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# On the Exponential Stability of a State-Dependent Delay Equation

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*This paper is dedicated to Professor Ferenc Móricz on the occasion of his 60th birthday.*

**Abstract**

In this paper we investigate the exponential stability of the trivial solution of the state-dependent delay differential equation $\dot{x}(t) = a(t)x(t - \tau(t, x(t)))$. It is shown that, under some conditions, this state-dependent equation is exponentially stable, if the trivial solution of $\dot{y}(t) = a(t)y(t - \tau(t, 0))$ is exponentially stable. Assuming the existence of bounded partial derivatives of the delay function, the reverse statement will also be proved.

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## 1 Introduction

In this paper we study the asymptotic behavior of the state-dependent delay equation

$$\dot{x}(t) = a(t)x(t - \tau(t, x(t))). \tag{1.1}$$

Similar questions have been studied in [5]–[7], [13]–[22] for various classes of state-dependent equations. [5] and [6] show that the asymptotic behavior of the autonomous version of (1.1) for $a(t) = a < 0$ is equivalent to that of the corresponding ODE $\dot{x}(t) = ax(t)$, i.e., the trivial solution of both equations are exponentially stable. [14] proves that equation (1.1) with $a(t) = a > 0$ and $\tau(t, u) = |u|$ is unstable, but the speed of the convergence of the solution to $\infty$ is not necessary exponential. [22] relates the asymptotic behavior of (1.1) to the ODE $\dot{x}(t) = a(t)x(t)$, assuming $a(t) \le 0$, $0 \le \tau(t, u) \le K|u|$. [7] and [13] compares the asymptotics of some classes of autonomous state-dependent equations with distributed delays to certain constant delay equations (to the so-called "linearized" equations). [12] shows that, under certain conditions, the trivial solution of a time-, but not state-dependent delay equation is exponentially stable if and only if the trivial solution of a certain constant delay equation is exponentially stable.

Motivated by [12], and based on a technique applied for the investigation of the asymptotic stability problem in delay perturbed equations in [10], we investigate the exponential stability of (1.1) through that of the linear equation

$$\dot{y}(t) = a(t)y(t - \tau(t, 0)). \tag{1.2}$$

We will show (see Theorem 2.2 below) that if the trivial solution of the "linearized" equation (1.2) is exponentially stable, then so is the trivial solution of (1.1). For equations with some extra smoothness on the delay we will prove that the statement can be reversed, i.e., (1.1) is exponentially stable if and only if (1.2) is exponentially stable (see Theorem 2.3 below). As a consequence of our theorems, we can give explicit sufficient conditions for the exponential stability of (2.1) (see Corollary 2.4 below), and necessary and sufficient conditions in the cases the equations are autonomous (see Corollary 2.5 below) or the linearized equation (1.2) is an ordinary differential equation (see Corollary 2.6 below).

Finally, we refer the interested reader (without completeness) to [1]–[4], [15]–[19], [23] for some recent applications and general theory of state-dependent differential equations from the recent mathematical literature.

## 2 Main results

Consider the scalar state-dependent delay equation

$$\dot{x}(t) = a(t)x(t - \tau(t, x(t))), \qquad t \geq t_0, \tag{2.1}$$

with initial condition

$$x(t) = \varphi(t), \qquad t \in [t_0 - r, t_0]. \tag{2.2}$$

We assume that $t_0 \geq 0$ and $r > 0$ are fixed, and

(H1) $a\colon [t_0, \infty) \to \mathbb{R}$ is continuous, and $|a(t)| \leq a_0$, $t \in [t_0, \infty)$ for some constant $a_0$;

(H2) the delay function $\tau\colon [t_0, \infty) \times \mathbb{R} \to [0, r]$ is continuous;

(H3) there exist a constant $\gamma > 0$ and a continuous function $\omega : (-\gamma, \gamma) \to [0, \infty)$, such that

$$|\tau(t, u) - \tau(t, 0)| \leq \omega(u), \qquad t \in [t_0, \infty), \quad u \in (-\gamma, \gamma),$$

and $\omega(0) = 0$.

Note that for autonomous equations (H1) and (H3) are automatically satisfied, assuming $\tau\colon \mathbb{R} \to [0, r]$ is continuous.

Throughout this paper we use the notation $\|\varphi\| \equiv \max\{|\varphi(s)|\colon\, t_0 - r \leq s \leq t_0\}$. This notation does not emphasize the dependence of $\|\varphi\|$ on $t_0$, because we can consider $t_0$ to be fixed.

