Application of Delay Equations

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Consider the autonomous linear delay equation

$$\dot{x}(t) = -\sum_{k=0}^{m} a_k x(t - \tau_k) + f(t), \qquad t \geq 0. \tag{24}$$

We assume $a_0 \in \mathbb{R}$ $(k = 1, \ldots, m)$, and $0 = \tau_0 < \tau_1 < \ldots < \tau_m$.
It is easy to prove by the method of steps that (24) has a unique solution corresponding to any initial condition

$$x(t) = \varphi(t), \qquad t \in [-\tau, 0]. \tag{25}$$

As for ODEs, there is a variation-of-constants formula which expresses the solution of the inhomogeneous equation in terms of the the solution of the associated homogeneous equation

$$\dot{y}(t) = -\sum_{k=0}^{m} a_k y(t - \tau_k), \qquad t \geq 0, \tag{26}$$

and the so-called fundamental solution of (26).

$$\dot{y}(t) = -\sum_{k=0}^{m} a_k y(t - \tau_k), \qquad t \geq 0, \tag{26}$$

The fundamental solution of (26) is the solution of the IVP

$$\dot{v}(t) = -\sum_{k=0}^{m} a_k v(t - \tau_k), \qquad t \geq 0 \tag{27}$$

$$v(t) = \begin{cases} 1, & t = 0, \\ 0, & t \in [-\tau_m, 0). \end{cases} \tag{28}$$

Then we have

$$x(t) = y(t) + \int_0^t v(t-s) f(s)\, ds, \qquad t \geq 0.$$

Here y is the solution of the homogeneous equation (26) which can be written in the form

$$y(t) = v(t)\varphi(0) - \sum_{k=0}^{m} a_k \int_{-\tau_k}^{0} v(t - s - \tau_k)\varphi(s)\, ds, \qquad t \geq 0.$$

$$\dot{y}(t) = -\sum_{k=0}^{m} a_k y(t-\tau_k), \qquad t \geq 0, \tag{26}$$

Seeking the solution of (26) in the form of $y(t) = e^{\lambda t}$, we get

$$\lambda e^{\lambda t} = -\sum_{k=0}^{m} a_k e^{\lambda(t-\tau_k)}.$$

Therefore the characteristic equation of (26) is

$$\lambda = -\sum_{k=0}^{m} a_k e^{-\lambda \tau_k}. \tag{29}$$

$$\dot{y}(t) = -\sum_{k=0}^{m} a_k y(t - \tau_k), \qquad t \geq 0, \tag{26}$$

Theorem 7.

Let $a_k \in \mathbb{R}$ *and* $\tau_k \geq 0$ *be fixed. The following statements are equivalent:*

(i) *The trivial solution of (26) is asymptotically stable.*

(ii) *For any solution* $y : [-\tau, \infty) \to \mathbb{R}$ *of (26),* $y(t) \to 0,\ t \to +\infty$.

(iii) *Any characteristic root* $\lambda \in \mathbb{C}$ *of (29) satisfies* $\mathrm{Re}\,\lambda < 0$.

(iv) *The fundamental solution* $v(t)$ *of* (26) *tends to 0 exponentially as* $t \to \infty$.

(v) *The fundamental solution* $v(t)$ *of* (26) *is in* $L^1[0, \infty)$*, i.e.,*

$$\int_0^\infty |v(t)|\, dt < \infty.$$

$$\dot{y}(t) = -\sum_{k=0}^{m} a_k y(t-\tau_k), \qquad t \geq 0, \tag{26}$$

Remark 8.

If the trivial solution of (26) *is asymptotically stable, then*

$$\int_0^\infty v(t)\, dt = \frac{1}{\sum_{k=0}^m a_k}.$$

Theorem 9.

The fundamental solution v *of* (26) *is positive on* $[0,\infty)$*, if and only if the characteristic equation* (29) *has a real root.*

$$\dot{y}(t) = -\sum_{k=0}^{m} a_k y(t - \tau_k), \qquad t \geq 0, \tag{26}$$

Theorem 10.

Suppose $a_k \geq 0$ $(k = 0, \dots, m)$. *Then*

(i) *If*

$$\sum_{k=1}^{m} a_k \tau_k > \frac{1}{e},$$

then all solutions of (26) *(including the fundamental solution) are oscillatory, i.e., have arbitrary large zeros.*

(ii) *If*

$$\left(\sum_{k=0}^{m} a_k\right) \tau_m \leq \frac{1}{e},$$

then there exists a nonoscillatory solution of (26), *in particular, the fundamental solution* $v(t)$ *of* (26) *is positive for* $t > 0$.

