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Global attractivity in $x'(t) = -\delta x(t) + pf(x(t-\tau))$

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Abstract. We study the attractivity properties of equilibriums of the scalar delay differential equation $x'(t) = -\delta x(t) + pf(x(t-\tau))$ which arises in many contexts in the medicine, ecology, and environment sciences. New sufficient conditions for the global stability of the unique positive steady state are obtained. These conditions contain earlier results [4], [7], [8], [9], [11], [17] as particular cases. Some persistence results for this equation are also proved.

1. Introduction

1.1. The problem and the main assumptions. Here we shall be interested in the study of delay differential equations of the following form

$$\begin{aligned} &x'(t) = -\delta x(t) + pf(x(t-\tau)),\ p,\ \delta,\ \tau > 0, \\ &\text{with initial condition } x(s) = \gamma(s),\ \gamma \in C([-\tau, 0], R_+) \end{aligned} \tag{1}$$

which arise in many contexts in the medicine, ecology, and environment sciences etc. (see [1], [18], [19], [20] for motivations). Recently, this equation have attracted a lot of interest. There are a number of results concerning oscillations, global attractivity, periodicity and bifurcations of solutions to (1), see for example [7], [8], [9], [11], [16],

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[17], [21]. Here we shall investigate the attractivity properties of equilibrium points of this delay differential equation. Obviously, these points are determined from the equation

$$x = \frac{p}{\delta} f(x) \stackrel{def}{=} \varphi(x).$$

To specify the nature of (1), we shall make several standing assumptions on $f(x)$. We will suppose that $f : [0, +\infty) \to [0, +\infty)$ is a continuous map and $f(+\infty) = 0$. For easy reference in the sequel, we also list the following hypotheses on f which will be assumed only whenever this is explicitely indicated:

H1 $f : [0, +\infty) \to (0, +\infty)$ is strictly decreasing function;

H2 $f : (0, +\infty) \to (0, +\infty)$ is a unimodal function (that is, it has at most one critical point) and $f(0) = 0$.

H3 $f : (0, +\infty) \to (0, +\infty)$ is a unimodal function and $f(x) = xf_1(x)$, $pf_1(0) > \delta$, where $f_1 : [0, +\infty) \to (0, +\infty)$ is continuous.

H4 $f : (0, +\infty) \to (0, +\infty)$ is unimodal function and $f(x) = xf_1(x)$, $pf_1(0) \leq \delta$, where $f_1 : [0, +\infty) \to (0, +\infty)$ is a continuous map.

Thus $f(x)$ can have at most one critical point, it will be denoted by x^*. We shall write also $x^* = 0$ if H1 is fulfilled. We suppose hat τ is fixed positive real number characterizing delay in the system.

Taking into account the unimodal character of $f(x)$ we can obtain at most two constant solutions $x_1(t) \equiv 0$ and $x_2(t) \equiv x_2 > 0$ to (1). Below we will study the global asymptotic stability property of $x_1(t)$ and, mainly, of $x_2(t)$.

It should be noted that this problem has been investigated in the papers [4], [7], [8], [9], [11], [13], [16], [17], [22]. The delay-difference equations which are discrete analogues of the (1) were investigated in [2], [3], [6], [10], [17].

Let us denote by $\psi : (0, \varphi(x^*)] \to [x^*, +\infty)$ the inverse of $\varphi(x)$ on $[x^*, +\infty)$, and set $\theta(x) = x - \exp(-\delta\tau)\psi(x)$. We will prove that, under one of the conditions H1 or H3, the map

$$\xi(x) = \theta^{-1}((1 - \exp(-\delta\tau))\varphi(x)) : [\mathbf{a}, \mathbf{b}] \to [\mathbf{a}, \mathbf{b}],$$

is well defined and unimodal for some $\mathbf{a}, \mathbf{b} > 0 : \mathbf{a} \leq x_2 \leq \mathbf{b}$, and $\xi(x_2) = x_2$. The main result of this paper asserts that, if x_2 is a globally attracting and locally stable point of ξ, then the equilibrium x_2 of the equation (1) is globally attracting.

Moreover, if $f(x)$ is three times continuously differentiable, and satisfies H1 or H3, and

1. $2f'(x)f'''(x) < 3(f''(x))^2$ for all $x > 0$, $x \neq x^*$ with

2. $0 > (1-\exp(-\delta\tau))\frac{p}{\delta}f'(x_2) > -\frac{1}{2}\left(1+\sqrt{1+4\exp(-\delta\tau)(1-\exp(-\delta\tau))}\right)$,

then the equilibrium x_2 of the (1) is globally asymptotically stable.

The paper is organized as follows. The subsection 1.2 contains some preliminary facts, notations and definitions. In the next section we prove some persistence results necessary for the third section, where the main theorem is proved and some applications are given.

1.2. Some preliminary facts, notations, definitions and results. In the paper we shall use the following notations and definitions:

$R_+ = [0, +\infty)$, $C_+ = C([-\tau, 0], R_+)$, $Z_+ = \{0, 1, 2, ...\}$.