**Lemma 2.1** *Assume (H1)–(H2), and let $\varphi$ be continuous on $[t_0 - r, t_0]$. Then the initial value problem* (2.1)-(2.2) *has a solution, $x$, which is defined for all $t \geq t_0$, and satisfies*

$$|x(t)| \leq e^{a_0(t-t_0)}\|\varphi\| \tag{2.3}$$

*for all $t \geq t_0$.*

**Proof** The existence of solution of (2.1)-(2.2) on an interval $[t_0 - r, T)$ for some $T > t_0$ follows, e.g., from [8] or [9]. Here we prove that (2.3) is satisfied for $t \in [t_0, T)$, which easily yields that the solution can be extended for all $t \geq t_0$. Integrating (2.1) from $t_0$ to $t > t_0$ we get

$$x(t) = \varphi(t_0) + \int_{t_0}^{t} a(s)x(s - \tau(s, x(s)))\, ds,$$

therefore for $t \in [t_0, T)$

$$|x(t)| \leq \|\varphi\| + a_0 \int_{t_0}^{t} \max_{t_0 - r \leq u \leq s} |x(u)|\, ds.$$

The right-hand-side is monotone in $t$ and $|x(t)| \leq \|\varphi\|$ for $t \in [t_0 - r, t_0]$, therefore

$$\max_{t_0 - r \leq u \leq t} |x(u)| \leq \|\varphi\| + a_0 \int_{t_0}^{t} \max_{t_0 - r \leq u \leq s} |x(u)|\, ds, \qquad t \in [t_0, T),$$

which proves the statement, using Gronwall's inequality. □

Note that the uniqueness of the solution of initial value problem (2.1)-(2.2) (which is not necessary to have to discuss stability) does not follow from our assumptions (H1)–(H3). See [13] for a counterexample, and [8], [9] or [13] for conditions implying existence and uniqueness of solutions for more general state-dependent delay equations.

We associate the linear delay equation

$$\dot{y}(t) = a(t)y(t - \tau(t, 0)), \qquad t \geq t_0 \tag{2.4}$$

to (2.1). This equation can be considered as the "linearization" of (2.1), since, as Theorem 2.2 shows, the exponential stability of the trivial solution of (2.4) implies that of equation (2.1).

The trivial solution of the linear equation (2.4) is exponentially stable, if there exist constants $\alpha > 0$ and $K > 0$ such that any solution of (2.4) corresponding to initial time $t_0$ satisfies

$$|x(t)| \leq Ke^{-\alpha(t-t_0)}\|\varphi\|, \qquad t \geq t_0. \tag{2.5}$$

It is known (see, e.g., [11]) that, under our assumptions, the exponential stability of the trivial solution of (2.4) is equivalent to the uniform asymptotic stability of the trivial solution. The trivial solution of the nonlinear equation (2.1) is called exponentially stable, if there exist positive constants $K$, $\alpha$ and $\sigma$, such that (2.5) holds for any solution $x$ of (2.1) corresponding to any initial time $t_0 \geq 0$ and initial function satisfying $\|\varphi\| < \sigma$.

**Theorem 2.2** *Suppose (H1)–(H3). If the trivial solution of* (2.4) *is exponentially stable, then the trivial solution of* (2.1) *is exponentially stable, as well.*

**Proof** We can rewrite (2.1) in the form

$$\dot{x}(t) = a(t)x(t - \tau(t,0)) + f(t),$$

where

$$f(t) \equiv a(t)\Big(x(t - \tau(t, x(t))) - x(t - \tau(t,0))\Big).$$

This equation can be considered as a perturbation of (2.4) by the forcing term $f(t)$, therefore the variation-of-constants formula (see, e.g., [11]) yields

$$x(t) = y(t) + \int_{t_0}^{t} v(t,s) f(s)\, ds, \qquad t \geq t_0, \tag{2.6}$$

where $y$ is the solution of (2.4) associated to the initial condition (2.2), and $v$ is the fundamental solution of (2.4), i.e., the solution of the initial value problem

$$\frac{\partial v}{\partial t}(t,s) = a(t)v(t - \tau(t,0), s), \qquad t \geq s, \tag{2.7}$$

$$v(t,s) = \begin{cases} 1, & t = s, \\ 0, & t < s. \end{cases} \tag{2.8}$$

It is known (see, e.g., [11]) that the assumed exponential stability of (2.4) is equivalent to that there exist constants $K_1, K_2 \geq 1$ and $\alpha > 0$ that the solution $y$ and the fundamental solution $v$ of (2.4) satisfy the exponential estimates

$$|y(t)| \leq K_1 e^{-\alpha(t-t_0)}\|\varphi\|, \quad t \geq t_0, \qquad \text{and} \qquad |v(t,s)| \leq K_2 e^{-\alpha(t-s)}, \qquad t \geq s.$$

Therefore it follows from (2.6) that

$$|x(t)| \leq K_1 e^{-\alpha(t-t_0)}\|\varphi\| + K_2 \int_{t_0}^{t} e^{-\alpha(t-s)}|f(s)|\, ds. \tag{2.9}$$

