Application of Delay Equations

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Consider the ODE

$$\dot{x}(t) = f(t, x(t)), \qquad x(0) = x_0.$$

Pick a step-size $h > 0$, and consider the mesh points $t_i = ih$, $i = 0, 1, \ldots, N$. Let $T = Nh$, and we look for numerical solution on the interval $[0, T]$.

Euler-method:

$$\begin{aligned} z_{k+1} &= z_k + hf(kh, z_k), \qquad k = 0, 1, \ldots, N-1 \\ z_0 &= x_0 \end{aligned}$$

local error:

$$e_k = \frac{x(t_{k+1}) - x(t_k)}{h} - f(t_k, x(t_k))$$

$$|e_k| \le K_1 h \Longrightarrow |x(t_k) - z_k| \le Kh, \qquad k = 0, 1, \ldots, N-1.$$

Consider a nonliear delay equation

$$\dot{x}(t) = f(t, x(t), x(t-\tau)),$$

and an initial condition

$$x(t) = \varphi(t), \qquad t \in [-\tau, 0].$$

Euler-method:

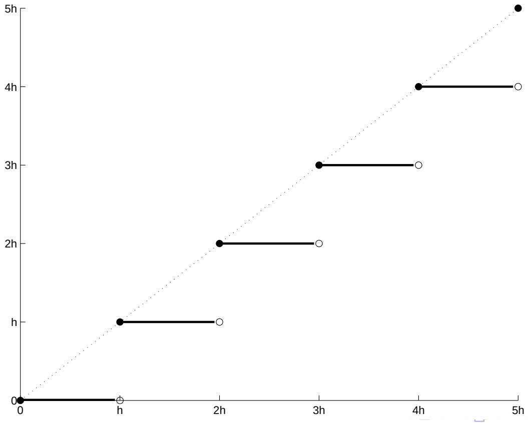
$$\begin{aligned} h &= \frac{\tau}{m} \\ z_{k+1} &= z_k + hf(kh, z_k, z_m), \qquad k = 0, 1, \ldots, \\ z_k &= \varphi(kh), \qquad k = 0, -1, \ldots, \quad -r \leq kh \leq 0. \end{aligned}$$

$\tau = \tau(t)$ time-dependent delay
$\tau = \tau(t, x(t))$ time- and state-dependent delay

$x(t_k - \tau(t_k)) \approx z_\ell, \qquad \ell = ?$

We introduce the notation

$$[t]_h = \left[\frac{t}{h}\right] h$$



Properties:

1. $[t]_h$ is a piecewise-constant function;
2. has jumps at the points $\{kh\colon\ k \in \mathbb{Z}\}$, where it is right-continuous;
3. domain of $[t]_h$ is the whole real line, the range is the discrete values $\{kh\colon\ k \in \mathbb{Z}\}$;
4. $|t - [t]_h| \leq h$;
5. $\lim\limits_{t \to 0+} [t]_h = t$.

Consider the nonlinear, time-dependent delay system

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x(t-\tau(t))), \qquad t \geq 0 & (65)\\ x(t) &= \varphi(t), \qquad t \in [-r, 0]. & (66)\end{aligned}$$

We associate the following EPCA to the IVP (65)-(66)

$$\begin{aligned} \dot{y}_h(t) &= f\Big([t]_h, y_h([t]_h), y_h\Big([t]_h - [\tau([t]_h)]_h\Big)\Big), \quad t \geq 0 & (67)\\ y_h(t) &= \varphi(t), \qquad t \in [-r, 0]. & (68)\end{aligned}$$

The right-hand-side of (74) is constant on the intervals $[kh, (k+1)h)$, so the solution of the IVP (74)-(75) is a continuous function which is linear in between the mesh points $\{kh\colon\ k \in \mathbb{N}\}$.