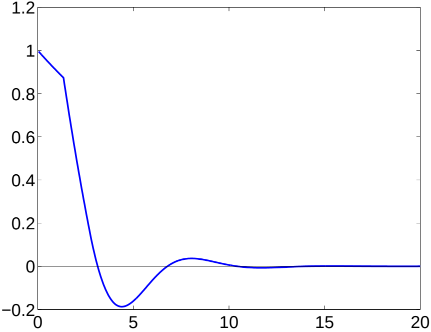
Consider the homogeneous equation

$$\dot{x}(t) = -ax(t) - bx(t-\tau), \qquad t \geq 0, \tag{30}$$

where $a, b \in \mathbb{R}$ and $\tau \geq 0$.
The fundamental solution of (30) is the solution of the initial value problem

$$\dot{v}(t) = -av(t) - bv(t-\tau), \qquad t \geq 0 \tag{31}$$

$$v(t) = \begin{cases} 1, & t = 0, \\ 0, & t \in [-\tau, 0). \end{cases} \tag{32}$$



$$\dot{v}(t) = -0.1v(t) - 0.5v(t - 1.4)$$

Seeking the solution of (30) in the form of $y(t) = e^{\lambda t}$, we get the characteristic equation of (30):

$$\lambda = -a - be^{-\lambda\tau}. \tag{33}$$

Introduce the new variable

$$y(t) = x(\tau t).$$

Then

$$\dot{y}(t) = \tau\dot{x}(\tau t) = \tau\Big(-ax(\tau t) - bx(\tau t - \tau)\Big) = -a\tau y(t) - b\tau y(t-1),$$

i.e., y is the solution of

$$\dot{y}(t) = -Ay(t) - By(t-1), \tag{34}$$

where $A = a\tau$, $B = b\tau$.
The characteristic equation of (34) is

$$\lambda = -A - Be^{-\lambda}. \tag{35}$$

$$\dot{x}(t) = -ax(t) - bx(t-\tau) \tag{30}$$

$$\dot{y}(t) = -Ay(t) - By(t-1), \tag{34}$$

Remark 11.

(i) $x(t) \to 0$ *as* $t \to \infty \iff y(t) \to 0$ *as* $t \to \infty$

(ii) $x(t)$ *oscillates* $\iff y(t)$ *oscillates*

$$\dot{y}(t) = -Ay(t) - By(t-1) \tag{34}$$

Introduce the new variable $z(t) = e^{At}y(t)$. Then

$$\begin{aligned} \dot{z}(t) &= Ae^{At}y(t) + e^{At}\dot{y}(t) \\ &= Ae^{At}y(t) + e^{At}\Big(-Ay(t) - By(t-1)\Big) \\ &= -Be^{At}\tau z(t-1), \end{aligned}$$

i.e., z is the solution of

$$\dot{z}(t) = -cz(t-1), \tag{36}$$

where $c = Be^{At}$.
The characteristic equation of (36) is

$$\lambda = -ce^{-\lambda}. \tag{37}$$

$$\dot{x}(t) = -ax(t) - bx(t-\tau) \tag{30}$$

$$\dot{y}(t) = -Ay(t) - By(t-1) \tag{34}$$

$$\dot{z}(t) = -cz(t-1) \tag{36}$$

Remark 12.

$x(t)$ *oscillates* $\iff$ $y(t)$ *oscillates* $\iff$ $z(t)$ *oscillates*

$$\lambda = -ce^{-\lambda} \tag{37}$$

Lemma 13.

(i) *If* $0 < c < \frac{1}{e}$, *then* (37) *has exactly two real roots,* λ_1, λ_2, *which satisfy*

$$\lambda_1 < \lambda_2 < 0.$$

(ii) *If* $c = \frac{1}{e}$, *then* (37) *has a unique real root,* $\lambda_0 = -1$, *which is a double root.*

(iii) *If* $c > \frac{1}{e}$, *then* (37) *has no real root.*

(iv) *If* $c < 0$, *then* (37) *has a unique real root* $\lambda_0 > 0$.

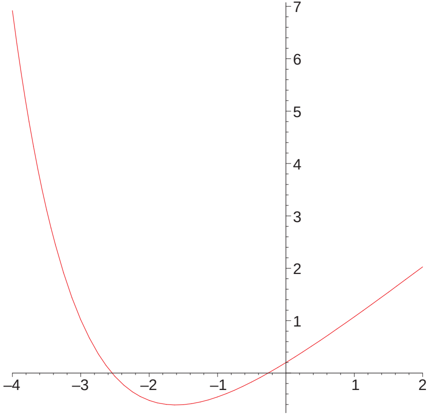
Moreover, in Case (i) and (iv) all real roots are simple, and all complex roots have smaller real part than the largest real root.

$$\lambda = -ce^{-\lambda} \tag{37}$$

We say that $\tilde{\lambda} = \tilde{\alpha} + i\tilde{\beta}$ is a leading characteristic root of (37), if it is a solution of the characteristic equation (37) and for any other root $\lambda = \alpha + i\beta$ of (37), $\alpha < \tilde{\alpha}$ holds, assuming $\lambda \neq \tilde{\alpha} \pm i\tilde{\beta}$.
Consider the function

$$g(\lambda) = \lambda + ce^{-\lambda}.$$

Then the characteristic equation (37) can be re-written as $g(\lambda) = 0$.