Let $t_1 \equiv \min\{t - \tau(t, x(t)), t - \tau(t,0)\}$ and $t_2 \equiv \max\{t - \tau(t, x(t)), t - \tau(t,0)\}$. Then $|f(t)| = |a(t)||x(t_2) - x(t_1)|$. We consider three cases: (i) $t_2 \leq 0$, (ii) $t_1 \geq 0$ and (iii) $t_1 < 0 < t_2$. In case (i) we have the estimate $|f(t)| \leq 2a_0\|\varphi\|$. In case (ii) equation (2.1) implies that

$$|f(t)| = |a(t)| \left| \int_{t_1}^{t_2} x(s - \tau(s, x(s)))\, ds \right| \leq a_0 |\tau(t, x(t)) - \tau(t,0)| \Big( \|\varphi\| + \max_{t_0 \leq u \leq t} |x(u)| \Big).$$

In case of (iii) we have by cases (i) and (ii) that

$$|f(t)| \leq a_0|x(t_2) - x(0)| + a_0|x(0) - x(t_1)| \leq a_0 t_2 \Big( \|\varphi\| + \max_{t_0 \leq u \leq t} |x(u)| \Big) + 2a_0\|\varphi\|.$$

Therefore, using (H3), for all $t \geq 0$

$$|f(t)| \leq 2a_0\|\varphi\| + a_0\omega(x(t))\Big(\|\varphi\| + \max_{t_0\leq u\leq t}|x(u)|\Big) \tag{2.10}$$

holds. It follows from (2.9) and (2.10) that

$$|x(t)| \leq K_1e^{-\alpha(t-t_0)}\|\varphi\|+K_2a_0e^{-\alpha t}(\|\varphi\|+\max_{t_0\leq u\leq t}|x(u)|)\int_{t_0}^{t} e^{\alpha s}\Big(2\|\varphi\|+\omega(x(s))\Big)\,ds. \tag{2.11}$$

Assumption (H3) implies that there exists $0 < \varepsilon_0 < \gamma$ such that

$$\frac{K_2a_0}{\alpha}\max_{|u|\leq\varepsilon_0}\omega(u) \leq \frac{1}{4}. \tag{2.12}$$

Let $0 < \varepsilon \leq \varepsilon_0$ be arbitrary, and define

$$\delta \equiv \min\left\{\frac{\varepsilon}{3(K_1+\frac{1}{2})}, \frac{\alpha}{8K_2a_0}\right\}. \tag{2.13}$$

Fix an initial function satisfying $\|\varphi\| \leq \delta$, and let $x$ be any solution of (2.1) corresponding to this initial function. Then $|\varphi(0)| \leq \delta < \varepsilon$, therefore there exists $T > t_0$ such that $|x(t)| < \varepsilon$ for $t \in [t_0, T)$. Suppose $|x(T)| = \varepsilon$. Then (2.11), (2.12) and (2.13) imply that

$$\begin{aligned}
\varepsilon &\leq K_1e^{-\alpha(T-t_0)}\delta + K_2a_0e^{-\alpha T}(\delta+\varepsilon)(2\delta + \max_{|u|\leq\varepsilon}\omega(u))\int_{t_0}^{T} e^{\alpha s}\,ds \\
&\leq K_1\delta + \frac{K_2a_0}{\alpha}(\delta+\varepsilon)(2\delta + \max_{|u|\leq\varepsilon}\omega(u)) \\
&= K_1\delta + \frac{K_2a_0}{\alpha}2\delta^2 + \frac{K_2a_0}{\alpha}\max_{|u|\leq\varepsilon}\omega(u)\delta + \frac{K_2a_0}{\alpha}2\delta\varepsilon + \frac{K_2a_0}{\alpha}\max_{|u|\leq\varepsilon}\omega(u)\varepsilon \\
&\leq K_1\delta + \frac{\delta}{4} + \frac{\delta}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4},
\end{aligned}$$

which, together with (2.13), yields

$$\frac{\varepsilon}{2} \leq K_1\delta + \frac{\delta}{2} \leq \frac{\varepsilon}{3}.$$

This contradiction means that $|x(t)| < \varepsilon$ is satisfied for all $t > 0$, i.e., the trivial solution of (2.1) is (uniformly) stable.