We integrate both sides of (74) from kh to t, where $kh < t < (k+1)h$:

$$\int_{kh}^{t} \dot{y}_h(s)\, ds = \int_{kh}^{t} f\Big([s]_h, y_h([s]_h), y_h\Big([s]_h - [\tau([s]_h)]_h\Big)\Big)\, ds$$

Using that the integrand on the right-hand-side is constant, we get

$$y_h(t) - y_h(kh) = f\Big(kh, y_h(kh), y_h\Big(kh - [\tau(kh)]_h\Big)\Big)(t - kh).$$

So taking the limit $t \to (k+1)h-$ we get

$$y_h((k+1)h) - y_h(kh) = hf\Big(kh, y_h(kh), y_h\Big(kh - [\tau(kh)]_h\Big)\Big).$$

Since y_h is linear between the mesh points, the values $a(k) = y_h(kh)$ uniquely determine the solution. The sequence $a(k)$ satisfies the difference equation

$$\begin{aligned} a(k+1) &= a(k) + f\Big(kh, a(k), a(k-d_k)\Big)\cdot h, \qquad k = 0,1,2,\dots, \\ a(-k) &= \varphi(-kh), \qquad k = 0,1,2,\dots, \quad -r \le -kh \le 0, \end{aligned}$$

where $d_k \equiv \Big[\tau(kh)/h\Big]$.

Consider the delayed logistic equation

$$\dot{N}(t) = N(t)\Big(r - mN(t-\tau)\Big), \qquad t \geq 0, \tag{69}$$

with initial condition

$$N(t) = \varphi(t), \qquad -\tau \leq t \leq 0, \tag{70}$$

Pick a discretization parameter $h > 0$, and consider the approximating EPCA

$$\dot{y}_h(t) = y_h([t]_h)\Big(r - my_h([t]_h - [\tau]_h)\Big), \qquad t \geq 0,$$

therefore the sequence $a(k) = y_h(kh)$ is defined by

$$\begin{aligned} a(k+1) &= a(k) + ha(k)\Big(r - ma(k-\ell)\Big), \qquad k = 0, 1, 2, \dots, \\ a(-k) &= \varphi(-kh), \qquad k = 0, 1, 2, \dots, \quad -r \leq -kh \leq 0, \end{aligned}$$

where $\ell \equiv \Big[\tau/h\Big]$.

We assume f is Lipschitz-continuous: there exists $L > 0$ such that

$$|f(t,u,v) - f(\tilde{t},\tilde{u},\tilde{v})| \le L_1(|t-\tilde{t}| + |u-\tilde{u}| + |v-\tilde{v}|)$$

The delay function τ is Lipschitz-continuous:

$$|\tau(t) - \tau(\tilde{t})| \le L_2|t-\tilde{t}|.$$

Theorem 31.

If f and τ are Lipschitz-continuous, then for any $T > 0$ there exists a constant $K > 0$ such that

$$|x(t) - y_h(t)| \le Kh, \qquad t \in [0, T].$$

Corollary 32.

$$\lim_{h\to 0+} y_h(t) = x(t)$$

uniformly for $t \in [0, T]$.

Proof of Theorem 31: Integrating (65) and (74) we get

$$x(t) = \varphi(0) + \int_0^t f\Big(s, x(s), x(s-\tau(s))\Big)\,ds$$

and similarly,

$$y_h(t) = \varphi(0) + \int_0^t f\Big([s]_h, y_h([s]_h), y_h\Big([s]_h - [\tau([s]_h)]_h\Big)\Big)\,ds.$$

Taking the difference of the two equations yields

$$\begin{aligned} x(t) - y_h(t) &= \int_0^t \Bigg(f\Big(s, x(s), x(s-\tau(s))\Big) \\ &\quad - f\Big([s]_h, y_h([s]_h), y_h\Big([s]_h - [\tau([s]_h)]_h\Big)\Big)\Bigg)\,ds, \end{aligned}$$

and so

$$
\begin{aligned}
|x(t)-y_h(t)| &\le \int_0^t \Big| f\Big(s, x(s), x(s-\tau(s))\Big) \\
&\qquad -f\Big([s]_h, y_h([s]_h), y_h\Big([s]_h-[\tau([s]_h)]_h\Big)\Big)\Big| ds \\
&\le L_1 \int_0^t \Big(|s-[s]_h| + |x(s)-y_h([s]_h)| \\
&\qquad + \Big| x(s-\tau(s)) - y_h\Big([s]_h-[\tau([s]_h)]_h\Big)\Big| \Big) ds.
\end{aligned}
\tag{71}
$$