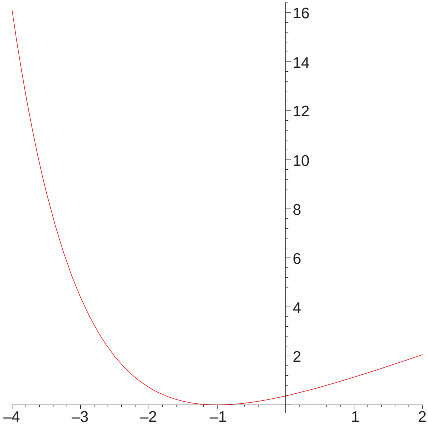


Graph of $g(\lambda) = \lambda + 0.2e^{-\lambda}$

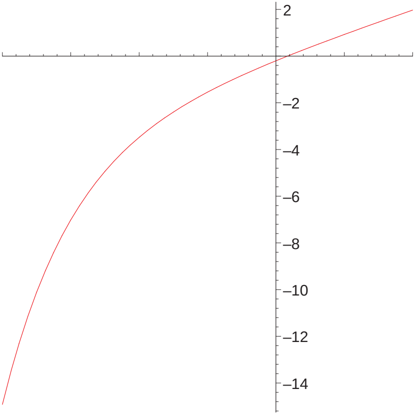
λ_0 is a double root of g, if and only if

$$g(\lambda_0) = \lambda_0 + ce^{-\lambda_0} = 0 \quad \text{and} \quad g'(\lambda_0) = 1 - ce^{-\lambda_0} = 0.$$

Then $\lambda_0 = -1$ and $c = \frac{1}{e} \approx 0.3679$.



Graph of $g(\lambda) = \lambda + \frac{1}{e}e^{-\lambda}$



Graph of $g(\lambda) = \lambda - 0.2e^{-\lambda}$

$$\lambda = -ce^{-\lambda} \tag{37}$$

Now look for complex roots of (37). Let $\lambda = \alpha + i\beta$. Then (37) yields

$$\alpha + i\beta = -ce^{-(\alpha+i\beta)},$$

and so

$$\alpha + i\beta = -ce^{-\alpha}(\cos\beta - i\sin\beta).$$

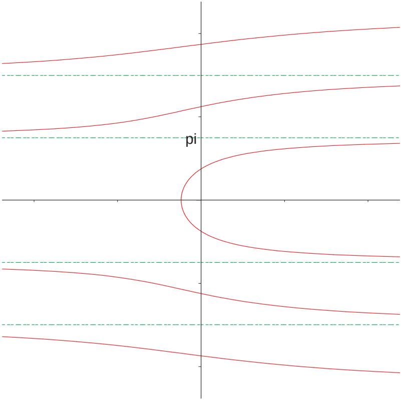
Comparing the real and complex parts of the two sides we get

$$\begin{aligned} \alpha &= -ce^{-\alpha}\cos\beta \\ \beta &= ce^{-\alpha}\sin\beta \end{aligned}$$

Since we investigate the case of the complex roots of (37), we can assume that $\beta \neq 0$. In this case simple algebraic manipulation of these equations yields the equivalent system

$$\alpha = -\beta\cot\beta, \tag{38}$$

$$\beta^2 = c^2e^{-2\alpha} - \alpha^2. \tag{39}$$

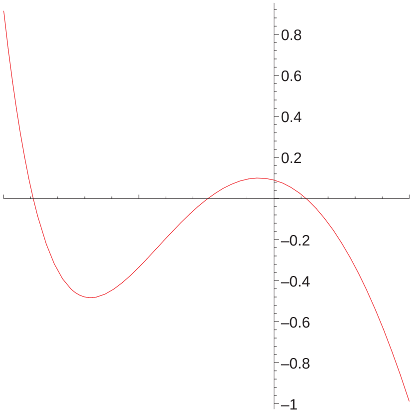


The graph of $\alpha = -\beta \cot \beta$.

