constants $K_\alpha$ and $\alpha$ in the definition of the exponential stability can be selected independently of the parameters. This is the result we will need in Section 4.

In the proof of our main theorem we need the following estimate which can be proved easily by using Gronwall's inequality (see, e.g., Lemma 2.1 in [8]).

**Lemma 2.1.** *Assume* (H1) *and* (H2). *Then the solution $x$ of the initial value problem* (2.1)-(2.3) *satisfies*

$$|x(t)| \le e^{\int_0^t |A(s)|\,ds}\|\varphi\| \tag{2.6}$$

*for all $t \ge 0$.*

Next we prove the main result of this section.

**Theorem 2.2.** *Assume* (H1) *and* (H2), *and the trivial solution of* (2.2) *is exponentially stable with order* $\alpha > 0$. *Then for any* $0 < \beta < \alpha$ *there exists* $\varepsilon > 0$ *such that if*

$$\overline{\lim_{t\to\infty}} \left( |A(t) - B(t)|e^{\beta\gamma(t)} + |B(t)|e^{\beta\gamma(t)} \left| \int_{t-\sigma(t)}^{t-\eta(t)} |B(s)|e^{\beta\gamma(s)}\, ds \right| \right) < \varepsilon, \quad (2.7)$$

*then the trivial solution of the corresponding equation* (2.1) *is exponentially stable with order* $\beta$, *i.e., there exists* $K_\beta \geq 1$ *such that*

$$|x(t;\varphi)| \leq K_\beta e^{-\beta t}\|\varphi\|, \qquad t \geq 0. \quad (2.8)$$

*Proof.* We can rewrite (2.1) in the form

$$\dot{x}(t) = B(t)x(t - \eta(t)) + f(t),$$

where

$$f(t) \equiv A(t)x(t - \sigma(t)) - B(t)x(t - \eta(t)).$$

Let $V$ be the fundamental solution of (2.2), i.e., the matrix valued solution of the initial value problem

$$\begin{aligned} \frac{\partial V}{\partial t}(t,s) &= B(t)V(t - \eta(t), s), \qquad t \geq s, \\ V(t,s) &= \begin{cases} I, & t = s, \\ 0, & t < s, \end{cases} \end{aligned}$$

where $I$ and $0$ is the identity and the zero matrix, respectively. Then the variation-of-constants formula (see, e.g., [13]) implies

$$x(t) = y(t) + \int_0^t V(t,s)f(s)\, ds, \qquad t \geq 0. \quad (2.9)$$

It is known (see, e.g., [13]) that the assumed exponential stability with order $\alpha$ of the trivial solution of (2.2) implies that there exist constants $\alpha > 0$, $K_\alpha \geq 1$ and $\tilde{K}_\alpha \geq 1$ such that $y$ and $V$ satisfy

$$|y(t;\varphi)| \leq K_\alpha e^{-\alpha t}\|\varphi\|, \quad \text{and} \quad |V(t,s)| \leq \tilde{K}_\alpha e^{-\alpha(t-s)} \qquad \text{for } t \geq s. \quad (2.10)$$

Therefore we get from (2.9) for any $t_1 > 0$ that

$$
\begin{aligned}
|x(t)| &\le |y(t)| + \int_0^t |V(t,s)||f(s)|\,ds \qquad (2.11)\\
&\le \begin{cases} K_\alpha e^{-\alpha t}\|\varphi\| + \tilde{K}_\alpha e^{-\alpha t}\int_0^{t_1} e^{\alpha s}|f(s)|ds, & t\in[0,t_1]\\ K_\alpha e^{-\alpha t}\|\varphi\| + \tilde{K}_\alpha e^{-\alpha t}\left(\int_0^{t_1} e^{\alpha s}|f(s)|ds + \int_{t_1}^t e^{\alpha s}|f(s)|ds\right), & t>t_1. \end{cases}
\end{aligned}
$$

Let $0<\beta<\alpha$ be fixed,

$$\varepsilon \equiv \frac{\alpha-\beta}{\tilde{K}_\alpha}, \qquad (2.12)$$

and let $A$ and $\sigma$ be such that (2.7) holds. We introduce the simplifying notation

$$d \equiv \overline{\lim_{t\to\infty}} \left( |A(t)-B(t)|e^{\beta\gamma(t)} + |B(t)|e^{\beta\gamma(t)} \left| \int_{t-\sigma(t)}^{t-\eta(t)} |B(s)|e^{\beta\gamma(s)}\,ds \right| \right),$$

and let $\delta>0$ be such that $d+\delta<\varepsilon$, and let $t_1>0$ be such that the inequalities

$$t-\sigma(t)\ge 0, \quad t-\eta(t)\ge 0, \qquad t\ge t_1 \qquad (2.13)$$

and

$$|A(t)-B(t)|e^{\beta\gamma(t)} + |B(t)|e^{\beta\gamma(t)} \left| \int_{t-\sigma(t)}^{t-\eta(t)} |B(s)|e^{\beta\gamma(s)}ds \right| < d+\delta,\ t\ge t_1 \qquad (2.14)$$

hold. Let $t>t_1$. Then (2.11) and the definition of $f$ yield

$$
\begin{aligned}
|x(t)| \le{}& K_\alpha e^{-\alpha t}\|\varphi\|\\
&+ \tilde{K}_\alpha e^{-\alpha t}\int_0^{t_1} e^{\alpha s}(|A(s)||x(s-\sigma(s))| + |B(s)||x(s-\eta(s))|)\,ds\\
&+ \tilde{K}_\alpha e^{-\alpha t}\int_{t_1}^{t} e^{\alpha s}|A(s)-B(s)||x(t-\sigma(s))|\,ds\\
&+ \tilde{K}_\alpha e^{-\alpha t}\int_{t_1}^{t} |B(s)||x(s-\sigma(s))-x(s-\eta(s))|\,ds. \qquad (2.15)
\end{aligned}
$$

The first integral of the right-hand-side of (2.15) can be estimated using Lemma 2.1 as

$$
\begin{aligned}
&\tilde{K}_\alpha e^{-\alpha t}\int_0^{t_1} e^{\alpha s}(|A(s)||x(s-\sigma(s))| + |B(s)||x(s-\eta(s))|)ds\\
&\le Ce^{-\alpha t}\|\varphi\|,
\end{aligned} \qquad (2.16)
$$

where $C$ is defined by

$$C \equiv \tilde{K}_\alpha \left( \max_{s\in[0,t_1]} |A(s)| e^{\int_0^{t_1} |A(s)|\,ds} + \max_{s\in[0,t_1]} |B(s)| e^{\int_0^{t_1} |B(s)|\,ds} \right) \times \frac{e^{\alpha t_1} - 1}{\alpha}. \tag{2.17}$$

