Application of Delay Equations

István Győri and Ferenc Hartung

Department of Mathematics and Computing
University of Pannonia
Veszprém, Hungary

Sarajevo
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Stability of several classes of difference equations has been studied extensively in the recent literature. Without completeness, we refer to

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V. L. Kocic, G. Ladas, Kluwer Aacademic Publishers (1993)
I. Kovácsvölgyi, Appl. Math. Letters 13 (2000)
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R. Ogita, H. Matsunaga, T. Hara,JMAA 248 (2000)

Consider

$$x(n+1) - x(n) = -ax(n-k), \qquad n = 0, 1, \dots \tag{83}$$

where $a \in \mathbb{R}$, $k \geq 0$. Look for solutions of the form

$$x(n) = \lambda^n.$$

Then

$$\lambda^{n+1} - \lambda^n = -a\lambda^{n-k},$$

or equivalently,

$$\lambda^{k+1} - \lambda^k = -a. \tag{84}$$

Theorem 34 (Levin and May, 1976).

The following statements are equivalent

(i) (83) *is asymptotically stable;*

(ii) *all roots of* (84) *satisfy* $|\lambda| < 1$;

(iii) $0 < ak < 2k \cos \frac{k\pi}{2k+1}$.

Consider the linear delay difference equation

$$x(n+1) - x(n) = -\sum_{i=1}^{m} a_i x(n - k_i(n)), \qquad n \in \mathbb{Z}^+, \tag{85}$$

where $a_i > 0$ and $k_i \colon \mathbb{Z}^+ \to \mathbb{Z}^+$, $(i = 1, \ldots, m)$, and there exists $r > 0$ such that $k_i(n) \leq r$ for $n \in \mathbb{Z}^+$ and $i = 1, \ldots, m$. Equation (85) has a unique solution, assuming that

$$x(n) = \varphi(n), \qquad \varphi\colon [-r, 0] \to \mathbb{R}. \tag{86}$$

We compare the stability of the discrete equation (85) to that of a differential equation. We associate the linear delay differential equation with piecewise constant argument

$$\dot{y}(t) = -\sum_{i=1}^{m} a_i y\Big([t] - k_i([t])\Big), \qquad t \geq 0, \tag{87}$$

and the initial condition

$$y(t) = \varphi(t), \qquad t \in [-r, 0] \tag{88}$$

to (85)-(86), where $[\cdot]$ is the greatest integer function.
Integrating both sides of (87) from n to $t \in [n, (n+1))$, we get

$$y(t) - y(n) = -\sum_{i=1}^{m} a_i y\Big(n - k_i(n)\Big)(t-n).$$

Therefore IVP (87)-(88) has a unique solution, which is piecewise linear between nonnegative integers, and

$$y(n+1) - y(n) = -\sum_{i=1}^{m} a_i y\Big(n - k_i(n)\Big), \qquad n \in \mathbb{N}_0. \tag{89}$$

We can observe that the solutions of (85) and (87) are related by $y(n) = x(n)$. Therefore the trivial solution of (85) is asymptotically stable, if and only if, so is the trivial solution of (87).

$$x(n+1) - x(n) = -\sum_{i=1}^{m} a_i x(n - k_i(n)), \qquad n \in \mathbb{Z}^+, \tag{85}$$

$$\dot{y}(t) = -\sum_{i=1}^{m} a_i y\Big([t] - k_i([t])\Big), \qquad t \geq 0, \tag{87}$$

Rewrite (87) as

$$\dot{y}(t) = -\sum_{i=1}^{m} a_i y\Big(t - \sigma_i(t)\Big), \qquad t \geq 0, \tag{90}$$

where

$$\sigma_i(t) \equiv k_i([t]) + t - [t].$$

Define the function

$$\Phi(r) = \int_0^\infty |v(t)|\, dt,$$

where v is the fundamental solution of the equation

$$\begin{aligned} \dot{v}(t) &= -v(t-r), \qquad t \geq 0, \\ v(t) &= \begin{cases} 1, & t = 0, \\ 0, & t < 0. \end{cases} \end{aligned}$$

$$\dot{y}(t) = -\sum_{i=1}^{m} a_i y\Big(t-\tau\Big) + \sum_{i=1}^{m} a_i \Bigg(y(t-\tau) - y\Big(t-\sigma_i(t)\Big)\Bigg),$$