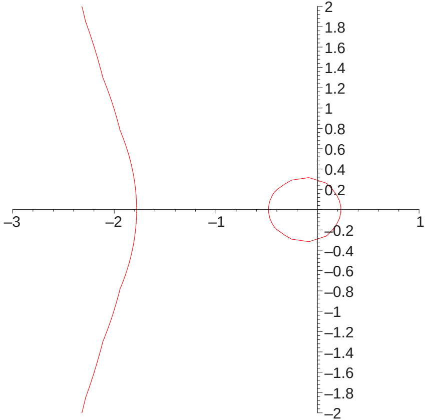
We define the function

$$h(u) = c^2 e^{-2u} - u^2.$$

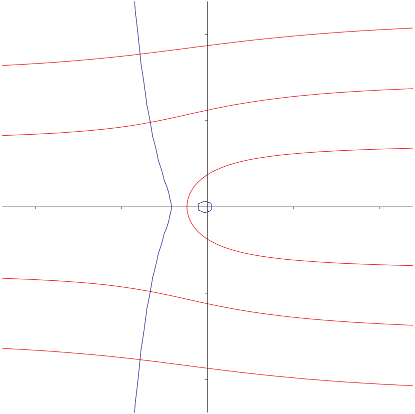
Then (39) is equivalent to $\beta^2 = h(\alpha)$ or $\beta = \pm\sqrt{h(\alpha)}$.



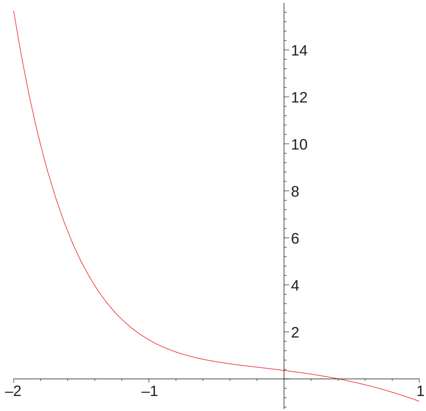
Graph of $h(u)$, $c = 0.3$



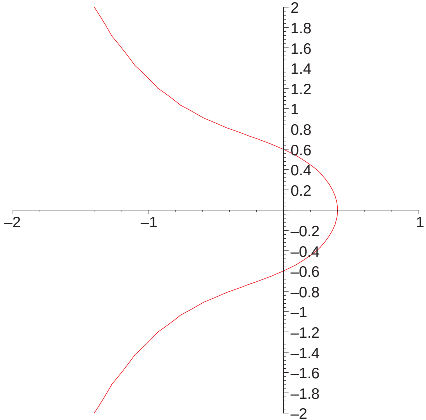
Graph of $\pm\sqrt{h(u)}$, $c = 0.3$



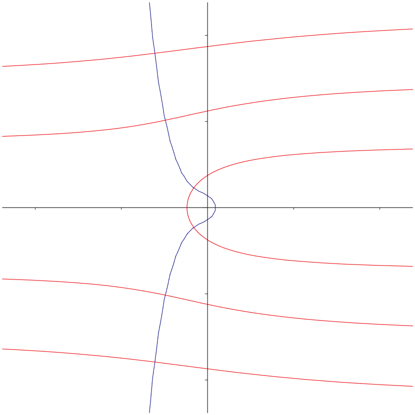
Characteristic roots, $c = 0.3$.



Graph of $h(u)$, $c = 0.6$



Graph of $\pm\sqrt{h(u)}$, $c = 0.6$



Characteristic roots, $c = 0.6$.

$$\lambda = -ce^{-\lambda} \tag{37}$$

Remark 14.

It is known that (37) *always has a leading root. Moreover, if* (λ_n) *is a sequence of characteristic roots of* (37) *such that* $|\lambda_n| \to \infty$ *as* $n \to \infty$*, then* $\mathrm{Re}\,\lambda_n \to \infty$ *as* $n \to \infty$*. There are only a finite number of characteristic roots of* (37) *in any vertical strip in the complex plane.*

$$\dot{y}(t) = -by(t-\tau) \tag{40}$$

$$\lambda = -be^{-\lambda\tau} \tag{41}$$

Theorem 15.

Let $b > 0$ *and* $\tau > 0$ *be fixed. The following statements are equivalent:*

(i) *The trivial solution of (40) is asymptotically stable.*

(ii) *For any solution* $y : [-\tau, \infty) \to \mathbb{R}$ *of (40),* $y(t) \to 0,\ t \to +\infty$.

(iii) *Any characteristic root* $\lambda \in \mathbb{C}$ *of (41) satisfies* $\mathrm{Re}\,\lambda < 0$.

(iv) *The fundamental solution* $v(t)$ *of* (40) *tends to 0 exponentially as* $t \to \infty$.

(v) *The fundamental solution* $v(t)$ *of* (40) *is in* $L^1[0,\infty)$, *i.e.,*

$$\int_0^\infty |v(t)|\, dt < \infty.$$

(vi) $b\tau < \frac{\pi}{2}$.

$$\dot{y}(t) = -by(t-\tau) \qquad (40)$$

Theorem 16.

The following statements are equivalent:

(i) *The solutions of* (40) *are non-oscillatory.*

(ii) *The fundamental solution* v *of* (40) *is positive on* $[0,\infty)$.

(iii) *The leading characteristic root of* (41) *is real.*

(iv) $0 < b\tau \le \frac{1}{e}$.