We have $x(s-\eta(s)) - x(s-\sigma(s)) = \int_{s-\sigma(s)}^{s-\eta(s)} \dot{x}(u)\,du$ for $s \geq t_1$ by (2.13). Therefore the third integral of the right-hand-side of (2.15) can be rewritten as

$$\begin{aligned} &\tilde{K}_\alpha e^{-\alpha t} \int_{t_1}^{t} |B(s)||x(s-\sigma(s)) - x(s-\eta(s))|\,ds \\ &\quad = \tilde{K}_\alpha e^{-\alpha t} \int_{t_1}^{t} e^{\alpha s} |B(s)| \left| \int_{s-\sigma(s)}^{s-\eta(s)} \dot{x}(u)\,du \right| ds \\ &\quad \leq \tilde{K}_\alpha e^{-\alpha t} \int_{t_1}^{t} e^{\alpha s} |B(s)| \left| \int_{s-\sigma(s)}^{s-\eta(s)} |A(u)||x(u-\sigma(u))|\,du \right| ds. \end{aligned} \tag{2.18}$$

Multiplying both sides of (2.15) by $e^{\beta t}$, using the estimates (2.16) and (2.18), and introducing $z(t) \equiv e^{\beta t}|x(t)|$ we get for $t > t_1$

$$\begin{aligned} z(t) &\leq (K_\alpha + C)\|\varphi\| \\ &\quad + \tilde{K}_\alpha e^{-\alpha t + \beta t} \int_{t_1}^{t} e^{\alpha s} |A(s) - B(s)| z(s-\sigma(s)) e^{-\beta(s-\sigma(s))} ds \\ &\quad + \tilde{K}_\alpha e^{-\alpha t + \beta t} \int_{t_1}^{t} e^{\alpha s} |B(s)| \left| \int_{s-\sigma(s)}^{s-\eta(s)} |B(u)| z(u-\sigma(u)) e^{-\beta(u-\sigma(u))} du \right| ds \\ &\leq (K_\alpha + C)\|\varphi\| \\ &\quad + \tilde{K}_\alpha e^{-(\alpha-\beta)t} \max_{-r\leq u\leq t} z(u) \int_{t_1}^{t} e^{(\alpha-\beta)s} |A(s) - B(s)| e^{\beta\sigma(s)} ds \\ &\quad + \tilde{K}_\alpha e^{-(\alpha-\beta)t} \max_{-r\leq u\leq t} z(u) \\ &\quad \times \int_{t_1}^{t} e^{(\alpha-\beta)s} |B(s)| \left| \int_{s-\sigma(s)}^{s-\eta(s)} |B(u)| e^{-\beta(u-s-\sigma(u))} du \right| ds. \end{aligned} \tag{2.19}$$

Suppose $\eta(s) \leq \sigma(s)$. Then $-\sigma(s) \leq u - s \leq -\eta(s)$ for $u \in [s-\sigma(s), s-\eta(s)]$, and so $s - u \leq \sigma(s)$. In the case when $\sigma(s) \leq \eta(s)$ relation $s - u \leq \eta(s)$

follows similarly for $u \in [s-\eta(s), s-\sigma(s)]$, hence in both cases $s-u \le \gamma(s)$. Therefore (2.19) and (2.14) imply

$$\begin{aligned}
z(t) &\le (K_\alpha + C)\|\varphi\| \\
&\qquad + \tilde{K}_\alpha e^{-(\alpha-\beta)t} \max_{-r\le u\le t} z(u) \int_{t_1}^{t} e^{(\alpha-\beta)s} |A(s)-B(s)| e^{\beta\gamma(s)}\, ds \\
&\qquad + \tilde{K}_\alpha e^{-(\alpha-\beta)t} \\
&\qquad \times \max_{-r\le u\le t} z(u) \int_{t_1}^{t} e^{(\alpha-\beta)s} |B(s)| \left| \int_{s-\sigma(s)}^{s-\eta(s)} |B(u)| e^{\beta(\gamma(s)+\gamma(u))}\, du \right| ds \\
&\le (K_\alpha + C)\|\varphi\| + \tilde{K}_\alpha e^{-(\alpha-\beta)t} \max_{-r\le u\le t} z(u)(d+\delta) \int_{t_1}^{t} e^{(\alpha-\beta)s}\, ds \\
&= (K_\alpha + C)\|\varphi\| + \tilde{K}_\alpha e^{-(\alpha-\beta)t} \max_{-r\le u\le t} z(u)(d+\delta) \frac{e^{(\alpha-\beta)t} - e^{(\alpha-\beta)t_1}}{\alpha-\beta} \\
&\le (K_\alpha + C)\|\varphi\| + \frac{d+\delta}{\varepsilon} \max_{-r\le u\le t} z(u). \qquad (2.20)
\end{aligned}$$

It is easily follows from (2.11) that (2.20) holds for $t \in [0, t_1]$, as well. Since the right-hand-side of (2.20) is monotone in $t$, and $z(t) = e^{\beta t}|\varphi(t)| \le \|\varphi\| \le K_\alpha\|\varphi\|$ for $t \le 0$, therefore (2.20) yields

$$\max_{-r\le u\le t} z(u) \le (K_\alpha + C)\|\varphi\| + \frac{d+\delta}{\varepsilon} \max_{-r\le u\le t} z(u),$$

and hence $z(t) \le \max_{-r\le u\le t} z(u) \le K_\beta\|\varphi\|$, where

$$K_\beta \equiv \frac{K_\alpha + C}{1 - \frac{d+\delta}{\varepsilon}}. \qquad (2.21)$$

This implies that $|x(t)| \le K_\beta e^{-\beta t}\|\varphi\|$ for $t \ge 0$. □

Next we give conditions when the constant $K_\beta$ in (2.8) is independent of the selection of the coefficient matrix $A$ and the delay $\sigma$ satisfying (2.7), i.e., the trivial solution of (2.1) is exponentially equi-stable with respect to $A$ and $\sigma$ satisfying (2.7) (see the formal definition in Section 3 below).

**Theorem 2.3.** *Assume* (H1) *and* (H2), *and the trivial solution of* (2.2) *is exponentially stable with the order* $\alpha > 0$. *Let* $\tilde{K}_\alpha$ *be such that the fundamental solution* $V$ *of* (2.2) *satisfies* $|V(t,s)| \le \tilde{K}_\alpha e^{-\alpha(t-s)}$ *for* $t \ge s$, *and let* $0 < \beta <$

*$\alpha$ be fixed. Suppose the functions $\Gamma^+, \Gamma^- : [0,\infty) \to \mathbb{R}^{n\times n}$ and $\Delta^+, \Delta^- : [0,\infty) \to \mathbb{R}$ are such that*

$$0 \le \Gamma^+(t), \quad 0 \le \Gamma^-(t), \quad 0 \le \Delta^-(t) \le \eta(t), \quad 0 \le \Delta^+(t) \le \gamma(t) - \eta(t) \tag{2.22}$$

*for $t \ge 0$, and*

$$\overline{\lim_{t\to\infty}} \left( |\max(\Gamma^+(t), \Gamma^-(t))| e^{\beta\gamma(t)} + |B(t)| e^{\beta\gamma(t)} \int_{t-\eta(t)-\Delta^+(t)}^{t-\eta(t)+\Delta^-(t)} |B(s)| e^{\beta\gamma(s)} ds \right) < \frac{\alpha - \beta}{\tilde{K}_\alpha}. \tag{2.23}$$