Next we show that the trivial solution of (2.1) is exponentially stable, as well. Let $0 < \beta < \alpha$ be arbitrary, and $0 < \varepsilon < \gamma$ be such that

$$\frac{K_2a_0^2e^{2\beta r}}{\alpha-\beta}\max_{|u|\leq\varepsilon}\omega(u) < 1, \tag{2.14}$$

and $0 < \sigma \leq \varepsilon$ be such that $|x(t)| < \varepsilon$ for $t \geq t_0$ and for $\|\varphi\| < \sigma$. Fix any initial function satisfying $\|\varphi\| < \sigma$, and let $x$ be any corresponding solution of (2.1). Multiplying both sides of (2.9) by $e^{\beta(t-t_0)}$ we get

$$
\begin{aligned}
e^{\beta(t-t_0)}|x(t)| &\le K_1 e^{-(\alpha-\beta)(t-t_0)}\|\varphi\| + K_2 e^{\beta(t-t_0)} \int_{t_0}^{t} e^{-\alpha(t-s)}|f(s)|\,ds \\
&\le K_1 e^{-(\alpha-\beta)(t-t_0)}\|\varphi\| \\
&\quad + K_2 a_0 e^{\beta(t-t_0)-\alpha t} \int_{t_0}^{t_0+r} e^{\alpha s}|x(s-\tau(s,x(s))) - x(s-\tau(s,0))|\,ds \\
&\quad + K_2 a_0 e^{\beta(t-t_0)-\alpha t} \int_{t_0+r}^{t} e^{\alpha s} \left| \int_{s-\tau(s,0)}^{s-\tau(s,x(s))} a(u)x(u-\tau(u,x(u)))du \right| ds. \qquad (2.15)
\end{aligned}
$$

It follows from (2.3) that

$$
|x(t)| \le e^{a_0 r}\|\varphi\|, \qquad t \in [t_0 - r, t_0 + r]. \tag{2.16}
$$

Introduce the function $z(t) \equiv e^{\beta(t-t_0)}|x(t)|$. With this notation we have from (2.15), (2.16), the assumptions and the Mean Value Theorem that

$$
\begin{aligned}
z(t) &\le K_1\|\varphi\| + K_2 a_0 e^{\beta(t-t_0)-\alpha t} 2e^{a_0 r}\|\varphi\| \int_{t_0}^{t_0+r} e^{\alpha s}\,ds \\
&\quad + K_2 a_0^2 e^{\beta(t-t_0)-\alpha t} \int_{t_0+r}^{t} e^{\alpha s} \left| \int_{s-\tau(s,0)}^{s-\tau(s,x(s))} e^{-\beta(u-\tau(u,x(u))-t_0)} z(u-\tau(u,x(u)))\,du \right| ds \\
&\le K_1\|\varphi\| + K_2 a_0 e^{\beta(t-t_0)-\alpha t} 2e^{a_0 r}\|\varphi\| \frac{e^{\alpha(t_0+r)} - e^{\alpha t_0}}{\alpha} \\
&\quad + K_2 a_0^2 e^{\beta(t-t_0)-\alpha t} e^{\beta r} \max_{t_0-r\le u\le t} z(u) \int_{t_0+r}^{t} e^{\alpha s} \left| \int_{s-\tau(s,0)}^{s-\tau(s,x(s))} e^{-\beta(u-t_0)}\,du \right| ds \\
&\le K_1\|\varphi\| + \frac{2K_2 a_0}{\alpha} e^{-(\alpha-\beta)(t-t_0)} e^{(a_0+\alpha)r}\|\varphi\| \\
&\quad + \frac{K_2 a_0^2}{\beta} e^{\beta(t-t_0)-\alpha t+\beta r} \max_{t_0-r\le u\le t} z(u) \int_{t_0+r}^{t} e^{\alpha s} \left| e^{-\beta(s-\tau(s,x(s))-t_0)} - e^{-\beta(s-\tau(s,0)-t_0)} \right| ds \\
&\le K_1\|\varphi\| + \frac{2K_2 a_0}{\alpha} e^{(a_0+\alpha)r}\|\varphi\| \\
&\quad + \frac{K_2 a_0^2}{\beta} e^{\beta(t-t_0)-\alpha t+\beta r} \max_{t_0-r\le u\le t} z(u) \int_{t_0+r}^{t} e^{(\alpha-\beta)s} e^{\beta(r+t_0)} \beta|\tau(s,x(s)) - \tau(s,0)|\,ds \\
&\le K_1\|\varphi\| + \frac{2K_2 a_0}{\alpha} e^{(a_0+\alpha)r}\|\varphi\| + \frac{K_2 a_0^2}{\alpha-\beta} e^{2\beta r} \max_{|u|\le\varepsilon} \omega(u) \max_{t_0-r\le u\le t} z(u).
\end{aligned}
\tag{2.17}
$$

The right-hand-side of (2.17) is monotone in $t$, and $z(t) \le |\varphi(t)| \le \|\varphi\|$ for $t \in [t_0 - r, t_0]$, therefore (2.17) yields

$$
\left(1 - \frac{K_2 a_0^2}{\alpha-\beta} e^{2\beta r} \max_{|u|\le\varepsilon} \omega(u)\right) \max_{t_0-r\le u\le t} z(u) \le K_1\|\varphi\| + \frac{2K_2 a_0}{\alpha} e^{(a_0+\alpha)r}\|\varphi\|. \tag{2.18}
$$