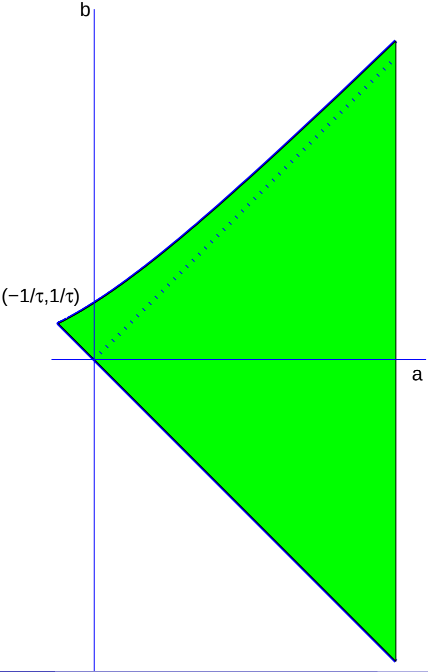
$$\dot{x}(t) = -ax(t) - bx(t-\tau) \tag{30}$$

The asymptotic behavior of the solution of equation (30) is well-known.

Theorem 17.

The trivial (zero) solution of equation (30) is asymptotically stable independently of the delay if and only if $a > |b|$. Moreover, the exact stability region of the trivial solution of (30) is bounded by the line $b = -a$ and by the curve

$$a = -s\cot(\tau s), \quad b = \frac{s}{\sin(\tau s)}, \qquad s \in \left[0, \frac{\pi}{\tau}\right].$$


b
(−1/τ,1/τ)
a

$$\dot{y}(t) = -ay(t) - by(t-\tau) \tag{30}$$
$$\lambda = -a - be^{-\lambda\tau}. \tag{33}$$

Theorem 18.

Let $a, b \in \mathbb{R}$, $\tau > 0$.

(i) *A leading characteristic root* $\lambda_0 = \alpha_0 + i\beta_0$ *of* (33) *is a real number, if and only if*

$$b\tau e^{a\tau} \leq \frac{1}{e}.$$

(ii) *The fundamental solution* $v(t)$ *of* (30) *is positive for* $t \geq 0$ *and* $v(t) \to 0$ *as* $t \to \infty$*, if and only if*

$$-a < b \leq \frac{1}{e\tau} e^{-a\tau}.$$

Moreover, in this case $\int_0^\infty v(t)\,dt = \frac{1}{a+b}$.

$$\lambda = -a - be^{-\lambda\tau}. \tag{33}$$

Lemma 19.

Let $a, b \in \mathbb{R}$, $\tau > 0$.

(i) *If* $0 < b\tau e^{a\tau} < \frac{1}{e}$, *then* (33) *has exactly two real roots,* λ_1, λ_2, *which satisfy*

$$\lambda_1 < -a + \frac{1}{\tau}\log(b\tau) < -a - \frac{1}{\tau} < \lambda_2 < -a.$$

(ii) *If* $b\tau e^{a\tau} = \frac{1}{e}$, *then* (33) *has a unique real root,* $\lambda_0 = -a - \frac{1}{\tau}$, *which is a double root.*

(iii) *If* $b\tau e^{a\tau} > \frac{1}{e}$, *then* (33) *has no real root.*

(iv) *If* $b < 0$, *then* (33) *has a unique real root* $\lambda_0 > -a$.

Moreover, in Case (i) and (iv) all real roots are simple, and all complex roots have smaller real part than the largest real root.

Consider a nonautonomous linear homogeneous delay differential equation

$$\dot{y}(t) = \sum_{k=0}^{m} a_k(t) y(t - \tau_k(t)), \qquad t \geq 0, \tag{42}$$

Theorem 20.

(i) *If*

$$\sum_{k=0}^{m} \sup_{t\geq 0} |a_k(t)| \sup_{t\geq 0} \tau_k(t) < 1,$$

then the trivial solution of (42) *is asymptotically stable.*

(ii) *If* $a_k(t) \equiv a_k$ $(k = 0, 1, \ldots, m)$, *and*

$$\sum_{k=0}^{m} a_k \sup_{t\geq 0} \tau_k(t) < \frac{3}{2},$$

then the trivial solution of (42) *is asymptotically stable.*

(iii) *If* $a_k(t) \equiv a_k$ *and* $\tau_k(t) \equiv \tau_k$ $(k = 0, 1, \ldots, m)$, *and*

$$\sum_{k=0}^{m} a_k \tau_k < \frac{\pi}{2},$$

then the trivial solution of (42) *is asymptotically stable.*

$$\dot{x}(t) = -\sum_{i=1}^{m} a_i x(t - \tau_i - \eta_i(t)), \qquad t \geq 0, \tag{43}$$

$$\dot{y}(t) = -\sum_{i=1}^{m} a_i y(t - \tau_i), \qquad t \geq 0. \tag{44}$$

Here $\eta_i \colon [0,\infty) \to [0,\infty)$ are piecewise continuous bounded functions.