*Suppose the parameters $A\colon [0,\infty) \to \mathbb{R}^{n\times n}$ and $\sigma\colon [0,\infty) \to \mathbb{R}$ belong to the set*

$$\begin{aligned} \Pi \equiv \Big\{ (A,\sigma)\colon\ & B(t) - \Gamma^-(t) \le A(t) \le B(t) + \Gamma^+(t) \quad \textit{and} \\ & \eta(t) - \Delta^-(t) \le \sigma(t) \le \eta(t) + \Delta^+(t) \quad \textit{for} \quad t \ge 0 \Big\}. \end{aligned} \tag{2.24}$$

*Then there exists $K_\beta \ge 1$ such that*

$$|x(t;\varphi,A,\sigma)| \le K_\beta e^{-\beta t} \|\varphi\|, \qquad t \ge 0, \quad (A,\sigma) \in \Pi,$$

*i.e., for any $(A,\sigma) \in \Pi$ the trivial solution of the corresponding equation* (2.1) *is exponentially stable with order $\beta$, and the constant $K_\beta$ is independent of the parameters $(A,\sigma) \in \Pi$.*

*Proof.* If $(A,\sigma) \in \Pi$ then $\sigma$ satisfies (H2). We denote $ij$-th element of the matrices $A(t)$, $B(t)$, $\Gamma^+(t)$ and $\Gamma^-(t)$ by $a_{ij}(t)$, $b_{ij}(t)$, $\gamma^+_{ij}(t)$ and $\gamma^-_{ij}(t)$, respectively. The definition of $\Pi$ yields that $|a_{ij}(t) - b_{ij}(t)| \le \max(\gamma^+_{ij}(t), \gamma^-_{ij}(t))$ for all $i$ and $j$, therefore the assumed properties of the matrix norm implies $|A(t) - B(t)| \le |\max(\Gamma^+(t), \Gamma^-(t))|$ for all $t \ge 0$. Therefore we have

$$\begin{aligned} & |A(t) - B(t)| e^{\beta\gamma(t)} + |B(t)| e^{\beta\gamma(t)} \left| \int_{t-\sigma(t)}^{t-\eta(t)} |B(s)| e^{\beta\gamma(s)}\, ds \right| \\ & \le |\max(\Gamma^+(t), \Gamma^-(t))| e^{\beta\gamma(t)} + |B(t)| e^{\beta\gamma(t)} \int_{t-\eta(t)-\Delta^+(t)}^{t-\eta(t)+\Delta^-(t)} |B(s)| e^{\beta\gamma(s)} ds, \end{aligned} \tag{2.25}$$

which, together with (2.12) and (2.23), implies that $A$ and $\sigma$ satisfy (2.7). Therefore the constant $K_\beta$ defined by (2.21) in the proof of Theorem 2.2

satisfies (2.8). We have to show that $K_\beta$ can be defined independently of the particular choice of $(A,\sigma)\in\Pi$. For this (see (2.21) and (2.17)) it is enough to prove that $C$ can be selected independently of $(A,\sigma)\in\Pi$. In view of the inequality $|A(t)|\le|B(t)|+|\max(\Gamma^+(t),\Gamma^-(t))|$ $(t\ge 0)$ and (2.17), we have to show only that $t_1$ can be independent of $A$ and $\sigma$. We recall that $t_1$ is defined by inequalities (2.13) and (2.14). Assumption (H2) yields that $t-\sigma(t)\ge t-\gamma(t)$ and $t-\eta(t)\ge t-\gamma(t)$ for $t\ge 0$, therefore $t_1$ can be chosen so that (2.13) be satisfied for any selection of the delays. It follows from (2.25) that $t_1$ can be such that (2.14) holds for any $(A,\sigma)\in\Pi$, which completes the proof of this theorem. ☐

**Remark 2.4.** *It is easy to see that the function*

$$\Delta_\varepsilon(t) \equiv \begin{cases} 1, & |B(t)|=0,\\ \min\left(1, \frac{\varepsilon}{3|B(t)|e^{\beta\gamma(t)}\max\{|B(s)|e^{\beta\gamma(s)}:\, s\in[t-\eta(t)-1,t-\eta(t)+1]\}}\right), & |B(t)|\ne 0 \end{cases} \tag{2.26}$$

*satisfies*

$$\overline{\lim_{t\to\infty}}\, |B(t)|e^{\beta\gamma(t)}\int_{t-\eta(t)-\Delta_\varepsilon(t)}^{t-\eta(t)+\Delta_\varepsilon(t)} |B(s)|e^{\beta\gamma(s)}\,ds<\varepsilon.$$

*Therefore if the trivial solution of* (2.2) *is exponentially stable, there always exists a "neighborhood" of* $(B,\eta)$ *of the form* (2.24) *such that the trivial solution of* (2.1) *corresponding to coefficient* $A$ *and delay* $\sigma$ *from this neighborhood is exponentially stable, as well.*

**Corollary 2.5.** *If the delay functions are bounded, i.e.,* $\gamma$ *in (H2) is* $\gamma(t)\equiv r$, *then the statement of Theorem* 2.3 *remains valid when condition* (2.23) *is replaced by*

$$\overline{\lim_{t\to\infty}}\left(|\max(\Gamma^+(t),\Gamma^-(t))|e^{\beta r}+e^{2\beta r}|B(t)|\int_{t-\eta(t)-\Delta^+(t)}^{t-\eta(t)+\Delta^-(t)}|B(s)|ds\right)<\frac{\alpha-\beta}{\tilde K_\alpha}.$$

*If, in addition,* $|B(t)|\le b_0$ *for* $t\ge 0$, *then* $\Gamma^+,\Gamma^-,\Delta^+$ *and* $\Delta^-$ *can be selected as* $\Gamma^+(t)=\Gamma^-(t)=\Gamma$ *is a componentwise nonnegative constant matrix,* $\Delta^+(t)=\Delta^-(t)=\Delta$ *is a nonnegative constant satisfying*

$$|\Gamma|e^{\beta r}+2\Delta b_0^2e^{2\beta r}<\frac{\alpha-\beta}{\tilde K_\alpha}. \tag{2.27}$$

We note that if

$$|\Gamma| + 2\Delta b_0^2 < \frac{\alpha}{\tilde{K}_\alpha},$$

then relation (2.27) holds, as well, for some $0 < \beta < \alpha$.

The results of this section can be immediately generalized to linear equations of the form

$$\dot{x}(t) = \sum_{k=1}^{m} A_k(t)x(t - \sigma_k(t)), \qquad t \geq 0, \tag{2.28}$$

where the functions $A_k$ and $\sigma_k$ satisfy conditions (H1) and (H2), respectively, for all $k = 1, \dots, m$ with bounds $\gamma_k$. We formulate the generalization of Theorem 2.3 for this equation. Theorem 2.2 can be stated similarly.