Theorem 19 yields that the trivial solution of (90) (i.e., that of (87)) is asymptotically stable, if for some $\tau \in [0, \pi/(2a))$ it follows

$$\tau a - \frac{1}{\Phi(\tau a)} < \sum_{i=1}^{m} a_i \varliminf_{t\to\infty} \sigma_i(t) \leq \sum_{i=1}^{m} a_i \varlimsup_{t\to\infty} \sigma_i(t) < \tau a + \frac{1}{\Phi(\tau a)}, \qquad (91)$$

where $a \equiv \sum_{i=1}^{m} a_i$. Since

$$\varliminf_{t\to\infty} \sigma_i(t) = \varliminf_{t\to\infty} \Bigg(k_i([t]) + t - [t]\Bigg) \geq \varliminf_{n\to\infty} k_i(n)$$

and

$$\varlimsup_{t\to\infty}\sigma_i(t)=\varlimsup_{t\to\infty}\Big(k_i([t])+t-[t]\Big)\le\varlimsup_{n\to\infty}k_i(n)+1,$$

we get the following result.

$$x(n+1) - x(n) = -\sum_{i=1}^{m} a_i x(n - k_i(n)), \qquad n \in \mathbb{Z}^+, \qquad (85)$$

Theorem 35 (I. Győri, F. Hartung (2000)).

Suppose $a_i > 0$ $(i = 1, \ldots, m)$, $a \equiv \sum_{i=1}^{m} a_i$, and for some $\tau \in [0, \pi/(2a))$

$$\tau a - \frac{1}{\Phi(\tau a)} < \sum_{i=1}^{m} a_i \varliminf_{n\to\infty} k_i(n) \le \sum_{i=1}^{m} a_i \varlimsup_{n\to\infty} k_i(n) < (\tau - 1)a + \frac{1}{\Phi(\tau a)}$$

holds. Then the trivial solution of (85) is asymptotically stable.

Let $\tau = \frac{1}{ae}$. Then $\Phi(a\tau) = 1$, and we get

Corollary 36.

Suppose $0 < a_i$ $(i = 1, \dots, m)$, *and*

$$\sum_{i=1}^{m} a_i \varlimsup_{n\to\infty} k_i(n) < 1 + \frac{1}{e} - \sum_{i=1}^{m} a_i.$$

Then the trivial solution of (85) *is asymptotically stable.*

Consider the time-dependent scalar linear delay difference equation

$$x(n+1) - x(n) = -a(n)x(n-k(n)), \qquad n \in \mathbb{Z}^+, \tag{92}$$

where $a\colon \mathbb{Z}^+ \to [0,\infty)$, $k\colon \mathbb{Z}^+ \to \mathbb{Z}^+$.

Theorem 37.

Assume $\sum_{n=0}^{\infty} a(n) = \infty$, and there exists $\tau \in [0, \pi/2)$ such that

$$\tau - \frac{1}{\Phi(\tau)} < \underline{\lim}_{n\to\infty} \sum_{i=n-k(n)}^{n-1} a(i) \le \overline{\lim}_{n\to\infty} \sum_{i=n-k(n)}^{n} a(i) < \tau + \frac{1}{\Phi(\tau)}.$$

Then the trivial solution of (92) is asymptotically stable.

If $\tau = \frac{1}{e}$ we get $\Phi(\tau) = 1$, and

Corollary 38.