Inequality (2.14) implies that the constant

$$K \equiv \frac{K_1 + \frac{2K_2 a_0}{\alpha} e^{(a_0+\alpha)r}}{1 - \frac{K_2 a_0^2}{\alpha-\beta} e^{2\beta r} \max_{|u|\le\varepsilon} \omega(u)}$$

is positive. Hence it follows from (2.18) that $z(t) \le K\|\varphi\|$, and so $|x(t)| \le Ke^{-\beta(t-t_0)}\|\varphi\|$ for $t \ge t_0$ and for $\|\varphi\| < \sigma$. □

Next we show that, assuming some extra conditions on the delay function, Theorem 2.2 can be reversed. In addition to (H1)–(H3) we assume

(H4) there exists $\delta_0 > 0$ such that the delay function $\tau$ is continuously differentiable on $[t_0, \infty) \times [-\delta_0, \delta_0]$;

(H5) there exist constants $0 \le c < 1$ and $0 \le d$ such that

$$\left|\frac{\partial \tau}{\partial t}(t,u)\right| \le c \qquad \text{and} \qquad \left|\frac{\partial \tau}{\partial u}(t,u)\right| \le d \qquad \text{for} \quad t \in [t_0, \infty), \quad |u| \le \delta_0.$$

**Theorem 2.3** *Suppose (H1)–(H5). Then the trivial solution of* (2.1) *is exponentially stable if and only if the trivial solution of* (2.4) *is exponentially stable.*

**Proof** It was shown in Theorem 2.2 that the exponential stability of the trivial solution of (2.4) is sufficient for that of (2.1). We have to show that this is also necessary. It is known (see, e.g., [11]) that the trivial solution of (2.4) is exponentially stable if and only if the fundamental solution of (2.4) (i.e., the solution of the initial value problem (2.7)-(2.8)) satisfies an estimate of the form $|v(t,s)| \le K_0 e^{-\alpha_0(t-s)}$ for some positive constants $K_0$ and $\alpha_0$. The exponential stability of (2.1) yields that there exist $K, \alpha, \sigma > 0$ such that any solution of (2.1) satisfies $|x(t)| \le Ke^{-\alpha(t-t_0)}\|\varphi\|$ for $t \ge t_0$, assuming $\|\varphi\| < \sigma$.

Fix a continuous initial function $\varphi$ defined on $[t_0 - r, t_0]$ such that

$$0 < \|\varphi\| < \min\left\{ \frac{(1-c)\alpha}{3a_0 d K^2 (2\alpha + (1-c)a_0)}, \frac{1-c}{2a_0 dK}, \frac{\delta_0}{K}, \frac{\gamma}{K}, \sigma \right\}, \tag{2.19}$$

and

$$\varphi(t_0) = \|\varphi\|, \quad \text{and} \quad \frac{4a_0}{(1-c)\|\varphi\|} \int_{t_0-r}^{t_0} |\varphi(u)|\, du \le \frac{1}{3}. \tag{2.20}$$

Let $x$ and $y$ be a solution of (2.1) and (2.4), respectively, both corresponding to this initial function, $\varphi$. Then the variation-of-constants formula (see [11]) yields

$$x(t) = y(t) + \int_{t_0}^{t} v(t,s) g(s)\, ds, \qquad t \ge t_0, \tag{2.21}$$

where

$$g(t) \equiv a(t)\Big(x(t - \tau(t, x(t))) - x(t - \tau(t, 0))\Big), \qquad t \ge t_0.$$

Assumptions (H4) and (H5) imply

$$\frac{d}{dt}\Big(t-\tau(t,0)\Big) = 1-\frac{\partial\tau}{\partial t}(t,0)\ge 1-c>0. \tag{2.22}$$

It follows from (2.22) that there exists $t_1\ge t_0$ such that

$$t-\tau(t,0)\begin{cases} <t_0, & t\in[t_0,t_1),\\ =t_0, & t=t_1,\\ >t_0, & t>t_1.\end{cases}$$

Then Theorem 1.2 from Section 6.1 of [11] yields the following relation (which can be checked by direct calculation, as well)

$$y(t)=v(t,t_0)\varphi(t_0)+\int_{t_0}^{t_1} v(t,u)a(u)\varphi(u-\tau(u,0))\,du, \qquad t\ge t_0. \tag{2.23}$$