If $G, F : X \to X$, then $F^j(x) = \underbrace{F \circ \ldots \circ F}_{j \text{ times}}(x)$, where $F \circ G(x) = F(G(x))$;

Let X be metric space, $B, A \subset X, \varepsilon > 0$. The $\varepsilon-$neighborhood of A is denoted by $U_\varepsilon(A)$ and $F^j(A) \to B$ as $j \to \infty$ means that $\forall\ \varepsilon > 0\ \exists\ j_0 : F^j(A) \subset U_\varepsilon(B)\ \forall j \geq j_0$.

Definition 1. *Let $T = R_+$ or Z_+. The semidynamical system $\Phi^t x : T \times X \to X$ has a global attractor A if $\Phi^t x \to A$ as $t \to +\infty$ for every $x \in X$. Equation (1) is said to be globally attracting if there exists $a \geq 0$ such that $x(t) \to a$, $t \to +\infty$ for each nonzero solution $x(t)$, $t \geq 0$ of (1).*

We note that if equation (1) is globally attracting, then either $a = x_1 = 0$ or $a = x_2 > 0$.

We shall relate the global attractivity of delay differential equation (1) to the global attractivity properties of some one-dimensional maps. To do this we will use the following simple propositions.

Example 2. *Let $T = Z_+$, $g : [a, b] \to [a, b]$ is continuous and $g^j(x)$ have a globally attracting set $[\alpha, \beta] \subseteq [a, b]$. Then*

$$\alpha \leq \liminf_{j\to\infty} g^j(x) \leq \limsup_{j\to\infty} g^j(x) \leq \beta, \text{ for every } x \in [a, b].$$

Lemma 3. *Let X be a compact metric space, K be a nonempty, closed, subset of X and $T : X \to X$ be a continuous map such that $T^j(x) \to K$, $j \to \infty$ for any $x \in X$ (in other words, invariant set K is a global attractor for this system). Suppose that the set K is stable. Then $T^j(X) \to K$ as $j \to \infty$.*

Proof. Since K is stable, for every $\varepsilon-$neighborhood $U_\varepsilon(K)$ of K we can find a $\delta-$neigbourhood $U_\delta(K)$, such that $T^j(z) \in U_\varepsilon(K)$ for every $j \geq 0, z \in U_\delta(K)$. Because of the global attractivity of K, for every $w \in X$ we can find a minimal number $j(w)$ such that $T^{j(w)}(w) \in U_\delta(K)$. It is clear, that there exist a neighborhood $U_{\mu(w)}(w)$ with property $T^{j(w)}(U_{\mu(w)}(w)) \in U_\delta(K)$. The family of open sets $\{(U_{\mu(w)}(w), w \in X\}$ covers the compact set X and therefore there exists a finite subcover $\{(U_{\mu(k)}(w_k), k = 1, \ldots, r\}$. Let $s = \max\limits_{k=1,\ldots,r} j(w_k)$. Obviously, $T^j(X) \in U_\varepsilon(K)$ for every $j \geq s$. ■

Remark 1. *The stability assumption for K is essential in the formulation of this Lemma.*

Corollary 4. *Let the map $g : [a,b] \to [a,b]$ have a globally attracting stable fixed point x_1. Then $g^j[a,b] \to x_1$ as $j \to \infty$.*

In our work we shall use some facts from the theory of one-dimensional maps with negative Schwarzian derivative.

Definition 5. *Let f be a real function having at least three continuous derivatives. The Schwarzian derivative of f at point x, denoted by $(Sf)(x)$, is given by*

$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

The following formula can be checked by direct computation:

$$S(f \circ g)(x) = (Sf)(g(x))(g'(x))^2 + (Sg)(x). \tag{2}$$

Using the indentities $f \circ f^{-1}(x) = x$, $(Sx)(x) \equiv 0$ and (2), we obtain

Lemma 6. *1. If $(Sf)(x) > 0$ (resp., < 0) for all x, then $(Sf^{-1})(x) < 0$ (resp., > 0) for all x;*

2. If $(Sf)(x) < 0$, $(Sg)(x) \leq 0$, then $S(g \circ f)(x) < 0$.

The next theorem, which can be easily deduced from the Singer's results ([15], see also [12]), will play a crucial role in the sequel.

Proposition 7. *Let the function $g : [a,b] \to [a,b]$, $g(x) \in C^3[a,b]$, have at most one critical point in $[a,b]$. If a unique fixed point $x_2 \in [a,b]$ of g is locally asymptotically stable and the Schwarzian derivative $(Sg)(x) < 0$, then x_2 is a global attractor of g.*

2. Persistence.

First we introduce the concept of uniform persistence. In recent years this concept has attracted a lot of interest and has played an important role in mathematical ecology (see, for example, [5]). Since we intend to study only those solutions $x(t)$ of (1), which are biologically meaningful, in the sequel, we will assume that $x(t) \geq 0$ for all t.

Definition 8. *System (1) is said to be uniformly persistent if there exists a positive number α such that*

$$\lim_{t \to +\infty} \inf x(t) \geq \alpha$$

for every trajectory with positive initial values.

For the equation (1) we indicate below the conditions of uniform persistence and the method to obtain the lower survival bounds α, while in the next section some global attractivity results for this equation will be established.