Assume $\sum_{n=0}^{\infty} a(n) = \infty$, *and*

$$\overline{\lim_{n\to\infty}} \sum_{i=n-k(n)}^{n} a(i) < 1 + \frac{1}{e}.$$

Then the trivial solution of (92) *is asymptotically stable.*

$$x(n+1) - x(n) = -a(n)x(n-k), \qquad \sum_{n=0}^{\infty} a(n) = \infty, \quad a(n) \geq 0$$

Condition of Ladas, Qian, Vlahos, Yan (1991):

$$\overline{\lim_{n\to\infty}} \sum_{i=n-k}^{n} a(i) < 1$$

Condition of Győri and Pituk (1997):

$$\overline{\lim_{n\to\infty}} \sum_{i=n-k}^{n-1} a(i) < 1$$

Our condition:

$$\overline{\lim_{n\to\infty}} \sum_{i=n-k}^{n} a(i) < 1 + \frac{1}{e}.$$

$$x(n+1) - x(n) = -\sum_{i=1}^{m} a_i x(n-k_i), \qquad a_i > 0$$

Condition of Cooke, Győri (1994):

$$\sum_{i=1}^{m} a_i k_i < 1,$$

Our condition

$$\sum_{i=1}^{m} a_i k_i < 1 + \frac{1}{e} - \sum_{i=1}^{m} a_i$$

$$x(n+1)-x(n)=-a(n)x(n-k), \qquad \sum_{n=0}^{\infty} a(n)=\infty, \quad a(n)\geq 0$$

Condition of Erbe, Xia and Yu (1995):

$$\overline{\lim_{n\to\infty}} \sum_{i=n-k}^{n} a(i) < \frac{3}{2}+\frac{1}{2(k+1)}$$

Applying this for equation

$$x(n+1)-x(n)=-ax(n-k)$$

we get

$$ak \leq \frac{3}{2}\frac{k}{k+1}+\frac{k}{2(k+1)^2}$$

Our condition is

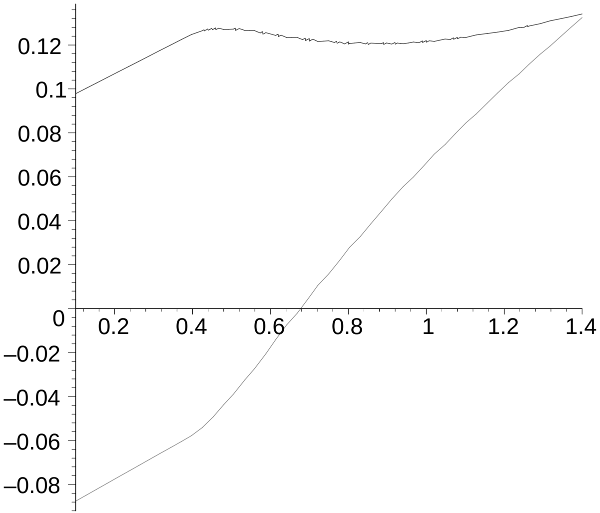
$$\tau - \frac{1}{\Phi(\tau)} < ak < \frac{k}{k+1}\left(\tau + \frac{1}{\Phi(\tau)}\right) \quad \text{for some } \tau \in [0, \pi/2)$$

Exact condition (Levin and May (1976)):

$$ak < 2k \cos \frac{k\pi}{2k+1}$$

$$\begin{aligned} \tau_k - \frac{1}{\Phi(\tau_k)} &= \frac{k}{k+1}\left(\tau_k + \frac{1}{\Phi(\tau_k)}\right) \\ \tau_k \Phi(\tau_k) &= 2k+1 \\ \tau_k &\longrightarrow \frac{\pi}{2} \end{aligned}$$

k	exact	Erbe	our	τ_k
10	1.495	1.405	1.354	1.419
15	1.519	1.436	1.416	1.157
20	1.532	1.451	1.454	1.489
25	1.540	1.461	1.473	1.499
30	1.545	1.467	1.488	1.509
35	1.549	1.472	1.500	1.519
40	1.551	1.475	1.511	1.529



graphs of $\tau - 1/\Phi(\tau)$ and $\frac{10}{11}(\tau + 1/\Phi(\tau))$