**Theorem 2.6.** *Assume $B_k$ and $\eta_k$ satisfy conditions* (H1) *and* (H2) *with $\gamma_k$, respectively, and suppose the trivial solution of*

$$\dot{x}(t) = \sum_{k=1}^{m} B_k(t)x(t - \eta_k(t)), \qquad t \geq 0 \tag{2.29}$$

*is exponentially stable with the order $\alpha$. Let $\tilde{K}_\alpha$ be such that the fundamental solution $V$ of* (2.29) *satisfies $|V(t,s)| \leq \tilde{K}_\alpha e^{-\alpha(t-s)}$ for $t \geq s$, let $0 < \beta < \alpha$ be fixed. Suppose the functions $\Gamma_k^+, \Gamma_k^- \colon [0,\infty) \to \mathbb{R}^{n\times n}$ and $\Delta_k^+, \Delta_k^- \colon [0,\infty) \to \mathbb{R}$ are such that*

$$0 \leq \Gamma_k^+(t), \quad 0 \leq \Gamma_k^-(t), \quad 0 \leq \Delta_k^-(t) \leq \eta_k(t), \quad 0 \leq \Delta_k^+(t) \leq \gamma_k(t) - \eta_k(t)$$

*for $t \geq 0$, $k = 1, \dots, m$, and*

$$\varlimsup_{t\to\infty} \Bigg( \sum_{k=1}^{m} |\max(\Gamma_k^+(t), \Gamma_k^-(t))| e^{\beta\gamma_k(t)} + \sum_{k=1}^{m} |B_k(t)| e^{\beta\gamma_k(t)} \int_{t-\eta_k(t)-\Delta_k^+(t)}^{t-\eta_k(t)+\Delta_k^-(t)} |B_k(s)| e^{\beta\gamma_k(s)}\, ds \Bigg) < \frac{\alpha-\beta}{\tilde{K}_\alpha}.$$

*Define the parameter set*

$$\Pi \equiv \Big\{ (A_1, \dots, A_m, \sigma_1, \dots, \sigma_m) : B_k(t) - \Gamma_k^-(t) \leq A_k(t) \leq B_k(t) + \Gamma_k^+(t) \text{ and } \eta_k(t) - \Delta_k^-(t) \leq \sigma_k(t) \leq \eta_k(t) + \Delta_k^+(t) \text{ for } t \geq 0, \quad k = 1, \dots, m \Big\}.$$

*Then there exists $K_\beta \geq 1$ such that*

$$|x(t; \varphi, A_1, \dots, A_m, \sigma_1, \dots, \sigma_m)| \leq K_\beta e^{-\beta t} \|\varphi\|,$$

*for $t \geq 0$, $(A_1, \dots, A_m, \sigma_1, \dots, \sigma_m) \in \Pi$.*

## 3. Equi-Stability with respect to a Set of Parameters

In Section 2 we studied a linear delay equation where we considered the coefficient and the delay function as parameters in the equation. In Theorem 2.3 we gave conditions when the solution tends to zero exponentially, and when the constants in the exponential estimate can be selected independently of the particular choice of the parameters. In this section we study this "independence from the parameters" in a more general form. We introduce the notion of equi-stability with respect to a set of parameters, and then prove our comparison principle for a certain class of functional differential equations. Consider

$$\dot{y}(t) = g(t, y_t, p), \qquad t \geq 0 \tag{3.1}$$

with initial condition

$$y(t) = \varphi(t), \qquad t \in [-r, 0], \tag{3.2}$$

where $g:\ [0,\infty) \times \Omega \times \mathcal{U} \to \mathbb{R}^n$, $\Omega \subset C$ including the zero function 0, the parameter $p$ belongs to a certain parameter set $\mathcal{U}$, and $g(t,0,p) = 0$ for all $t \geq 0$ and $p \in \mathcal{U}$. Note that in the applications we will show in Section 4 and in the next theorem the set $\mathcal{U}$ will be a subset of a function space, but for the sake of the following definitions $\mathcal{U}$ can be an arbitrary set without any structure in it. A solution of (3.1)–(3.2) corresponding to initial function $\varphi$ and parameter $p \in \mathcal{U}$ is denoted by $y(t) = y(t;\varphi,p)$.

We say that the trivial ($y = 0$) solution of (3.1)–(3.2) is *equi-stable* with respect to $\mathcal{U}$, if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|y(t;\varphi,p)| < \varepsilon$ for any $t \geq 0$, $\|\varphi\| < \delta$ and $p \in \mathcal{U}$. We say that the trivial solution of (3.1)–(3.2) is *asymptotically equi-stable* with respect to $\mathcal{U}$, if it is equi-stable with respect to $\mathcal{U}$, and there exists $\theta > 0$ that $\lim_{t\to\infty} y(t;\varphi,p) = 0$ for $\|\varphi\| < \theta$ and $p \in \mathcal{U}$. We say that the trivial solution of (3.1)–(3.2) is *exponentially equi-stable* with respect to $\mathcal{U}$, if for any $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$, $K = K(\varepsilon) \geq 1$ and $\alpha = \alpha(\varepsilon) > 0$ such that $|y(t;\varphi,p)| < Ke^{-\alpha t}\|\varphi\|$ for any $t \geq 0$, $\|\varphi\| < \delta$ and $p \in \mathcal{U}$.

Consider the functional differential equation

$$\dot{x}(t) = f(t, x_t, x_t), \qquad t \geq 0, \tag{3.3}$$

with initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0], \tag{3.4}$$

where

(A) $f\colon\ [0,\infty) \times \Omega_1 \times \Omega_2 \to \mathbb{R}^n$ is continuous, $\Omega_1$ and $\Omega_2$ are open subsets of $C$ both containing the identically zero function 0, and $f(t,0,u) = 0$ for $t \in [0,\infty)$ and $u \in \Omega_2$.

Let $\varrho > 0$ be fixed, and $\mathcal{S}(\varrho)$ denote the set of continuous functions $u : [-r,\infty) \to \mathbb{R}^n$ satisfying $|u(t)| \le \varrho$ for $t \ge -r$. Suppose $\varrho$ is small enough to satisfy $\mathcal{S}(\varrho) \subset \Omega_2$, and fix a function $u \in \mathcal{S}(\varrho)$. We associate the equation

$$\dot{y}(t) = f(t, y_t, u_t), \qquad t \ge 0, \tag{3.5}$$

to the function $u$ and to Equation (3.3) with the initial condition (3.2) corresponding to (3.4). A solution of (3.5)–(3.2) corresponding to initial function $\varphi$ and the function $u \in \mathcal{S}(\varrho)$ is denoted by $y(t) = y(t;\varphi,u)$. Assumption (A) yields that the identically zero function is a solution of both initial value problems (3.3)–(3.4) and (3.5)–(3.2).

The next theorem shows that the equi-stability of the trivial solution of (3.5) implies the stability of the trivial solution of (3.3).