Let us consider the map $\varphi : [0, +\infty) \to [0, +\infty)$ when either the condition H3 or H1 holds. In both these cases there is an invariant attracting set $[\mathbf{a}, \mathbf{b}]$ with $0 < \mathbf{a} \leq \mathbf{b} < +\infty$ for the map $\varphi(x) : (0, +\infty) \to (0, +\infty)$.

Particularly, under condition H3, it is sufficient to consider the following particular cases F1-F3 depending on the mutual disposition of $\varphi(x^*)$, $\varphi^2(x^*)$, x^*, x_2:

F1. $\varphi : [\mathbf{a}, \mathbf{b}] \to [\mathbf{a}, \mathbf{b}]$ with $\mathbf{b} = \varphi(x^*)$ and $0 < \mathbf{a} = \varphi^2(x^*) \leq x^* \leq x_2 \leq \mathbf{b}$;

Fig. F1

F2. $\varphi : [\mathbf{a}, \mathbf{b}] \to [\mathbf{a}, \mathbf{b}]$ with $x^* \leq \mathbf{a} \leq x_2 \leq \mathbf{b} \leq \varphi(x^*)$ and $\varphi(\mathbf{a}) = \mathbf{b},\ \varphi(\mathbf{b}) = \mathbf{a}$:

Fig. F2

F3. $\varphi : [\mathbf{a}, \mathbf{b}] \to [\mathbf{a}, \mathbf{b}]$ with $0 < \mathbf{a} \leq x_2 \leq x^* = \mathbf{b}$ and $\varphi(x)$ increases on $[\mathbf{a}, \mathbf{b}]$:

Fig. F3

It should be noted that case F3 is trivial (see proposition 9 below) and, mainly, we shall consider the cases F1 and F2. Moreover, the diagram F2 is appropriate to describe φ when assumption H1 is fulfilled (supposing $x^* = 0$). The assertion below follows immediately from the graphical analysis of diagrams F1-F3.

Proposition 9. *Let us consider the map $\varphi : [\mathbf{a}, \mathbf{b}] \to [\mathbf{a}, \mathbf{b}] \ni x_2$, where x_2 is the unique fixed point of $\varphi(x)$ in $[\mathbf{a}, \mathbf{b}]$. Then the following statements hold:*

1. Assume $x^* < x_2$. If $m = \inf\limits_{n \geq 0} \varphi^n(z),\ M = \sup\limits_{n \geq 0} \varphi^n(z)$ for some z and $x_2 \leq m \leq M$ or $x_2 \geq M \geq m$, then $x_2 = m = M$.
2. If $x_2 \leq x^*$, then x_2 is global attractor.

Corollary 10. *If $\varphi(x) \in C^1[\mathbf{a}, \mathbf{b}]$ and $\varphi'(x_2) \geq 0$, then x_2 is a global attractor.*

Proof. Under our assumptions, $\varphi'(x_2) \geq 0$ implies the estimate $\varphi'(x_2) < 1$. ■

Now we will analyze (1) under the condition H2 in more detail.

Evidently, the solution $x(t, \gamma)$ of the problem (1) exists for $t \geq 0$ and satisfies the integral equation

$$x(t, \gamma) = \exp(-\delta(t-s))x(s, \gamma) + p\int_s^t \exp(-\delta(t-u))f(x(u-\tau, \gamma))du$$

for all $t \geq s \geq 0$. Since $f(x) \geq 0$ is bounded, $\delta > 0$, it follows that $x(t, \gamma) \geq 0$ is bounded. Moreover, if $\gamma(s) \not\equiv 0$, then $x(t) > 0$ for $t \geq \tau$. Thus (1) generates semidynamical system $\Phi^s\gamma : R_+ \times C_+ \to C_+$ by the relation: $(\Phi^s\gamma)(u) = x(s+u, \gamma),\ u \in [-\tau, 0]$. We indicate that all positive trajectories of this system are bounded and, consequently, precompact. Thus the $\omega-$ limit set $\omega(\gamma) \in C_+$ is compact, invariant and $\Phi^s\gamma \to \omega(\gamma)$ as $s \to +\infty$ (see lemmas 4.1.3 and 4.1.4 in [14]). We leave for the reader to verify that the following proposition holds:

Lemma 11. *Let $z(t)$, $t \in R$ be a trajectory of (1). Then $z(t)$ satisfies the equation*

$$z(t) = p\int_{-\infty}^t \exp(-\delta(t-u))f(z(u-\tau))du. \tag{3}$$

Corollary 12. *Let the assumption (H) be fulfilled. Then*

1. *If $z(t)$, $t \in R$ is a nonnegative solution of (1) and $z(s) = 0$ for some $s \in R$, then $z(t) \equiv 0$ for all $t \in R$.*
2. *Let H4 hold and $f_1(x) < \max\limits_{x \geq 0} f_1(x) = f_1(0) = M < \infty$ for all $x > 0$. Then $\omega(\gamma) = 0$ for every γ (i.e. the steady state $x_1 \equiv 0$ is a global attractor of the semidynamical system Φ^s independently on τ).*
3. *If H3 holds, then the system (1) is uniformly persistent: namely, if $\gamma \not\equiv 0$, then $0 \not\subset \omega(\gamma)$ and there exists $m > 0$, which is independent on τ, such that*

$$\liminf_{t \to +\infty} x(t) > m \ \forall\ x(t) \not\equiv 0$$

Proof. Let us prove the last statement, the two others being left to the reader. By contrary, suppose that $x(t) \not\equiv 0$, $t \geq 0$ is a solution of (1) such that $\liminf\limits_{t \to +\infty}$