One of the basic propety of the function $[s]_h$ is the estimate

$$|s-[s]_h| \le h.$$

Define

$$z_h(t) = |x(t)-y_h(t)|$$

and

$$M = \max_{0 \le s \le T} |\dot{x}(s)|.$$

Then the Mean Value Theorem yields

$$|x(t) - x(\tilde{t})| \le M|t - \tilde{t}|, \qquad t, \tilde{t} \in [0, T].$$

Therefore

$$\begin{aligned} |x(s) - y_h([s]_h)| &\le |x(s) - x([s]_h)| + |x([s]_h) - y_h([s]_h)| \\ &\le M|s - [s]_h| + z_h([s]_h) \\ &\le Mh + z_h([s]_h). \end{aligned}$$

Similarly,

$$\begin{aligned}
&|x(s-\tau(s)) - y_h([s]_h - [\tau([s]_h)]_h)| \\
&\quad \le |x(s-\tau(s)) - x([s]_h - [\tau([s]_h)]_h)| \\
&\qquad +|x([s]_h - [\tau([s]_h)]_h) - y_h([s]_h - [\tau([s]_h)]_h)| \\
&\quad \le M|s-\tau(s) - ([s]_h - [\tau([s]_h)]_h)| + z_h([s]_h - [\tau([s]_h)]_h) \\
&\quad \le M|s-\tau(s) - ([s]_h - [\tau([s]_h)]_h)| + z_h([s]_h - [\tau([s]_h)]_h) \\
&\quad \le M\Big(|s-[s]_h| + |\tau(s)-\tau([s]_h)| + |\tau([s]_h) - [\tau([s]_h)]_h\Big) \\
&\qquad +z_h([s]_h - [\tau([s]_h)]_h) \\
&\quad \le M\Big(2h + L_2|s-[s]_h|\Big) + z_h([s]_h - [\tau([s]_h)]_h) \\
&\quad \le M(2+L_2)h + z_h([s]_h - [\tau([s]_h)]_h).
\end{aligned}$$

Hence (71) implies

$$z_h(t) \le L_1 \int_0^t \Big(h + Mh + M(2+L_2)h + z_h([s]_h) + z_h([s]_h - [\tau([s]_h)]_h)\Big)\, ds.$$

Let

$$w_h(t) = \max_{-r \le s \le t} z_h(s) = \max_{0 \le s \le t} z_h(s).$$

Then

$$z_h(t) \le \int_0^t \Big(K_1 h + 2L_1 w_h(s) \Big) ds, \qquad t \in [0, T],$$

where $K_1 = 1 + 3M + ML_2$. We have for $u \le t$

$$z_h(u) \le L_1 \int_0^u \Big(K_1 h + 2L_1 w_h(s) \Big) ds \le L_1 \int_0^t \Big(K_1 h + 2L_1 w_h(s) \Big) ds,$$

so

$$w_h(t) \le \int_0^t \Big(K_1 h + 2L_1 w_h(s) \Big) ds \le K_1 Th + \int_0^t 2L_1 w_h(s)\, ds, \qquad t \in [0, T]$$

also holds. Then Gronwall's Lemma

$$u(t) \le A + \int_0^t u(s)v(s)\, ds \Longrightarrow u(t) \le A e^{\int_0^t v(s)\, ds}$$

yields

$$z_h(t) \le w_h(t) \le K_1 Th e^{\int_0^t 2L_1\, ds} \le K_1 The^{2L_1 T}, \qquad t \in [0, T].$$

Consider the nonlinear, time-dependent delay system

$$\dot{x}(t) = f(t, x(t), x(t - \tau(t, x(t)))), \qquad t \geq 0 \tag{72}$$
$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{73}$$

We associate the following EPCA to the IVP (72)-(73)

$$\dot{y}_h(t) = f\Big([t]_h, y_h([t]_h), y_h\Big([t]_h - \Big[\tau([t]_h, y_h([t]_h))\Big]_h\Big)\Big), \qquad t \geq 0 \tag{74}$$
$$y_h(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{75}$$