**Theorem 3.1.** *Assume* (A), *let* $\varrho > 0$ *be such that* $\mathcal{S}(\varrho) \subset \Omega_2$, *and* $u \in \mathcal{S}(\varrho)$. *Then*

(i) *if the trivial solution of* (3.5) *is equi-stable with respect to* $\mathcal{S}(\varrho)$, *then the trivial solution of* (3.3) *is stable, as well;*

(ii) *if the trivial solution of* (3.5) *is asymptotically equi-stable with respect to* $\mathcal{S}(\varrho)$, *then the trivial solution of* (3.3) *is asymptotically stable, as well;*

(iii) *if the trivial solution of* (3.5) *is exponentially equi-stable with respect to* $\mathcal{S}(\varrho)$, *then the trivial solution of* (3.3) *is exponentially stable, as well.*

*Proof.* (i) Fix any $0 < \varepsilon < \varrho$, and let $0 < \delta < \varrho$ be a constant corresponding to $\varepsilon$ in the definition of equi-stability with respect to $\mathcal{S}(\varrho)$ of the trivial solution of (3.5). Let $\varphi$ satisfy $\|\varphi\| < \delta$, and let $x(t) = x(t;\varphi)$ be any corresponding solution of (3.3)–(3.4). Since, by assumption, $|x(0)| < \varrho$, the continuity of $x$ yields that $|x(t)| < \varrho$ for $t > 0$ close to 0. Suppose there exists $T > 0$ such that $|x(t)| < \varrho$ for $t \in [0,T)$ and $|x(T)| = \varrho$. Define

$$u(t) = \begin{cases} x(t), & t \in [-r,T), \\ x(T), & t \ge T. \end{cases}$$

Then $u \in \mathcal{S}(\varrho)$. Let $y(t) = y(t;\varphi,u)$ be the solution of the corresponding (3.5)–(3.2). By the equi-stability with respect to $\mathcal{S}(\varrho)$ of (3.5), $|y(t;\varphi,u)| < \varepsilon < \varrho$ for $t \ge 0$. On the other hand, $y(t) = x(t)$ for $t \in [0,T)$. Therefore, by continuity, $|x(T)| = |y(T)| = \varrho$ gives a contradiction to the definition of $T$. Hence $|x(t)| = |y(t)| < \varepsilon$ for $t \ge 0$, which proves (i).

(ii) By part (i) the trivial solution of (3.3) is stable, therefore there exists $\delta > 0$ such that $|x(t;\varphi)| < \varrho$ for $t \geq 0$ and $\|\varphi\| < \delta$. The asymptotic equi-stability of (3.5) implies the existence of $\theta > 0$ that $\lim_{t\to\infty} y(t;\varphi,u) = 0$ for $\|\varphi\| < \theta$ and $u \in \mathcal{S}(\varrho)$. Let $u(t) = x(t;\varphi)$ for a fixed $\varphi$ satisfying $\|\varphi\| < \theta$, then $u \in \mathcal{S}(\varrho)$. Therefore $\lim_{t\to\infty} x(t) = 0$, as well, since $x(t;\varphi) = y(t;\varphi,u)$.

(iii) As in part (ii), there exists $\delta_0 > 0$ such that $|x(t;\varphi)| < \varrho$ for $t \geq 0$, $\|\varphi\| < \delta_0$. By assumption, there exist $\delta > 0$, $K \geq 1$ and $\alpha > 0$ such that $|y(t;\varphi,u)| \leq Ke^{-\alpha t}\|\varphi\|$ for $t \geq 0$, $\|\varphi\| < \delta$ and $u \in \mathcal{S}(\varrho)$. But for $\|\varphi\| < \min\{\delta_0,\delta\}$ and $u = x(\cdot;\varphi)$ we have $x(t;\varphi) = y(t;\varphi,u)$, and so $|x(t;\varphi)| \leq Ke^{-\alpha t}\|\varphi\|$, $t \geq 0$. □

Theorem 3.1 can be applied for example for state-dependent delay equations of the form

$$\dot{x}(t) = h(t, x(t), x(t - \tau(t, x_t))), \qquad t \geq 0, \tag{3.6}$$

where the delay function $\tau\colon [0,\infty) \times C \to \mathbb{R}$ is continuous, and $0 \leq \tau(t,\psi) \leq t + r$ for $t \geq 0$ and $\psi \in C$ and $h(t,0,0) = 0$, $t \geq 0$. The associated state-independent delay equation to (3.6) is

$$\dot{y}(t) = h(t, y(t), y(t - \tau(t, u_t))), \qquad t \geq 0. \tag{3.7}$$

Therefore some type of equi-stability of the trivial solution of (3.7) implies the same type of stability of the trivial solution of (3.6). We note that such results can be generalized for for state-dependent delay equations with multiple delays, and for other classes of differential equations, e.g., for equations with unbounded delays, (i.e., where the initial interval is $[-r,0] = (-\infty,0]$) or for neutral differential equations. The applicability of these theorems depends on if we can give conditions implying equi-stability of the associated equation. In the next section we will present such conditions for several classes of delay equations including equations with state-dependent delays.

## 4. Applications

In our first example we give a condition implying the equi-stability of a certain delay equation. Consider the linear delay equation

$$\dot{x}(t) = -\sum_{i=1}^{m} a_i(t,p)x(t - \tau_i(t)), \qquad t \geq 0, \tag{4.1}$$

where $p$ is a parameter in the equation belonging to a certain set $\mathcal{U}$.

**Theorem 4.1.** *Let $r > 0$, and assume $\tau_i \colon [0,\infty) \to [0,r]$ and $a_i \colon [0,\infty) \times \mathcal{U} \to [0,\infty)$ for $i = 1,\dots,m$, and there exist constants $0 \le d_{ik} < 1$ $(i,k = 1,\dots,m)$, $T \ge r$ and $K > 0$ such that*

$$\int_{t-\tau_i(t)}^{t} a_k(s,p)\,ds \le d_{ik}, \qquad t \ge T, \quad p \in \mathcal{U}, \quad i,k = 1\dots,m,$$

*where*

$$\sum_{i,k=1}^{m} d_{ik} < 1, \tag{4.2}$$

*and*

$$\int_0^T \sum_{i=1}^{m} a_i(s,p)\,ds \le K, \qquad p \in \mathcal{U}.$$

(i) *Then the trivial solution of* (4.1) *is equi-stable with respect to* $\mathcal{U}$.

(ii) *If we assume further that* $\displaystyle\int_0^\infty \sum_{j=1}^{m} a_j(s,p)ds = \infty$ *for* $p \in \mathcal{U}$, *then the trivial solution of* (4.1) *is asymptotically equi-stable with respect to* $\mathcal{U}$.

(iii) *If, moreover, there exists* $\alpha > 0$ *such that*

$$\int_s^t a_i(s,p)ds \ge \alpha(t-s) \quad \text{for} \quad t \ge s \ge 0, \ p \in \mathcal{U} \quad \text{and} \quad i = 1,\dots,m,$$

*then the trivial solution of* (4.1) *is exponentially equi-stable with respect to* $\mathcal{U}$.

*Proof.* (i) Fix $p \in \mathcal{U}$. We have

$$\dot{x}(t) = -\left(\sum_{i=1}^{m} a_i(t,p)\right) x(t) + \sum_{i=1}^{m} a_i(t,p)(x(t) - x(t-\tau_i(t))), \qquad t \ge 0.$$

Using the variation-of-constant formula for ODEs we get

$$\begin{aligned} x(t) &= e^{-\int_0^t \sum_{i=1}^m a_i(s,p)ds} x(0) \\ &\quad + \sum_{i=1}^{m} \int_0^t e^{-\int_s^t \sum_{j=1}^m a_j(u,p)du} a_i(s,p)(x(s) - x(s-\tau_i(s)))ds. \end{aligned}$$

Using $t-\tau_i(t)\geq 0$ for $t\geq T\geq r$ and $i=1,\dots,m$, Equation (4.1) yields for $t\geq T$

$$\begin{aligned}
x(t) &= e^{-\int_0^t \sum_{i=1}^m a_i(s,p)\,ds}x(0)\\
&\quad+\sum_{i=1}^m\int_0^T e^{-\int_s^t \sum_{j=1}^m a_j(u,p)\,du}a_i(s,p)(x(s)-x(s-\tau_i(s)))\,ds\\
&\quad-\sum_{i=1}^m\int_T^t e^{-\int_s^t \sum_{j=1}^m a_j(u,p)\,du}a_i(s,p)\int_{s-\tau_i(s)}^s\sum_{k=1}^m a_k(u,p)x(u-\tau_k(u))duds.
\end{aligned}$$