We have from (2.21) and (2.23) that

$$v(t,t_0)=\frac{1}{\varphi(t_0)}x(t)-\frac{1}{\varphi(t_0)}\int_{t_0}^{t_1} v(t,u)a(u)\varphi(u-\tau(u,0))\,du-\frac{1}{\varphi(t_0)}\int_{t_0}^{t} v(t,s)g(s)\,ds,$$

and so for $t\ge t_0$

$$\begin{aligned}|v(t,t_0)| &\le \frac{1}{\|\varphi\|}|x(t)|+\frac{a_0}{\|\varphi\|}\int_{t_0}^{t_1}|v(t,u)||\varphi(u-\tau(u,0))|\,du+\frac{1}{\|\varphi\|}\int_{t_0}^{t}|v(t,s)||g(s)|\,ds\\ &\le Ke^{-\alpha(t-t_0)}+\max_{t_0\le s\le t}|v(t,s)|\left(\frac{a_0}{\|\varphi\|}\int_{t_0}^{t_1}|\varphi(u-\tau(u,0))|\,du+\frac{1}{\|\varphi\|}\int_{t_0}^{t}|g(s)|\,ds\right),\end{aligned} \tag{2.24}$$

where in the second estimate we used that $v(t,u)=0$ for $u>t$. Using (2.22) and the definition of $t_1$ we have

$$\int_{t_0}^{t_1}|\varphi(u-\tau(u,0))|\frac{\frac{d}{du}(u-\tau(u,0))}{\frac{d}{du}(u-\tau(u,0))}\,du\le\frac{1}{1-c}\int_{t_0-r}^{t_0}|\varphi(s)|\,ds, \tag{2.25}$$

therefore (2.24) yields

$$|v(t,t_0)|\le Ke^{-\alpha(t-t_0)}+\max_{t_0\le s\le t}|v(t,s)|\left(\frac{a_0}{\|\varphi\|(1-c)}\int_{t_0-r}^{t_0}|\varphi(u)|\,du+\frac{1}{\|\varphi\|}\int_{t_0}^{t}|g(s)|\,ds\right). \tag{2.26}$$

Assumptions (H4), (H5), inequality $|x(t)|\le K\|\varphi\|$, and (2.19) imply that the time-lag function, $t-\tau(t,x(t))$, is monotone increasing in $t$, more precisely,

$$\begin{aligned}\frac{d}{dt}\Big(t-\tau(t,x(t))\Big) &= 1-\frac{\partial\tau}{\partial t}(t,x(t))-\frac{\partial\tau}{\partial u}(t,x(t))a(t)x(t-\tau(t,x(t)))\\ &\ge 1-c-a_0dK\|\varphi\|\\ &> \frac{1-c}{2}.\end{aligned} \tag{2.27}$$

Therefore there exists $t_2 \geq t_0$ such that

$$t - \tau(t, x(t)) \begin{cases} < t_0, & t \in [t_0, t_2), \\ = t_0, & t = t_2, \\ > t_0, & t > t_2. \end{cases}$$

It follows from (H4) and (H5), the definitions of $t_1$ and $t_2$, and from the Mean Value Theorem that

$$|t_2 - t_1| = |\tau(t_2, x(t_2)) - \tau(t_1, 0)| \leq c|t_2 - t_1| + d|x(t_2)| \leq c|t_2 - t_1| + dK\|\varphi\|,$$

hence

$$|t_2 - t_1| \leq \frac{dK}{1-c}\|\varphi\|. \tag{2.28}$$

Suppose $t > \max(t_1, t_2)$, and consider

$$\begin{aligned} \int_0^t |g(s)|\, ds &= \int_{t_0}^{\min(t_1,t_2)} |g(s)|\, ds + \int_{\min(t_1,t_2)}^{\max(t_1,t_2)} |g(s)|\, ds + \int_{\max(t_1,t_2)}^{t} |g(s)|\, ds \\ &\leq a_0 \int_{t_0}^{\min(t_1,t_2)} |x(s - \tau(s, x(s)))|\, ds + a_0 \int_{t_0}^{\min(t_1,t_2)} |x(s - \tau(s, 0))|\, ds \\ &\quad +2a_0|t_2 - t_1|K\|\varphi\| + a_0 \int_{\max(t_1,t_2)}^{t} \left| \int_{s-\tau(s,0)}^{s-\tau(s,x(s))} \dot{x}(u)\, du \right| ds \\ &\leq a_0 \int_{t_0}^{\min(t_1,t_2)} |\varphi(s - \tau(s, x(s)))|\, ds + a_0 \int_{t_0}^{\min(t_1,t_2)} |\varphi(s - \tau(s, 0))|\, ds \\ &\quad +\frac{2a_0 dK^2}{1-c}\|\varphi\|^2 + a_0^2 \int_{\max(t_1,t_2)}^{t} \left| \int_{s-\tau(s,0)}^{s-\tau(s,x(s))} |x(u - \tau(u, x(u)))|\, du \right| ds. \end{aligned} \tag{2.29}$$