Since y_h is linear between the mesh points, the values $a(k) = y_h(kh)$ uniquely determine the solution. The sequence $a(k)$ satisfies the difference equation

$$a(k+1) = a(k) + f\Big(kh, a(k), a(k - d_k)\Big) \cdot h, \qquad k = 0, 1, 2, \dots,$$
$$a(-k) = \varphi(-kh), \qquad k = 0, 1, 2, \dots, \qquad -r \leq -kh \leq 0,$$

where

$$d_k \equiv \Big[\tau(kh, a(k))/h\Big].$$

Equations with piecewise-constant arguments (EPCAs)

K. L. Cooke and J. Wiener, JMAA 99 (1984)
K. L. Cooke and J. Wiener, Comp. Math. Appl. 12A (1986)
K. L. Cooke and J. Wiener, Lecture Notes in Math 1475, (1991)

Numerical approximation with EPCAs

I. Győri, Int. J. Math. & Math. Sci. 14:1 (1991)
I. Győri, F. Hartung and J. Turi, Appl. Math. Letters 8:6 (1995)
F. Hartung, T. L. Herdman, and J. Turi, Appl. Numer. Math. 24 (1997)
I. Győri, F. Hartung, J. Difference Eqns., 14.8:11 (2002) 983-999.

Consider again the delayed logistic equation

$$\dot{N}(t) = N(t)\Big(r - mN(t-\tau)\Big), \qquad t \geq 0, \tag{76}$$

with initial condition

$$N(t) = \varphi(t), \qquad -\tau \leq t \leq 0, \tag{77}$$

Pick a discretization parameter $h > 0$, and consider a partially discretized approximating equation

$$\dot{y}_h(t) = y_h(t)\Big(r - my_h([t]_h - [\tau]_h)\Big), \qquad t \geq 0.$$

Then we get

$$\dot{y}_h(t) = y_h(t)\Big(r - my_h((k-\ell)h)\Big), \qquad t \in [kh, (k+1)h),$$

where $\ell \equiv \left[\tau/h\right]$. Therefore

$$y_h(t) = y_h(kh)e^{\left(r-my_h((k-\ell)h)\right)(t-kh)}, \qquad t \in [kh,(k+1)h).$$

We compute the sequence $a(k) = y_h(kh)$ by the explicit recursion

$$\begin{aligned} a(k+1) &= a(k)e^{(r-ma(k-\ell))h}, \qquad k = 0,1,2,\dots, \\ a(-k) &= \varphi(-kh), \qquad k = 0,1,2,\dots, \quad -r \le -kh \le 0, \end{aligned}$$

and then the solution is

$$y_h(t) = a(k)e^{\left(r-ma(k-\ell)\right)(t-kh)}, \qquad t \in [kh,(k+1)h].$$

$$\dot{x}(t) = \sum_{i=1}^{m} a_i x(t - \tau_i) \tag{78}$$

$$\dot{y}_h(t) = \sum_{i=1}^{m} a_i y_h\left([t]_h - [\tau_i]_h\right),$$

Theorem 33 (K. L. Cooke and I. Győri, 1994).

If the zero solution of (78) *is asymptotically stable, then*

$$\lim_{h\to 0+} \max_{t\geq 0} |x(t) - y_h(t)| = 0.$$

Consider the initial value problem corresponding to the neutral functional differential equation (NFDE)

$$\frac{d}{dt}\Big(x(t)+q(t)x(t-\tau(t))\Big)=f\Big(t,x(t),x(t-\sigma(t))\Big) \tag{79}$$

for $t\in[0,T]$, with initial condition

$$x(t)=\varphi(t),\qquad t\in[-r,0]. \tag{80}$$

Suppose τ and σ are continuous functions satisfying

$$0<r_0\le\tau(t)\le r,\qquad 0\le\sigma(t)\le r,\qquad t\ge 0,$$

q and f are continuous.