A simple generalization of Lemma 2.1 to Equation (4.1) implies

$$|x(t)|\leq e^{\int_0^T \sum_{j=1}^m a_j(s,p)\,ds}\|\varphi\|\leq e^K\|\varphi\|,\qquad t\in[0,T],\quad p\in\mathcal{U},$$

therefore, for $t\geq T$

$$\begin{aligned}
&|x(t)|\\
&\leq e^{-\int_0^t \sum_{j=1}^m a_j(s,p)ds}|x(0)|+2e^K\|\varphi\|\sum_{i=1}^m\int_0^T e^{-\int_s^t \sum_{j=1}^m a_j(u,p)du}a_i(s,p)ds\\
&\quad+\sum_{i=1}^m\int_T^t e^{-\int_s^t \sum_{j=1}^m a_j(u,p)du}a_i(s,p)\int_{s-\tau_i(s)}^s\sum_{k=1}^m a_k(u,p)|x(u-\tau_k(u))|duds \quad (4.3)\\
&\leq e^{-\int_0^t \sum_{j=1}^m a_j(s,p)ds}|x(0)|\\
&\quad+2e^K\|\varphi\|\left(e^{-\int_T^t \sum_{j=1}^m a_j(u,p)du}-e^{-\int_0^t \sum_{j=1}^m a_j(u,p)du}\right)\\
&\quad+\max_{-r\leq s\leq t}|x(s)|\sum_{i,k=1}^m\int_T^t e^{-\int_s^t \sum_{j=1}^m a_j(u,p)du}a_i(s,p)\int_{s-\tau_i(s)}^s a_k(u,p)duds\\
&\leq\left(1+2e^K\right)\|\varphi\|+\max_{-r\leq s\leq t}|x(s)|\sum_{i,k=1}^m d_{ik}\int_T^t e^{-\int_s^t a_i(u,p)du}a_i(s,p)ds\\
&=\left(1+2e^K\right)\|\varphi\|+\max_{-r\leq s\leq t}|x(s)|\sum_{i,k=1}^m d_{ik}\left(1-e^{-\int_T^t a_i(u,p)du}\right) \quad (4.4)\\
&\leq\left(1+2e^K\right)\|\varphi\|+\max_{-r\leq s\leq t}|x(s)|\sum_{i,k=1}^m d_{ik}.
\end{aligned}$$

Note that the last inequality holds for $t\in[-r,T]$, as well. It follows therefore

$$\max_{-r\leq s\leq t}|x(s)|\leq\left(1+2e^K\right)\|\varphi\|+d\max_{-r\leq s\leq t}|x(s)|,$$

where

$$d \equiv \sum_{i,k=1}^{m} d_{ik} < 1,$$

hence

$$|x(t)| \leq \max_{-r \leq s \leq t} |x(s)| \leq \frac{1+2e^K}{1-d}\|\varphi\|,$$

which yields the stability of the trivial solution of (4.1).

(ii) Let $x$ be any fixed solution of (4.1), then, by part (i), $\overline{\lim}_{t\to\infty}|x(t)|$ is finite. Let $\varepsilon > 0$ be fixed, and let $t_1 > T$ be such that $|x(t)| \leq \overline{\lim}_{t\to\infty}|x(t)|+\varepsilon$ for $t \geq t_1 - r$. Similarly to (4.4) one can easily obtain

$$\begin{aligned} |x(t)| &\leq e^{-\int_{t_1}^t \sum_{j=1}^m a_j(s,p)ds}|x(t_1)| \\ &\quad + (\overline{\lim_{s\to\infty}}|x(s)|+\varepsilon)\sum_{i,k=1}^m d_{ik}\left(1-e^{-\int_{t_1}^t a_i(s,p)ds}\right), \quad t \geq t_1. \end{aligned} \tag{4.5}$$

Then taking the limit as $t \to \infty$ we get

$$\overline{\lim_{s\to\infty}}|x(s)| \leq d(\overline{\lim_{s\to\infty}}|x(s)|+\varepsilon),$$

or equivalently,

$$\overline{\lim_{s\to\infty}}|x(s)| \leq \frac{d\varepsilon}{1-d},$$

which yields $\lim_{t\to\infty} x(t) = 0$, since $\varepsilon > 0$ was arbitrary.

To prove part (iii) fix $0 < \beta < \alpha$ such that

$$de^{2\beta r}\left(1+\frac{\beta}{\alpha-\beta}\right) < 1,$$

and introduce $z(t) = |x(t)|e^{\beta t}$. Multiplying both sides of (4.3) by $e^{\beta t}$ and using that

$$e^{-\int_0^t \sum_{j=1}^m a_j(s,p)\,ds} \leq e^{-\beta t}, \qquad t \geq 0, \quad p \in \mathcal{U},$$

we get

$$\begin{aligned} z(t) &\leq \|\varphi\| + 2e^K\|\varphi\|e^{\beta t}\left(e^{-\int_T^t \sum_{j=1}^m a_j(u,p)du} - e^{-\int_0^t \sum_{j=1}^m a_j(u,p)du}\right) \\ &\quad + e^{\beta t}\sum_{i,k=1}^m \int_T^t e^{-\int_s^t \sum_{j=1}^m a_j(u,p)du} a_i(s,p) \\ &\quad \times \int_{s-\tau_i(s)}^s a_k(u,p)|z(u-\tau_k(u))|e^{-\beta(u-\tau_k(u))}duds \end{aligned}$$

$$\le \|\varphi\| + 2e^K\|\varphi\| e^{\int_0^T \sum_{j=1}^m a_j(u,p)du}$$

$$+ e^{\beta r}\max_{-r\le s\le t} z(s)\sum_{i,k=1}^m \int_T^t e^{-\int_s^t a_i(u,p)du+\beta t} a_i(s,p)e^{-\beta(s-\tau_i(s))}\int_{s-\tau_i(s)}^s a_k(u,p)duds$$

$$\le (1+2e^{2K})\|\varphi\| + e^{2\beta r}\max_{-r\le s\le t} z(s)\sum_{i,k=1}^m d_{ik}\int_T^t e^{-\int_s^t a_i(u,p)du+\beta(t-s)}a_i(s,p)ds.$$

Integration by parts and inequality $e^{-\int_s^t a_i(s,p)\,ds} \le e^{-\alpha(t-s)}$ yield

$$z(t) \le (1+2e^{2K})\|\varphi\| + e^{2\beta r}\max_{-r\le s\le t} z(s)\sum_{i,k=1}^m d_{ik}\Big(1 - e^{-\int_T^t a_i(u,p)\,du+\beta(t-T)}$$

$$+\beta\int_T^t e^{-\int_s^t a_i(u,p)\,du} e^{\beta(t-s)}\,ds\Big)$$

$$\le (1+2e^{2K})\|\varphi\| + e^{2\beta r}\max_{-r\le s\le t} z(s)\sum_{i,k=1}^m d_{ik}\Big(1+\beta\int_T^t e^{(\alpha-\beta)(s-t)}\,ds\Big)$$

$$\le (1+2e^{2K})\|\varphi\| + e^{2\beta r}\max_{-r\le s\le t} z(s) d\Big(1+\frac{\beta}{\alpha-\beta}\Big),$$

which implies easily $z(t) \le M_\beta\|\varphi\|$, where

$$M_\beta \equiv \frac{1+2e^{2K}}{1-de^{2\beta r}\left(1+\frac{\beta}{\alpha-\beta}\right)}.$$