Relation (2.27) and the definition of $t_2$ yield

$$\int_{t_0}^{t_2} |\varphi(s - \tau(s, x(s)))| \frac{\frac{d}{ds}(s - \tau(s, x(s)))}{\frac{d}{ds}(s - \tau(s, x(s)))}\, ds \leq \frac{2}{1-c} \int_{t_0 - r}^{t_0} |\varphi(s)|\, ds,$$

hence it follows from (2.25) and (2.29) that

$$\begin{aligned} \int_{t_0}^{t} |g(s)|\, ds &\leq \frac{3a_0}{1-c} \int_{t_0-r}^{t_0} |\varphi(s)|\, ds + \frac{2a_0 dK^2}{1-c}\|\varphi\|^2 \\ &\quad + a_0^2 K\|\varphi\| \int_{\max(t_1,t_2)}^{t} |\tau(s, x(s)) - \tau(s, 0)|\, ds \\ &\leq \frac{3a_0}{1-c} \int_{t_0-r}^{t_0} |\varphi(s)|\, ds + \frac{2a_0 dK^2}{1-c}\|\varphi\|^2 + a_0^2 dK\|\varphi\| \int_{\max(t_1,t_2)}^{t} |x(s)| ds \\ &\leq \frac{3a_0}{1-c} \int_{t_0-r}^{t_0} |\varphi(s)|\, ds + \frac{2a_0 dK^2}{1-c}\|\varphi\|^2 + \frac{a_0^2 dK^2}{\alpha}\|\varphi\|^2. \end{aligned} \tag{2.30}$$

Combining (2.19), (2.20), (2.24) and (2.30) we get

$$
\begin{aligned}
|v(t,t_0)| &\le Ke^{-\alpha(t-t_0)} \\
&\quad + \max_{t_0\le s\le t}|v(t,s)|\left(\frac{4a_0}{(1-c)\|\varphi\|}\int_{t_0-r}^{t_0}|\varphi(s)|\,ds+\left(\frac{2}{1-c}+\frac{a_0}{\alpha}\right)a_0dK^2\|\varphi\|\right) &(2.31)\\
&\le K+\frac{2}{3}\max_{t_0\le s\le t}|v(t,s)|. &(2.32)
\end{aligned}
$$

Let $t_0 \le \bar{s} \le t$. Then (2.32) implies

$$|v(t,\bar{s})| \le K+\frac{2}{3}\max_{\bar{s}\le s\le t}|v(t,s)| \le K+\frac{2}{3}\max_{t_0\le s\le t}|v(t,s)|.$$

Therefore

$$\max_{t_0\le s\le t}|v(t,s)| \le 3K.$$

Substituting this back to (2.31) we get

$$|v(t,t_0)| \le Ke^{-\alpha(t-t_0)}+3K\left(\frac{4a_0}{(1-c)\|\varphi\|}\int_{t_0-r}^{t_0}|\varphi(s)|\,ds+\left(\frac{2}{1-c}+\frac{a_0}{\alpha}\right)a_0dK^2\|\varphi\|\right).$$

Since the second term on the right-hand-side can be made arbitrary small by selecting an appropriate initial function which satisfies (2.19) and (2.20) as well, we get that $|v(t,t_0)| \le Ke^{-\alpha(t-t_0)}$ holds for $t \ge t_0$, which proves the theorem. □

Note that Theorems 2.2 and 2.3 are straightforward to extend for the multiple delay case, i.e., for equations of the form

$$\dot{x}(t)=\sum_{i=0}^{m}a_i(t)x(t-\tau_i(t,x(t))).$$

Since sufficient conditions are known for the uniform asymptotic stability, and so for the exponential stability of the linear equation (2.4) (see, e.g., [20]), and a necessary and sufficient condition is known for the exponential stability in the case when (2.4) is an autonomous delay equation (see, e.g., [11]), and when it is an ODE, therefore Theorem 2.3 has the following corollaries.

**Corollary 2.4** *Suppose (H1)–(H3), and let* $q_0 \equiv \sup_{t\ge t_0}\tau(t,0)$. *Then the trivial solution of* (2.1) *is exponentially stable, if* $a(t)\le 0$, $t\ge t_0$, *and*

$$\inf_{t\ge t_0}\int_t^{t+q_0}(-a(s))\,ds>0, \qquad \textit{and} \qquad \sup_{t\ge t_0}\int_t^{t+q_0}(-a(s))\,ds<\frac{3}{2}. \qquad (2.33)$$

Lemma 2.2 of [21] proves that a condition of the form (2.33) where $q_0$ is replaced by $r$ implies the uniform asymptotic stability of the trivial solution of (2.1). Corollary 2.4 improves the above result of Yoneyama, since $q_0 < r$, in general, and in our result the exponential stability of (2.1) is obtained.