We associate the following NFDE with piecewise constant right-hand side to (79):

$$\begin{aligned}\frac{d}{dt}\Big(y_h(t)+q([t]_h)y_h(t-[\tau([t]_h)]_h)\Big)\\ = \; f\Big([t]_h,y_h([t]),y_h([t]_h-[\sigma([t]_h))]_h\Big).\end{aligned} \tag{81}$$

The associated initial condition to (81) is

$$y_h(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{82}$$

By a solution of the initial value problem (81)-(82) we mean a function $y_h : [-r, \infty) \to \mathbb{R}^n$, which is defined on $[-r, 0]$ by (82) and satisfies the following properties on $[0, \infty)$:

(i) the function $y_h(t) + q([t]_h)y_h(t - [\tau([t]_h)]_h)$ is differentiable at each point $t \in (0, \infty)$ with the possible exception of the points kh $(k = 0, 1, 2, \dots)$ where finite one-sided derivatives exist,

(ii) y_h satisfies (81) on each interval $[kh, (k+1)h)$ for $k = 0, 1, \dots$.

Let $(k+1)h < t$, and consider

$$\int_{kh}^{t} \frac{d}{ds}\Big(y_h(s) + q([s]_h) y_h(s - [\tau([s]_h)]_h) \Big)\, ds = \int_{kh}^{t} f\Big([s]_h, y_h([s]), y_h([s]_h - [\sigma([s]_h))]_h \Big)\, ds.$$

Then

$$y_h(t) = y_h(kh) + q(kh) y_h(kh - [\tau(kh)]_h) - q([t]_h) y_h(t - [\tau([t]_h)]_h) + \int_{kh}^{t} f\Big([s]_h, y_h([s]_h), y_h([s]_h - [\sigma([s]_h))]_h \Big)\, ds.$$

It is easy to see by the method of steps that for $0 < h < r_0$, y_h exists and it is right-continuous, and has jump discontinuities at the mesh points kh. Taking the limit $t \to (k+1)h+$ yields

$$\begin{aligned} y_h((k+1)h) = {} & y_h(kh) + q(kh) y_h(kh - [\tau(kh)]_h) \\ & - q((k+1)h) y_h((k+1)h - [\tau((k+1)h)]_h) \\ & + hf\Big(kh, y_h(kh), y_h(kh - [\sigma(kh))]_h \Big). \end{aligned}$$

Let $a(k) \equiv y_h(kh)$. Then

$$\begin{aligned} a(k+1) &= a(k) + q(kh)a\left(k - \left[\frac{\tau(kh)}{h}\right]\right) \\ &\quad - q((k+1)h)a\left(k+1 - \left[\frac{\tau((k+1)h)}{h}\right]\right)\Bigg) \\ &\quad + hf\left(kh, a(k), a\left(k - \left[\frac{\sigma(kh)}{h}\right]\right)\right) \\ &\qquad \text{for } k = 0, 1, \dots, \\ a(k) &= \varphi(kh), \quad \text{for } -r \le kh \le 0. \end{aligned}$$

Therefore computing $a(k)$ is a simple numerical task. Note, that this scheme uses approximate solution values only at mesh points. Introduce

$$b_h(k) \equiv \lim_{t \to kh-} y_h(t).$$

The continuity of $y_h(t) + q([t]_h)y_h(t - [\tau([t]_h)]_h)$ at $(k+1)h$ yields

$$
\begin{aligned}
b(k+1) + q(kh)b\left(k+1-\left[\frac{\tau(kh)}{h}\right]\right) \\
= \quad a(k+1) + q((k+1)h)a\left(k+1-\left[\frac{\tau((k+1)h)}{h}\right]\right),
\end{aligned}
$$

and so

$$
\begin{aligned}
b(k+1) &= a(k+1) + q((k+1)h)a\left(k+1-\left[\frac{\tau((k+1)h)}{h}\right]\right) \\
&\quad -q(kh)b\left(k+1-\left[\frac{\tau(kh)}{h}\right]\right), \qquad k = 0, 1, \ldots, \\
b(-k) &= \varphi(-kh), \qquad k \in \mathbb{N}, \quad kh \leq r.
\end{aligned}
$$