This therefore means that $|x(t)| \le M_\beta e^{-\beta t}\|\varphi\|$, i.e., the trivial solution of (4.1) is exponentially equi-stable with respect to $\mathcal{U}$. □

It has been shown in [14] by Krisztin (as a special case of a result proved for distributed delay case) that the trivial solution of the scalar equation

$$\dot{x}(t) = -\sum_{i=1}^m a_i(t)x(t-\tau_i(t)), \qquad t\ge 0 \tag{4.6}$$

is asymptotically stable, if $0\le a_i(t)\le \alpha_i$ and $0\le \tau_i(t)\le q_i$ for $t\ge 0$, and

$$\sum_{i=1}^m \alpha_i q_i < 1.$$

Yoneyama [20] proved the asymptotic stability of the trivial solution of the equation

$$\dot{x}(t) = -a(t)x(t-\tau(t)), \qquad t \geq 0 \tag{4.7}$$

under the integral condition that

$$0 < \inf_{t\geq 0} \int_{t-\tau_0}^{t} a(s)\,ds \leq \sup_{t\geq 0} \int_{t-\tau_0}^{t} a(s)\,ds < \frac{3}{2},$$

when $0 \leq a(t)$ and $0 \leq \tau(t) \leq \tau_0$ for $t \geq 0$. Our Theorem 4.1 was motivated by Yoneyama's condition and reformulates Krisztin's result using integral condition. Note that the upper limit $\frac{3}{2}$ in the above condition was increased in [9] (but at the same time the lower limit 0 had to be increased, as well), where it was shown that if $\int_0^\infty a(s)\,ds = \infty$ and the function $t \mapsto \int_0^t a(s)\,ds$ is monotone increasing, then for any $c \in (0, \pi/2)$ there exists $b \in (0, c)$ such that the trivial solution of (4.7) is asymptotically stable, assuming

$$b < \varliminf_{t\to\infty} \int_{t-\tau(t)}^{t} a(s)\,ds \leq \varlimsup_{t\to\infty} \int_{t-\tau(t)}^{t} a(s)\,ds < c.$$

In our next example we consider the scalar equation

$$\dot{x}(t) = -\sum_{i=1}^{m} a_i(t, x_t)x(t-\tau_i(t)), \qquad t \geq 0. \tag{4.8}$$

Theorems 3.1 and 4.1 have the following corollary.

**Theorem 4.2.** *Assume* $\tau_i\colon [0,\infty) \to [0,r]$, $a_i\colon [0,\infty)\times C \to [0,\infty)$, *there exist constants* $\varrho > 0$, $0 \leq d_{ik} < 1$ $(i,k = 1,\ldots,m)$, $T \geq r$ *and* $K > 0$ *such that*

$$\int_{t-\tau_i(t)}^{t} a_k(s, u_s)\,ds \leq d_{ik}, \qquad t \geq T, \quad u \in \mathcal{S}(\varrho), \quad i,k = 1,\ldots,m,$$

*where*

$$\sum_{i,k=1}^{m} d_{ik} < 1,$$

*and*

$$\int_0^T \sum_{i=1}^{m} a_i(s, u_s)\,ds \leq K, \qquad u \in \mathcal{S}(\varrho).$$

(i) *Then the trivial solution of* (4.8) *is stable.*

(ii) *If we assume further that* $\int_0^\infty \sum_{j=1}^m a_j(s,u_s)\,ds = \infty$ *for* $u \in \mathcal{S}(\varrho)$, *then the trivial solution of* (4.8) *is asymptotically stable.*

(iii) *If, moreover, there exists* $\alpha > 0$ *such that*

$$\int_s^t a_i(s,u_s)ds \ge \alpha(t-s) \quad \textit{for } t \ge s \ge 0,\ u \in \mathcal{S}(\varrho) \textit{ and } i = 1,\dots,m,$$

*then the trivial solution of* (4.8) *is exponentially stable.*

Next we study the exponential stability of the state-dependent delay system

$$\dot{x}(t) = B(t)x(t-\tau(t,x_t)), \qquad t \ge 0, \tag{4.9}$$

with the associated initial condition (2.3). We assume that $B$ satisfies (H1), and $\tau$ satisfies

(H3) $\tau\colon [0,\infty) \times C \to [0,\infty)$ is continuous, and there exist $\varrho > 0$ and a continuous function $\gamma\colon [0,\infty) \to \mathbb{R}$ such that $0 \le \gamma(t) \le t + r$ for $t \ge 0$, $\underline{\lim}_{t\to\infty}\, t - \gamma(t) > 0$, and
$$\tau(t,u_t) \le \gamma(t) \qquad \text{for} \quad t \ge 0, \quad u \in \mathcal{S}(\varrho).$$

Note that these conditions imply the local existence of solutions of (4.9)-(2.3), but not necessary the uniqueness of the solution (see, e.g., [4], [10]).

Remark 2.4 yields that for every $n,m \in \mathbb{N}$ there exist functions $\Delta^+_{n,m}$, $\Delta^-_{n,m}\colon [0,\infty) \to [0,\infty)$ satisfying

$$\overline{\lim_{t\to\infty}}\, |B(t)|e^{\frac{1}{m}\gamma(t)} \int_{t-\tau(t,0)-\Delta^-_{n,m}(t)}^{t-\tau(t,0)+\Delta^+_{n,m}(t)} |B(s)|e^{\frac{1}{m}\gamma(s)}\,ds < \frac{1}{n} \tag{4.10}$$

and

$$0 \le \Delta^-_{n,m}(t) \le \tau(t,0), \quad 0 \le \Delta^+_{n,m}(t) \le \gamma(t) - \tau(t,0) \qquad \text{for} \quad t \ge 0.$$

With the help of these functions we can test if the exponential stability of the trivial solution of

$$\dot{x}(t) = B(t)x(t-\tau(t,0)), \qquad t \ge 0 \tag{4.11}$$

is preserved for that of (4.9). In particular, assume that $\tau$ is such that

(H4) for every $n,m \in \mathbb{N}$ there exist $T = T_{n,m} > 0$ and $0 < \delta = \delta_{n,m} \le \varrho$ such that

$$\tau(t,0) - \Delta^-_{n,m}(t) \le \tau(t,u_t) \le \tau(t,0) + \Delta^+_{n,m}(t),\ t \ge T \text{ and } u \in \mathcal{S}(\delta). \tag{4.12}$$

Then we have the following result.