$x(t) = 0$ and $M_1 = \sup_{t>\tau} x(t) > 0$. It is clear that we can find $s, d > 0$ such that $m_1 = \min_{s\le t\le s+d} x(t) > 0$ satisfies the relations

$$f_1(m_1)p(1-\exp(-\delta d)) > \delta;\ \min_{t\in[m_1,M_1]}(tf_1(t)) = m_1 f_1(m_1).$$

Then for $t \ge s+d+\tau$ we obtain $\quad x(t) = x(s+\tau)\exp(-\delta(t-s-\tau))+$

$$+p\int_{s+\tau}^{t}\exp(-\delta(t-u))f(x(u-\tau))du \ge \exp(-\delta t)p\int_{s+\tau}^{s+\tau+d}\exp(\delta u)x(u-\tau)f_1(x(u-\tau))du \ge$$

$$\ge \exp(-\delta t)pm_1f_1(m_1).\int_{s+\tau}^{s+\tau+d}\exp(\delta u)du \ge \frac{1}{\delta}m_1f_1(m_1)p(1-\exp(-\delta d)) > m_1,$$

contradicting to $\liminf_{t\to+\infty} x(t) = 0$. Finally, we note that m_1 can be choosen independtly on the solution $x(t)$. ■

It is clear also that assumption H1 implies the uniform persistence of (1), and from now on we suppose that either H1 or H3 is fulfilled.

Now we can state how the lower survival bound α can be obtained:

Lemma 13. *Let $K = [\alpha_1, \beta_1]$ be the minimal invariant globally attracting set of the map $\varphi(x) : [\mathbf{a}, \mathbf{b}] \to [\mathbf{a}, \mathbf{b}]$. Then in Definition 8 we can choose $\alpha = \alpha_1$.*

Proof. Let $x(t)$ be a nonzero solution of the uniformly persistent system (1). Then $\omega(\gamma)$ is a closed invariant and compact set. We denote by $m = \min_{\beta\in\omega(\gamma)} \inf_{t\in R} x(t,\beta)$ and by $M = \max_{\beta\in\omega(\gamma)} \sup_{t\in R} x(t,\beta)$, where $x(t,\beta)$ is a solution of the initial value problem (1) with initial function $\beta \in C_+$.Since $\omega(\gamma)$ is a compact set, there exist a solution $y(t)$ and a real s such that $y(s) = M$. Then $y'(s) = 0$ and therefore $-\delta y(s)+pf(y(s-\tau)) = 0$. As a consequence, we obtain

$$M = y(s) = \frac{p}{\delta}f(y(s-\tau)) = \varphi(y(s-\tau)) \quad \textit{and} \tag{4}$$

$$M \le \max_{y\in[m,M]}\varphi(y). \tag{5}$$

Analogously, there exist a solution $z(t)$ and a real q such that $z(q) = m$. Then $z'(q) = 0$ and therefore $-\delta z(q) + pf(z(q-\tau)) = 0$. Hence

$$m = z(q) = \frac{p}{\delta} f(z(q-\tau)) = \varphi(z(q-\tau)) \text{ and} \tag{6}$$

$$m \geq \min_{x \in [m,M]} \varphi(x). \tag{7}$$

From (21), (7) we conclude that

$$[m, M] \subset \varphi([m, M]) \subset \varphi^2([m, M]) \subset ... \subset \varphi^j([m, M]) \subset ... \tag{8}$$

Now, a direct application of Lemma 3 completes the proof of Lemma 13. ■

3. Global Attractivity.

3.1. Main results. The following theorem is the main result of this paper.

Theorem 14. *Assume that one of the conditions H1 or H3 is fulfilled. Let* $\psi : [\mathbf{a}, \mathbf{b}] \to [x^*, \mathbf{b}]$ *be the inverse of* $\varphi(x)$ *on* $[x^*, \mathbf{b}]$, *and set* $\theta(x) = x - \exp(-\delta\tau)\psi(x)$. *If the fixed point* $x_2 > 0$ *is a globally attracting and locally stable point of the unimodal map*

$$\xi(x) = \theta^{-1}((1 - \exp(-\delta\tau))\varphi(x)) : [\mathbf{a}, \mathbf{b}] \to [\mathbf{a}, \mathbf{b}],$$

then the steady state x_2 *of the equation (1) is globally attracting.*

Corollary 15. *Let the map* $\xi(x)$ *has a negative Schwarzian derivative* $S\xi$ *on* R_+. *If* x_2 *is a locally asymptotically stable (LAS) fixed point of* $\xi(x)$, *then the equilibrium* x_2 *of (1) is globally attracting.*

Corollary 16. *Assume that one of the conditions H1 or H3 is fulfilled. If*

1. $(S\varphi)(x) = \frac{\varphi'''(x)}{\varphi'(x)} - \frac{3}{2}\left(\frac{\varphi''(x)}{\varphi'(x)}\right)^2 < 0$, $x \geq 0$
2. $0 > (1 - \exp(-\delta\tau))\varphi'(x_2) > -\frac{1}{2}\left(1 + \sqrt{1 + 4\exp(-\delta\tau)(1 - \exp(-\delta\tau))}\right)$,

then the equilibrium point x_2 of (1) is globally asymptotically stable.