Theorem 21 (Győri, Hartung, Turi, 1998).

Suppose that the trivial solution of (44) *is asymptotically stable, and*

$$\sum_{i=1}^{m} |a_i| \varlimsup_{t\to\infty} |\eta_i(t)| < \frac{1}{\left(\sum_{i=1}^{m} |a_i|\right) \int_0^\infty |v(t)|\, ds}, \tag{45}$$

where v *is the fundamental solution of* (44). *Then the trivial solution of* (43) *is asymptotically stable, as well.*

Idea of the proof:

$$\dot{x}(t) = -\sum_{i=1}^{m} a_i x(t-\tau_i) + \sum_{i=1}^{m} a_i \Big(x(t-\tau_i) - x(t-\tau_i-\eta_i(t)) \Big), \qquad t \geq 0,$$

Then

$$x(t) = y(t) + \sum_{i=1}^{m} a_i \int_0^t v(t-s) \Big(x(s-\tau_i) - x(s-\tau_i-\eta_i(s)) \Big) ds$$

For large s we have

$$\begin{aligned} x(s-\tau_i) - x(s-\tau_i-\eta_i(s)) &= \int_{s-\tau_i-\eta_i(s)}^{s-\tau_i} \dot{x}(u)\, du \\ &= -\sum_{j=1}^{m} a_j \int_{s-\tau_i-\eta_i(s)}^{s-\tau_i} x(u-\tau_j-\eta_j(u))\, du \end{aligned}$$

Let $\tau > 0$, and u be the solution of the initial value problem

$$\dot{u}(t) = -u(t-\tau), \qquad t \geq 0, \tag{46}$$

$$u(t) = \begin{cases} 1, & t = 0, \\ 0, & t < 0, \end{cases} \tag{47}$$

i.e., u is the fundamental solution of the scalar delay differential equation

$$\dot{x}(t) = -x(t-\tau), \qquad t \geq 0. \tag{48}$$

If we want to emphasize that the fundamental solution corresponds to delay τ, we use the notation $u(t;\tau)$.
Let $\lambda = \alpha_0 + i\beta_0$ be the leading root of the characteristic equation

$$\lambda = -e^{-\lambda\tau}. \tag{49}$$

It is known that $\alpha_0 < 0$ if and only if $\tau \in [0, \pi/2)$.

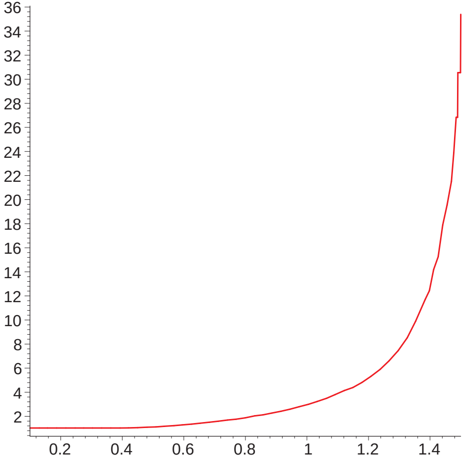
The trivial solution of (48) is asymptotically stable, if and only if $\int_0^\infty |u(t;\tau)|\,ds < \infty$. We introduce the function

$$\Phi(\tau) = \int_0^\infty |u(t;\tau)|\,dt. \tag{50}$$

Then $\Phi(\tau) = \infty$ for $\tau \geq \pi/2$. It is known that $u(t;\tau) > 0$ for $t > 0$, if and only if $\tau \leq 1/e$.
For $\tau \leq 1/e$ it follows

$$\Phi(\tau) = \int_0^\infty u(t;\tau)\,dt = \frac{1}{1} = 1.$$

For $1/e < \tau < \pi/2$ numerical estimate of Φ yields the next figure.



The graph of $\Phi(\tau)$

Open problem: Is Φ monotone increasing?

$$\lambda = -e^{-\lambda\tau}. \tag{49}$$

Theorem 22 (Győri, 1989).

For $\tau \in [0, \pi/2)$ the characteristic equation (49) has a root $\lambda_0 = \alpha_0 + i\beta_0$, such that $\alpha_0 < 0$, $\beta_0 \in [0, \pi/(2\tau))$, α_0 is the greatest real part of the roots of (49), and

$$\Phi(\tau) \le \frac{\alpha_0^2 + \beta_0^2}{\alpha_0^2}. \tag{51}$$

Note inequality (51) is exact for $\tau \in [0, 1/e]$, since then $\beta_0 = 0$, and $\Phi(\tau) = 1$.