**Theorem 4.3.** *Assume* (H1), (H3) *and* (H4), *and the trivial solution of* (4.11) *is exponentially stable. Then the trivial solution of* (4.9) *is exponentially stable, as well.*

*Proof.* For any $u \in \mathcal{S}(\varrho)$ we associate equation

$$\dot{y}(t) = B(t)y(t-\tau(t,u_t)), \qquad t \geq 0 \tag{4.13}$$

to (4.9). The assumptions imply that there exists $\tilde{K}_\alpha \geq 1$ and $\alpha > 0$ such that the fundamental solution $V$ of (4.11) satisfies $|V(t,s)| \leq \tilde{K}_\alpha e^{-\alpha(t-s)}$ for $t \geq s$. Fix $\frac{1}{\alpha} < m_0$, and let $n_0 \in \mathbb{N}$ be such that $\tilde{K}_\alpha/(\alpha - \frac{1}{m_0}) < n_0$, and let $T$ and $\delta$ be the corresponding constants from (H4). We define the functions

$$\Delta^+(t) \equiv \begin{cases} \Delta^+_{n_0,m_0}(t), & t \geq T, \\ \gamma(t) - \tau(t,0), & 0 \leq t < T \end{cases}$$

and

$$\Delta^-(t) \equiv \begin{cases} \Delta^-_{n_0,m_0}(t), & t \geq T, \\ \tau(t,0), & 0 \leq t < T \end{cases}$$

and the set

$$\Pi = \{\sigma\colon \tau(t,0) - \Delta^-(t) \leq \sigma(t) \leq \tau(t,0) + \Delta^+(t), \quad t \geq 0\}.$$

Then $\tau(\cdot,u_\cdot) \in \Pi$ for $u \in \mathcal{S}(\delta)$, and

$$\overline{\lim_{t\to\infty}}\, |B(t)| e^{\frac{1}{m_0}\gamma(t)} \int_{t-\tau(t,0)-\Delta^-(t)}^{t-\tau(t,0)+\Delta^+(t)} |B(s)| e^{\frac{1}{m_0}\gamma(s)}\, ds < \frac{\alpha - \frac{1}{m_0}}{\tilde{K}_\alpha}.$$

Hence Theorem 2.3 implies that the trivial solution of (4.13) is exponentially equi-stable with order $1/m_0$ with respect to the set $\mathcal{S}(\delta)$. Therefore Theorem 3.1 implies that the trivial solution of (4.9) is exponentially stable, as well. □

Note that if $|B(t)|e^{\beta\gamma(t)}$ is bounded for $t > 0$ and for some $\beta > 0$, then, for large enough $m$, $\Delta^+_{n,m}$ and $\Delta^-_{n,m}$ can be selected to be constants functions. If both $|B(t)|$ and $\gamma(t)$ are bounded, Corollary 2.5 and the last theorem imply immediately the next corollary, which slightly improves Theorem 2.2 of [8].

**Corollary 4.4.** *If* $|B(t)| \leq b_0$ *for* $t \geq 0$ *and* $\tau\colon [0,\infty) \times C \to [0,r]$, *then Theorem* 4.3 *remains true when assumption (H4) is replaced by*

(H4') *for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\overline{\lim}_{t\to\infty} |\tau(t,0) - \tau(t,u_t)| < \varepsilon, \qquad u \in \mathcal{S}(\delta). \tag{4.14}$$

We note that in [8] it was proved, that if we have more smoothness on the delay $\tau$, then the exponential stability of the trivial solution of (4.11) is not only sufficient, but also necessary for the exponential stability of the trivial solution of (4.9).

Our final result is formulated for the scalar delay equation

$$\dot{x}(t) = a(t)x(t - \tau(t, x_t)), \qquad t \geq 0, \tag{4.15}$$

where the delay function is defined by the threshold relation

$$\int_{t-\tau(t,x_t)}^{t} f(t, s, x_t)\, ds = m, \qquad t \geq 0 \tag{4.16}$$

for some $m > 0$. Recently such threshold-type delay equations have received considerable attention from modelling and theoretical point of view, as well (see, e.g., [1], [6], [7], [15]–[18]), but very little is known about the general stability theory of such equations (see [15]).

Let $F$ be a positive constant, $r \equiv m/F$, and we assume

(A1) $a\colon [0, \infty) \to \mathbb{R}$ is continuous and bounded,

(A2) $f\colon [0, \infty) \times [-r, \infty) \times C \to (0, \infty)$ is such that

- (i) for every $\varepsilon > 0$ there exist $\delta > 0$ and $T > 0$ such that $|f(t, s, \psi) - f(t, s, 0)| < \varepsilon$ for $t \geq T$, $s \geq T - r$ and $\psi \in \mathcal{S}(\delta)$,
- (ii) $f(t, s, 0) \geq F$ for $t \geq 0$ and $s \geq -r$.

Note that assumption (A2) (ii) implies $0 < \tau(t, 0) \leq r$ for $t \geq 0$.

**Theorem 4.5.** *Assume* (A1) *and* (A2), *and suppose the trivial solution of*

$$\dot{x}(t) = a(t)x(t - \tau(t, 0)), \qquad t \geq 0, \tag{4.17}$$

*is exponentially stable. Then the trivial solution of* (4.15) *is exponentially stable, as well.*

*Proof.* By Remark 4.4 it is enough to show that (4.14) holds. Assumption (A2)(i) yields that for any $\varepsilon > 0$ there exist $\delta > 0$ and $T > 0$ such that

$$\int_{t-\tau(t,u_t)}^{t} (f(t, s, 0) - \varepsilon)\, ds \leq \int_{t-\tau(t,u_t)}^{t} f(t, s, u_t)\, ds \leq \int_{t-\tau(t,u_t)}^{t} (f(t, s, 0) + \varepsilon)\, ds$$

for $t > T$ and any $u \in \mathcal{S}(\delta)$. On the other hand for such $u$ the definition of $\tau(t, u_t)$ implies

$$\int_{t-\tau(t,u_t)}^{t} f(t, s, u_t)\, ds = \int_{t-\tau(t,0)}^{t} f(t, s, 0)\, ds = m,$$

therefore

$$\int_{t-\tau(t,u_t)}^{t} (f(t,s,0)-\varepsilon)\,ds \le \int_{t-\tau(t,0)}^{t} f(t,s,0)\,ds \le \int_{t-\tau(t,u_t)}^{t} (f(t,s,0)+\varepsilon)\,ds,$$

and so

$$-\varepsilon\tau(t,u_t) \le \int_{t-\tau(t,0)}^{t-\tau(t,u_t)} f(t,s,0)\,ds \le \varepsilon\tau(t,u_t).$$

Hence

$$F|\tau(t,u_t)-\tau(t,0)| \le \left|\int_{t-\tau(t,0)}^{t-\tau(t,u_t)} f(t,s,0)\,ds\right| \le \varepsilon\tau(t,u_t) \tag{4.18}$$

for $t > T$ and $u \in \mathcal{S}(\delta)$. Assumption (A2) (i) yields

$$m = \int_{t-\tau(t,u_t)}^{t} f(t,s,u_t)\,ds \ge (F-\varepsilon)\tau(t,u_t).$$

Therefore it follows from (4.18) that

$$|\tau(t,u_t)-\tau(t,0)| \le \frac{\varepsilon m}{(F-m)F}, \qquad t > T, \quad u \in \mathcal{S}(\delta),$$

which implies property (H4'). $\square$

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