The following propositions are weaker than the above stated results, but their assumptions can easily be verified.

Corollary 17. *Let* x_2 *be a unique globally attracting and locally stable point of the map* $\zeta(x) = \exp(-\delta\tau)x_2 + (1 - \exp(-\delta\tau))\varphi(x) : [0, +\infty) \to (0, +\infty)$. *Then the steady state* x_2 *of (1) is globally stable.*

Corollary 18. *Let the map $\varphi(x)$ have the negative Schwarzian derivative $S\varphi$. If x_2 is a locally asymptotically stable fixed point of $\zeta(x)$, then the equilibrium x_2 of (1) is globally attracting.*

Remark. It should be noted that in these four assertions the conditions of global stability depend on the delay τ. We mention also that the condition $0 \geq (1 - \exp(-\delta\tau))\varphi'(x_2) > -1$ is sufficient for the point x_2 to be LAS.

Corollary 19. *[4]. Let x_2 be a unique globally attracting and locally stable point of the unimodal map $\varphi(x) : (0, +\infty) \to (0, +\infty)$. Then the steady state x_2 of the (1) is globally attracting.*

Corollary 20. *Assume that Schwarzian derivative $S\varphi$ of the map $\varphi(x)$ is negative. If x_2 is a locally asymptotically stable fixed point, then the steady state x_2 of (1) is globally attracting.*

Remark. It should be noted that the global stability proved in two previous Corollaries don't depend on the delay τ. Such behaviour has been called absolute stability in [23]. We mention that the condition $|\varphi'(x_2)| < 1$ is sufficient for the point x_2 to be LAS.

Proof of Theorem 16. Under the assumptions of the theorem, the map $\varphi : (0, +\infty) \to (0, +\infty)$ has a globally attracting set $[\mathbf{a}, \mathbf{b}] \ni x_2$ with $\mathbf{b} = \max \varphi(x) = \varphi(x^\bullet)$ and $0 < \mathbf{a} \leq \min\{\varphi^2(x^\bullet),\ x^\bullet\}$. Hence $[\mathbf{a}, \mathbf{b}] \to [\mathbf{a}, \mathbf{b}]$, and $\varphi(x_2) = x_2$. Let $x(t)$, $\gamma = x(s) \in C_+$ be a nonzero trajectory of uniformly persistent system (1). We know that according to our main assumption $\omega(\gamma)$ is a closed invariant and compact set. Thus, if we denote $m = m(\beta) = \min\limits_{\beta \in \omega(\gamma)} \inf\limits_{t \in R} x(t, \beta)$ and by $M = M(\beta) = \max\limits_{\beta \in \omega(\gamma)} \sup\limits_{t \in R} x(t, \beta)$, we have from Lemma 13 that $m, M \in [\mathbf{a}, \mathbf{b}]$. If either $x_2 \leq x^*$, or $x_2 \leq m \leq M$, or $m \leq M \leq x_2$ is satisfied, than the global asymptotic stability of the equation (1) follows from Proposition 9 and relation (8). So, we will investigate the case $x_2 < x^\bullet$ and $m \leq x_2 \leq M$ (therefore, $\varphi'(x_2) < 0$).

We now establish the conditions guarantying the equality $m(\beta) = M(\beta)$ for each $\beta \in \omega(\gamma)$. It is clear that these conditions will be sufficient for the global asymptotic stability of the steady state solution $x(t) \equiv x_2$.

We shall consider only the case F1 (with $x^* = x^\bullet$) from previous subsection, the analogous case F2 is being left to the reader.

Let $\varphi_1^{-1} : [\varphi^3(x^*), \mathbf{b}] \to [\mathbf{a}, x^*]$ be the inverse of $\varphi(x)$ on $[\mathbf{a}, x^*]$ and $\psi : [\mathbf{a}, \mathbf{b}] \to [x^*, \mathbf{b}] \ni x_2$ be the inverse of $\varphi(x)$ on $[x^*, \mathbf{b}]$.

Since $\omega(\gamma)$ is a compact invariant set, there exist a trajectory $y(t)$ and a real s such that $y(s) = M$. Since $\varphi(x) = \frac{p}{\delta} f(x)$ is a unimodal map and $x_2 \leq M \leq \mathbf{b} = \varphi(x^*)$, we obtain from (4) that

$$y(s-\tau) = \psi(M) \subset [\kappa, x_2], \tag{9}$$

where $\kappa = \min\{\varphi_1^{-1}(M), \mathbf{a}\}$ if $\varphi_1^{-1}(M) \neq \emptyset$ and $\kappa = \mathbf{a}$ in the opposite case.