Let $a_i > 0$ $(i = 1, \ldots, m)$, and consider the linear delay equation

$$\dot{x}(t) = -\sum_{i=1}^{m} a_i x(t - \sigma_i(t)), \qquad t \geq 0. \tag{52}$$

We can consider Equation (52) as the delay perturbation of

$$\dot{y}(t) = -\left(\sum_{i=1}^{m} a_i\right) y(t - \tau) \tag{53}$$

with the perturbations $\eta_i(t) = \sigma_i(t) - \tau$, where $\tau \geq 0$. Let v denote the fundamental solution of (53), then $\dot{v}(t) = -(\sum_{i=1}^{m} a_i)v(t - \tau)$. Therefore an application of Theorem 21 yields that if $0 \leq \tau \sum_{i=1}^{m} a_i < \pi/2$, and

$$\sum_{i=1}^{m} a_i \varlimsup_{t\to\infty} |\sigma_i(t) - \tau| < \frac{1}{(\sum_{i=1}^{m} a_i) \int_0^\infty |v(t)|\, dt}, \tag{54}$$

then the trivial solution of (52) is asymptotically stable. Introducing

$u(t) = v(t/\sum_{i=1}^m a_i)$ we get

$$\dot{u}(t) = \frac{1}{\sum_{i=1}^m a_i}\dot{v}\left(\frac{t}{\sum_{i=1}^m a_i}\right) = -v\left(\frac{t}{\sum_{i=1}^m a_i} - \tau\right) = -u\left(t - \tau\sum_{i=1}^m a_i\right).$$

On the other hand,

$$\begin{aligned}\Phi\left(\tau\sum_{i=1}^m a_i\right) &= \int_0^\infty |u(t)|\,dt = \int_0^\infty \left|v\left(\frac{t}{\sum_{i=1}^m a_i}\right)\right| dt \\ &= \left(\sum_{i=1}^m a_i\right)\int_0^\infty |v(t)|\,dt.\end{aligned}$$

Therefore, using the relation

$$\overline{\lim_{t\to\infty}}|f(t)| = \max\left\{\overline{\lim_{t\to\infty}} f(t), -\underline{\lim_{t\to\infty}} f(t)\right\}, \tag{55}$$

we get immediately the following result.

$$\dot{x}(t) = -\sum_{i=1}^{m} a_i x(t - \sigma_i(t)) \tag{52}$$

Theorem 23 (Győri, Hartung, 2001).

Suppose $a_i > 0$, $\sigma_i \colon [0, \infty) \to [0, \infty)$ is piecewise continuous $(i = 1, \dots, m)$, and there exists $\tau \in [0, \pi/(2a))$ such that

$$\tau a - \frac{1}{\Phi(\tau a)} < \sum_{i=1}^{m} a_i \varliminf_{t\to\infty} \sigma_i(t) \le \sum_{i=1}^{m} a_i \varlimsup_{t\to\infty} \sigma_i(t) < \tau a + \frac{1}{\Phi(\tau a)}, \tag{56}$$

where $a \equiv \sum_{i=1}^{m} a_i$. Then the trivial solution of (52) is asymptotically stable.

Note that the first inequality of (56) is automatically satisfied if $0 \le \tau a \le 1/e$, since then $\Phi(\tau a) = 1$. See the next figure for the numerically generated graph of the functions $\tau + 1/\Phi(\tau)$ and $\tau - 1/\Phi(\tau)$.

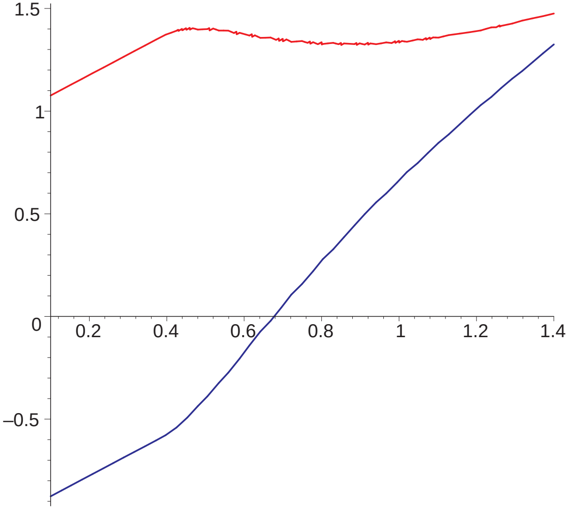


Figure: The graphs of $\tau + 1/\Phi(\tau)$ and $\tau - 1/\Phi(\tau)$

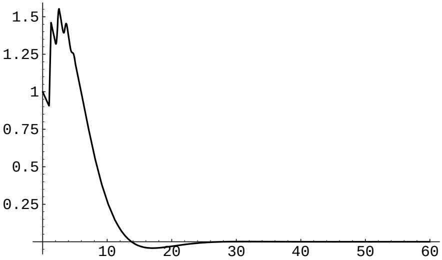
Consider the one-dimensional control system

$$\dot{x}(t) = -0.1x(t) + 2x(t-1) + Ku(t). \tag{57}$$

For $K = 0$ Eq. (57) is unstable. Let $K = -2$ and $u(t) = x(t-1.3)$ in (57). Numerical approximation of the fundamental solution of the corresponding equation

$$\dot{x}(t) = -0.1x(t) + 2x(t-1) - 2x(t-1.3). \tag{58}$$

yields



Fundamental solution of (58).