**Corollary 2.5** *Suppose $a(t) \equiv -a_0 < 0$ and $\tau \colon \mathbb{R} \to [0, r]$ is continuously differentiable in a neighborhood of 0. Then the trivial solution of* (2.1) *is exponentially stable if and only if $0 \le a_0 \tau(0) < \pi/2$.*

**Corollary 2.6** *Suppose (H1)–(H5), and $\tau(t, 0) = 0$ for all $t \ge t_0$. Then the trivial solution of* (2.1) *is exponentially stable if and only if there exists $\alpha > 0$ such that*

$$\frac{1}{t - t_0} \int_{t_0}^{t} a(s)\, ds \le -\alpha, \qquad t > t_0.$$

## References


[1] W. G. Aiello, H. I. Freedman and J. Wu, Analysis of a model representing stage-structured population growth with state-dependent time delay, SIAM J. Appl. Math, **52** (1992) 855–869.

[2] O. Arino, K. P. Hadeler and M. L. Hbid, Existence of periodic colutions for delay differential equations with state-dependent delay, J. Diff. Eqns, **144:2** (1998) 263–301.

[3] M. Bartha, On stability properties for neutral differential equations with state-dependent delays, to appear in Diff. Eqns. Dyn. Systems.

[4] J. Bélair, Population models with state-dependent delays, in Mathematical Population Dynamics, (O. Arino, D. E. Axelrod and M. Kimmel eds.), Marcel Dekker, (1991) 165–176.

[5] K. L. Cooke, Asymptotic theory for the delay-differential equation $u'(t) = -au(t - r(u(t)))$, J. Math. Anal. Appl., **19** (1967) 160–173.

[6] K. L. Cooke, Asymptotic equivalence of an ordinary and a functional differential equation, J. Math. Anal. Appl., **51:1** (1975) 187–207.

[7] K. L. Cooke and W. Huang, On the problem of linearization for state-dependent delay differential equations, Proc. Amer. Math. Soc., **124:5** (1996) 1417–1426.

[8] R. D. Driver, Existence theory for a delay-differential system, Contributions to Differential Equations, **1** (1961) 317–336.

[9] I. Győri, F. Hartung and J. Turi, Numerical approximations for a class of differential equations with time- and state-dependent delays, Appl. Math. Lett., **8:6** (1995) 19–24.

[10] I. Győri, F. Hartung and J. Turi, Preservation of Stability in Delay Equations under Delay Perturbations, J. Math. Anal. Appl., **220** (1998), 290–312.

[11] J. K. Hale and S. M. Verduyn Lunel, "Introduction to Functional Differential Equations", Spingler-Verlag, New York, 1993.

[12] T. Hara, R. Miyazaki, and Y. Matsumi, Exponential asymptotic stability in linear delay-differential equations with variable coefficients, Diff. and Int. Eqns, **11:2** (1998) 263–278.

[13] F. Hartung and J. Turi, Stability in a class of functional differential equations with state-dependent delays, in Qualitative Problems for Differential Equations and Control Theory, ed. C. Corduneanu, Word Scientific, 1995, 15–31.

[14] F. Hartung and J. Turi, On the asymptotic behavior of the solutions of a state-dependent delay equation, Diff. Int. Eqns., **8:7** (1995) 1867–1872.

[15] F. Hartung and J. Turi, On differentiability of solutions with respect to parameters in state-dependent delay equations, J. Diff. Eqns., **135:2** (1997) 192–237.

[16] T. Krisztin and O. Arino, The 2-dimensional attractor of a differential equation with state-dependent delay, prepript.

[17] J. Mallet-Paret, R. D. Nussbaum, Boundary layer phenomena for differential-delay equations with state-dependent time lags: I, Arc. Rational Mech. Anal., **120** (1992) 99–146.

[18] J. Mallet-Paret, R. D. Nussbaum, Boundary layer phenomena for differential-delay equations with state-dependent time lags: II, J. reine angew. Math., **477** (1996) 129–197.

[19] J. Mallet-Paret, R. D. Nussbaum and P. Paraskevopoulos, Periodic solutions for functional differential equations with multiple state-dependent time lags, Topol. Methods Nonlinear Anal., **3** (1994) 101–162.

[20] T. Yoneyama, On the $\frac{3}{2}$ stability theorem for one-dimensional delay-differential equations, J. Math. Anal. Appl., **125** (1987) 161–173.

[21] T. Yoneyama, Uniform stability for $n$-dimensional delay-differential equations, Funkcialaj Ekvacioj, **34** (1991) 495–504.

[22] E. Winston, Uniqueness of the zero solution for delay differential equations with state-dependence, J. Diff. Eqns., **7** (1970) 395–405.

[23] H. Z. Wu, A class of functional-differential equations with state-dependent deviating arguments, Acta Math. Sinica, **38** (1995) 803–809.