Since

$$M = y(s) = \exp(-\delta\tau) y(s-\tau) + p \int_{s-\tau}^{s} \exp(-\delta(s-u)) f(y(u-\tau)) du,$$

we have that

$$M \leq \exp(-\delta\tau)\psi(M) + (1 - \exp(-\delta\tau)) \max_{x\in[m,M]} \varphi(x),$$

$$\text{and, by (9), } M \leq \exp(-\delta\tau) x_2 + (1 - \exp(-\delta\tau)) \max_{x\in[m,M]} \varphi(x). \tag{10}$$

Finally, if we set $\theta(x) = x - \exp(-\delta\tau)\psi(x)$, we get

$$\theta(M) \leq (1 - \exp(-\delta\tau)) \max_{x\in[m,M]} \varphi(x). \tag{11}$$

Analogously, there exist a trajectory $z(t)$ and a real q such that $z(q) = m$. Then $m \in [\mathbf{a}, x_2]$ and, since $z(q-\tau) \geq m > \varphi_1^{-1}(m)$, we have from (6) that

$$z(q-\tau) = \psi(m).$$

Since

$$m = z(q) = \exp(-\delta\tau) z(q-\tau) + p \int_{q-\tau}^{q} \exp(-\delta(s-u)) f(z(u-\tau)) du,$$

we have:

$$m \geq \exp(-\delta\tau) x_2 + (1 - \exp(-\delta\tau)) \min_{x\in[m,M]} \varphi(x), \tag{12}$$

$$m \geq \exp(-\delta\tau)\psi(m) + (1 - \exp(-\delta\tau)) \min_{x\in[m,M]} \varphi(x). \tag{13}$$

Finally, (13) can be rewritten as

$$\theta(m) \geq (1 - \exp(-\delta\tau)) \min_{x\in[m,M]} \varphi(x). \tag{14}$$

It should be noted that

$$\begin{aligned}\theta(x) &= x - \exp(-\delta\tau)\psi(x) : [\mathbf{a},\mathbf{b}] \to [\mathbf{a} - \exp(-\delta\tau)\mathbf{b}, \mathbf{b} - \exp(-\delta\tau)x^*] \supseteq \\ &\supseteq [(1-\exp(-\delta\tau))\mathbf{a}, (1-\exp(-\delta\tau))\mathbf{b}] = (1-\exp(-\delta\tau))\varphi([\mathbf{a},\mathbf{b}])\end{aligned}$$

is an increasing function, since $\psi(x)$ is decreasing. Therefore,

$$\theta^{-1}((1-\exp(-\delta\tau))\varphi(x)) : [\mathbf{a},\mathbf{b}] \to [\mathbf{a},\mathbf{b}]$$

is well defined unimodal map with a unique positive fixed point x_2.

Finally, from the estimates (11), (14) we obtain that

$$[m,M] \subset \xi([m,M]) \subset \xi^2([m,M]) \subset ... \subset \xi^j([m,M]) \subset ...$$

The statement of Theorem 14 is now an immediate consequence of this relations and Lemma 3. ■

Now, we note that the proofs of Corollaries 19 and 20 immediately follow from the relation (8), Lemma 3 and Proposition 7. Next, $S\zeta = S\varphi$ and from the estimates (10), (12) we obtain that

$$[m,M] \subset \zeta([m,M]) \subset \zeta^2([m,M]) \subset ... \subset \zeta^j([m,M]) \subset ..,.$$

and proof of Corollary 18 is now an immediate consequence of this relations, Lemma 3 and Proposition 7.

Together with Proposition 7, Theorem 14 implies Corollary 15 and we have to prove only Corollary 16.

Proof of Corollary 16: the sufficient condition of the local exponential attractivity of x_2 in the form $|\xi'(x_2)| < 1$, can be checked easily, since

$$\xi'(x_2) = \tfrac{d}{dx}\left[\theta^{-1}\left((1-\exp(-\delta\tau))\varphi(x_2)\right)\right] = \tfrac{d}{dx}\left[\theta^{-1}(y)\right]_{y=\theta(x_2)}(1-\exp(-\delta\tau))\tfrac{d}{dx}\varphi(x_2)$$

$$= (\theta'(x_2))^{-1}(1-\exp(-\delta\tau))\varphi'(x_2) = \frac{(1-\exp(-\delta\tau))\varphi'(x_2)}{1-\exp(-\delta\tau)(\varphi'(x_2))^{-1}}$$

$$= \frac{(1-\exp(-\delta\tau))(\varphi'(x_2))^2}{\varphi'(x_2) - \exp(-\delta\tau)}.$$

Since $\varphi'(x_2) < 0$, the condition $|\xi'(x_2)| < 1$ has the following form:

$$\frac{(1-\exp(-\delta\tau))(\varphi'(x_2))^2}{\varphi'(x_2)-\exp(-\delta\tau)} > -1, \tag{15}$$

and solving (15), the second assertion of Corollary 16 is proved.