Therefore the feedback law $Ku(t) = -2x(t-1.3)$ stabilizes (57). Suppose that we sample the system only at the points $h, 2h, 3h, \ldots$, and use a piecewise constant feedback control $u(t) = x([(t-1.3)/h]h)$.

Objective: how large can h be that still the trivial solution of the resulting hybrid feedback system

$$\dot{x}(t) = -0.1x(t) + 2x(t-1) - 2x\left(\left[\frac{t-1.3}{h}\right]h\right). \qquad (59)$$

remains asymptotically stable? The piecewise constant delay in the last term in (59) can be considered as a perturbation of $t-1.3$ in (58) with

$$\eta(t) = t - 1.3 - \left[\frac{t-1.3}{h}\right]h.$$

Then we have that $|\eta(t)| \leq h$ for all $t \geq 0$. Numerical approximation gives that the fundamental solution of (58) satisfies $\int_0^\infty |v(t)|\,dt = 10.5914$. Therefore by Theorem 23 we have as a sufficient condition that $h < \frac{1}{10.5914 \cdot 8.2} = 0.0115$ guarantees that the trivial solution of (59) is asymptotically stable.

Consider the delay equation

$$\dot{x}(t) = -x(t - \sigma(t)), \qquad t \geq 0. \tag{60}$$

Theorem 24 (Myshkis, 1972).

If

$$\sup_{t \geq 0} \sigma(t) < 3/2,$$

then the trivial solution of (60) *is asymptotically stable.*

Myshkis gave an example, where $\sup_{t\geq 0} \sigma(t) \in (3/2, \pi/2)$ and the corresponding equation has unstable trivial solution. Note that in his example $\underline{\lim}_{t\to\infty} \sigma(t) = 0$.

$$\dot{x}(t) = -x(t - \sigma(t)) \tag{60}$$

Theorem 25 (Ladas, Sficas, Stavroulakis, 1983).

Suppose

$$\lim_{t\to\infty} \sigma(t) < \frac{\pi}{2},$$

then the trivial solution of (60) *is asymptotically stable.*

Theorem 23 has the following corollary:

Corollary 26.

For any $c \in (3/2, \pi/2)$ *there exists* $b < c$*, such that the trivial solution of* (60) *is asymptotically stable, if*

$$b < \varliminf_{t\to\infty} \sigma(t) \leq \varlimsup_{t\to\infty} \sigma(t) < c.$$

Consider

$$\dot{x}(t) = -a(t)x(t-\sigma(t)), \qquad t \geq 0, \tag{61}$$

where $a(t) \geq 0$ for $t \geq 0$, $\int_0^\infty a(t)\,dt = \infty$.

Theorem 27 (Yoneyama, 1987).

Suppose

$$0 < \inf_{t\geq 0} \int_{t-\sigma_0}^{t} a(s)\,ds \leq \sup_{t\geq 0} \int_{t-\sigma_0}^{t} a(s)\,ds < \frac{3}{2},$$

where $0 \leq \sigma(t) \leq \sigma_0$ *for* $t \geq 0$. *Then the trivial solution of* (61) *is asymptotically stable.*

Theorem 28 (Ladas, Sficas, Stavroulakis, 1983).

Suppose $\sigma(t) \equiv \sigma$,

$$\lim_{t\to 0} \int_{t-\sigma}^{t} a(s)\,ds < \frac{\pi}{2}.$$

Then the trivial solution of (61) *is asymptotically stable.*

$$\dot{x}(t) = -a(t)x(t - \sigma(t)) \tag{61}$$

Theorem 29 (Győri, Hartung, 2001).

Suppose $a\colon [0,\infty) \to [0,\infty)$ *is continuous, the function* $A(t) = \int_0^t a(s)\,ds$ *is strictly monotone increasing,* $\int_0^\infty a(t)\,dt = \infty$*, and* $\sigma\colon [0,\infty) \to [0,\infty)$ *is piecewise continuous and bounded, and assume there exists* $\tau \in [0, \pi/2)$ *such that*

$$\tau - \frac{1}{\Phi(\tau)} < \varliminf_{t\to\infty} \int_{t-\sigma(t)}^{t} a(s)\,ds \le \varlimsup_{t\to\infty} \int_{t-\sigma(t)}^{t} a(s)\,ds < \tau + \frac{1}{\Phi(\tau)}. \tag{62}$$

Then the trivial solution of (61) *is asymptotically stable.*