Now we have to evaluate the sign of $(S\xi)(x)$. We note, however, that the direct computations of the sign of $(S\xi)(x)$ is rather difficult. It is true, if $(S\theta)(x) > 0$ and $(S\varphi)(x) < 0$ because of Lemma 6. This is a rather surprising fact that $(S\xi)(x) < 0$ if $(S\varphi)(x) < 0$. Indeed, let us return to the condition $(S\theta)(x) > 0$. Since $(S\varphi)(x) < 0$ we have, by Lemma 6,

$$\frac{2\psi'''\psi' - 3(\psi'')^2}{2(\psi')^2} = (S\psi)(x) = (S\varphi^{-1})(x) > 0 \implies 2\psi'''\psi' - 3(\psi'')^2 > 0.$$

Since $\theta(x) = x - \exp(-\delta\tau)\psi(x)$, we obtain $\theta'(x) = 1 - \exp(-\delta\tau)\psi'(x)$, $\theta''(x) = -\exp(-\delta\tau)\psi''(x)$, $\theta'''(x) = -\exp(-\delta\tau)\psi'''(x)$ and

$$(S\theta)(x) = \frac{2\theta'''\theta' - 3(\theta'')^2}{2(\theta')^2} = \frac{\exp(-\delta\tau)}{2(\theta')^2}\left[\left\{2\psi'''\psi' - 3(\psi'')^2\right\}\exp(-\delta\tau) - 2\psi'''\right].$$

Hence $(S\theta)(x) > 0$, if $(S\varphi)(x) < 0$ and $\psi''' \le 0$. Let us now compute ψ''', we have:

$$\begin{aligned}\psi'(\varphi(x)) &= \frac{1}{\varphi'(x)};\ \psi''(\varphi(x)) = -\frac{\varphi''(x)}{[\varphi'(x)]^3};\\ \psi'''(\varphi(x)) &= \frac{3(\varphi''(x))^2 - \varphi'''(x)\varphi'(x)}{[\varphi'(x)]^5} < 0,\end{aligned}$$

since $\varphi'(x) < 0$ for $x > x^*$ and $3(\varphi''(x))^2 - \varphi'''(x)\varphi'(x) > 0$ because of

$$\frac{2\varphi'''\varphi' - 3(\varphi'')^2}{2(\varphi')^2} = (S\varphi)(x) < 0 \quad \blacksquare.$$

3.2. Application to Nicholson's blowflies model. Let us consider the Nicholson's blowflies delay-differential equation [1], [7], [8], [9], [11], [16], [17].

$$x'(t) = -\delta x(t) + px(t-\tau)\exp(-ax(t-\tau)),\ a > 0, \tag{16}$$

Here $x(t)$ is the size of the population at time t, p is the maximum per capita daily egg production rate, $\frac{1}{a}$ is the size at which the population reproduces at its maximum rate, δ is the per capita daily adult death and τ is the generation time.

In this case $f(x) = px\exp(-ax)$, $\varphi(x) = \frac{p}{\delta}xe^{-ax}$. If $p \le \delta$, we have only one constant solution $x_0 \equiv 0$. According to Corollary 12.1 it is global attractor independently

$\lvert\zeta'(x_2)\rvert < 1$	$(\exp(\delta\tau) - 1)\ln\frac{p}{\delta} < 2\exp(\delta\tau) - 1$	E_1
$\lvert\varphi'(x_2)\rvert < 1$	$\ln\frac{p}{\delta} < 2$	E_2
[11]	$(\exp(\delta\tau) - 1)\ln\frac{p}{\delta} \le 1$	E_3
[8]	$(\exp(\delta\tau) - 1)(\frac{p}{\delta} - 1) < 1$	E_4
[7]	$(\exp(\delta\tau) - 1)(\frac{p}{\delta} - 1) \le 1$	E_5
[16]	$\ln\frac{p}{\delta} \le 2$ or $\delta\tau e < p\tau < \exp(1 - \delta\tau)$	E_6

on τ. Let now $p > \delta$. In this case $x_0 \equiv 0$ is an unstable equilibrium (Corollary 12.2) and the second constant solution $x_2 = \frac{1}{a}\ln\frac{p}{\delta}$ appears. Since $\varphi(x) = \frac{p}{\delta}xe^{-ax}$ (and therefore $\zeta(x)$) has negative Schwarzian derivative $S\varphi = S\zeta$ (see [15]), we can apply Corollary 18. The conditions for the local exponential stability of the rest point x_2 of the maps $\varphi(x)$ and $\zeta(x)$ have the forms $|\varphi'(x_2)| < 1$ and $|\zeta'(x_2)| < 1$, respectively. Since $\varphi'(x_2) = 1 - \ln\frac{p}{\delta}$ and $\zeta'(x_2) = -(1 - \exp(-\delta\tau))\ln\frac{p}{\delta e}$, we obtain the new conditions for the global stability placed in Table (together with some previously known).

It is evident that estimate E_1 implies the others. It is interesting to note that the condition E_1 implies that, in the case $\ln\frac{p}{\delta} = 2$, equation (16) is globally asymptotically stable for all τ (this result seems to be new). We note that this result can be also deduced from Corollary 19, since in this case the fixed point $x_2 = \frac{2}{a}$ is still locally asymptotically (but not exponentially) stable of the map $\varphi(x) = \exp(2)x\exp(-ax)$

Applying Corollary 16 to equation (16), we get the following (best) condition for the global attractivity in the Nicholson's blowflies delay-differential equation:

$$(1-\alpha)\ln\frac{p}{\delta} < 1 - \alpha + \frac{1}{2}\left(1 + \sqrt{1 + 4\alpha(1-\alpha)}\right),\ \alpha = \exp(-\delta\tau). \tag{17}$$

The criterium of local asymptotic stability of the equilibrium x_2 of (16) can be obtained by direct computation of characteristic equation, which has the form

$$\lambda + \delta = (\delta\ln\frac{\delta e}{p})\exp(-\tau\lambda). \tag{18}$$

Obviously, x_2 will be locally exponentially stable, if all roots of this equation lie inside the half-plane $\{\mathrm{Re}\, z < 0\}$. Necessary and sufficient conditions for every root of (18) to have negative real part can be found in [16]: positive constant solution of (16) is exponentially locally stable if parameters $(\delta,\ p) \in R^2_+$ belong to the domain below the curve E_∞ given by

$$\delta = -t\cot(t),\ p = -t\cot(t)\exp(1 - \sec(t)),\ t \in (\pi/2, \pi).$$

It is interesting to study the following

Conjecture 21. *([16], p. 116). The local exponential stability of the positive equilibrium of (16) is sufficient for its global attractivity.*

Fig. 4 below shows the curves E_∞, $E_3 : (\exp(\delta\tau) - 1)\ln\frac{p}{\delta} = 1$; $E_2 : \ln\frac{p}{\delta} = 2$; $E_4 : (1 - \exp(-\delta\tau))\ln\frac{p}{\delta} = 1 - \exp(-\delta\tau) + \frac{1}{2}\left(1 + \sqrt{1 + 4\exp(-\delta\tau)(1 - \exp(-\delta\tau))}\right)$, when $\tau = 1$.

Fig. 4.

3.3. Application to a model of an erythropoietic system. Let us consider the delay differential equation

$$x'(t) = -\delta x(t) + p\exp(-ax(t-\tau)),\ p,\ \delta,\ a > 0. \tag{19}$$

This equation was presented in 1976 by Wazewska-Czyzewska and Lasota [18] to describe the erythropoietic (red-blood cell) system. Here $x(t)$ is the number of red blood cells at the time t, δ is the rate of death of red blood cells, $a > 0$ and $p > 0$ describe the generation of red blood cells per unit time, and τ is the time needed to produce a red blood cell. In [7], [18], [22] the following sufficient condition for the global attractivity of positive constant solution was established:

$$\frac{pa}{\delta} \le e, \tag{20}$$

and in [22] we can find the estimate

$$\tau pa\exp(\delta\tau) < \frac{1}{e}. \tag{21}$$

In [9] the authors proved *de facto* the following condition of global asymptotic stability of the positive solution:

$$\frac{pa(1-\alpha)}{\delta} < \ln 2\exp\left(\frac{\ln 2}{1-\alpha}\right), \tag{22}$$

where $\alpha = \exp(-\delta\tau)$.

Let us apply our results to (19). In this system we have only one equilibrium $x_2(t) \equiv x_2 > 0$ satisfying the transcendental equation $x_2 = \frac{p}{\delta}\exp(-ax_2)$. To show the dependence $x_2$on $\frac{p}{\delta}$, we shall write $x_2 = x_2(\frac{p}{\delta})$. Obviously, $x_2(z)$ is a function increasing in $z > 0$. Since the function $f(x) = \exp(-ax)$ has a negative Schwarzian derivative $(Sf)(x) = -\frac{a^2}{2} < 0$, we can apply Corollary 18. Therefore, if $(1-\exp(-\delta\tau))\,|\varphi'(x_2)| < 1$ then the equation (19) has the property of global attractivity. Since $\varphi'(x_2 = -ax_2$ and $x_2(z)$ is an increasing function, the last inequality will be fulfilled for all $\frac{p}{\delta} < z_0$ with $(1-\exp(-\delta\tau))ax_2(z_0) = 1$. Consequently,

$$z_0 = \frac{p_0}{\delta_0} = x_2(z_0)\exp(ax_2(z_0)) = (1-\exp(-\delta\tau))^{-1}a^{-1}\exp(1-\exp(-\delta\tau))^{-1}$$

and the global attractivity condition for the equation (19) has the form:

$$\frac{pa(1-\exp(-\delta\tau))}{\delta} < \exp(1-\exp(-\delta\tau))^{-1}. \tag{23}$$

Let us apply now the Corollary 16 to investigate (19). Proceeding analogously to the first part of this example, we obtain the following condition (we note that $\varphi'(x) < 0$ here)

$$\frac{pa(1-\alpha)}{\delta} < \frac{\left(1+\sqrt{1+4\alpha(1-\alpha)}\right)}{2}\exp\left(\frac{\left(1+\sqrt{1+4\alpha(1-\alpha)}\right)}{2(1-\alpha)}\right), \tag{24}$$

where $\alpha = \exp(-\delta\tau)$. It is evident that (20) implies (23) and (23) implies (24); and (21)⇒(22)⇒(23)⇒(24) so the estimate (24) is the sharpest here.

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