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ON THE ALGEBRAIC STRUCTURE OF PRIMITIVE RECURSIVE FUNCTIONS

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§ 0. In this paper we consider functions f from N to N. By $\boldsymbol{o}, \boldsymbol{s}, \boldsymbol{p}, sg$ we mean the functions which are given by

$$\boldsymbol{o}(n) = 0, \qquad \boldsymbol{s}(n) = n + 1, \qquad \boldsymbol{p}(n) = \begin{cases} 0 & \text{if } n = 0, \\ n - 1 & \text{if } n > 0, \end{cases} \qquad sg(n) = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \end{cases}$$

respectively. For any $c \in \mathsf{N}$ let $\tilde{c}$ be the function from N to N which is constant equal c and by $\boldsymbol{a}(n)$ we mean the quadratic residuum of n, i.e. the distance between n and the greatest square number not greater than n. By $\circ$ and $+$ we denote the operators of composition and addition of arithmetical functions respectively. For an arbitrary function $f\colon \mathsf{N} \to \mathsf{N}$ and a natural number m we denote by $f^{\square(m)}$ the iteration of f from place m, i.e. $f^{\square(m)}$ is inductively defined by

$$f^{\square(m)}(0) = m, \qquad f^{\square(m)}(n + 1) = f(f^{\square(m)}(n)).$$

Instead of $\square(0)$ we write $\square$.

The first characterizations of the class PR of all primitive recursive functions of only one variable ([9]) and of the class R of all general recursive functions of only one variable ([6]) were rather complicated. Gradually the characterizations and the proofs was simplified. The strongest result for PR seems to be the following, proved by JULIA ROBINSON in [7]: PR can be generated from two suitable functions u, v by the help of the operators $\circ$ and $\square$. However, there does not exist a single function which generates PR with these operators. This fact is my Theorem 3, a special case of my Theorem 2 or Theorem 2A. In [7] J. ROBINSON proves a similar result: There is no single function from which PR can be obtained by $\circ$ and $\square(m)$, where various values of m may be used. Her result is another generalization of my Theorem 3, but my proof seems easier to understand.

Similar results were proved by J. ROBINSON for R in [6] and [8] (some results of [6] can also be found in [4]). Namely she proves in [6] that there are two suitable complicated functions which generate R by the help of the operators $\circ$ and $^{-1}$ (where $f^{-1}(x) = \min\{y\colon f(y) = x\}$ for a surjective function f). For proving this fact she uses a certain operation $*$ "mirror", but my Theorem 1 says that her method is not applicable in the general case P, because this operation is an endomorphism on $\langle \mathrm{PR}, \circ, \square \rangle$. In [8] she examines finally the so-called generalized recursion scheme and proves that every general recursive function of one variable can be obtained from $\boldsymbol{o}$ and $\boldsymbol{s}$ by repeated compositions and general recursions from previously defined functions. If we allow only one of the operators $\circ$ and $\square$, it is easy to see that we need infinitely many initial functions but till now I have not found a really good set of such initial functions.

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§ 1. In this section we are dealing with the endomorphisms of the structure $\langle \mathrm{PR}, \circ, \square \rangle$. For every f we have $f^{\square\square} = \boldsymbol{o}$, therefore $\boldsymbol{o}$ is the only fixed point of $\square$. We denote by $\mathrm{End}(\mathfrak{A})$ the set of all endomorphisms of an algebraic structure $\mathfrak{A}$. It is obvious that $\boldsymbol{o}$ is the null-element of PR, i.e. $\boldsymbol{o} \circ \boldsymbol{o} = \boldsymbol{o}$ and $\boldsymbol{o}^{\square} = \boldsymbol{o}$. So we get that $\boldsymbol{Id}$ and $\boldsymbol{O}$ are elements of $\mathrm{End}(\langle \mathrm{PR}, \circ, \square \rangle)$, where $\boldsymbol{Id}(f) = f$ and $\boldsymbol{O}(f) = \boldsymbol{o}$ for every element f of PR. In Theorem 1 we prove that $\mathrm{End}(\langle \mathrm{PR}, \circ, \square \rangle) = \{\boldsymbol{Id}, \boldsymbol{O}\}$. It is easy to see that for $c \in \mathsf{N}$, $c \neq 0$, if $\boldsymbol{L}(f) = \tilde{c}$ for every $f \in \mathrm{PR}$, then $\boldsymbol{L} \in \mathrm{End}(\langle \mathrm{PR}, \circ \rangle)$ and $\boldsymbol{L} \notin \mathrm{End}(\langle \mathrm{PR}, \square \rangle)$. Conversely $\boldsymbol{L}_{sg}$, $\square \in \mathrm{End}(\langle \mathrm{PR}, \square \rangle) - \mathrm{End}(\langle \mathrm{PR}, \circ \rangle)$, where $\square(f) = f^{\square}$ and $\boldsymbol{L}_{sg}(f) = sg \circ f \circ sg$.

Lemma 1. *Let $u, v \in \mathrm{PR}$ be arbitrary functions such that $v \circ u \neq id$ and u is not constant. Let $\boldsymbol{L}(f) = u \circ f \circ v$ for every $f \in \mathrm{PR}$. Then $\boldsymbol{L} \notin \mathrm{End}\langle \mathrm{PR}, \circ \rangle$.*

Proof. Let x_1, x_2, y and z natural numbers such that $v \circ u(z) = y \neq z$ and $u(x_1) \neq u(x_2)$. Furthermore let $f, g \in \mathrm{PR}$ such that $g(v(0)) = z$ and $f(z) = x_1$, $f(y) = x_2$. Then

$$\boldsymbol{L}(f \circ g)(0) = (u \circ f \circ g \circ v)(0) = u(x_1) \neq u(x_2) = (u \circ f \circ v \circ u \circ g \circ v)(0) = [\boldsymbol{L}(f) \circ \boldsymbol{L}(g)](0). \quad \square$$

In consideration of this lemma the question arises whether there are functions u and v such that $f^{\square} = u \circ f \circ v$ for every $f \in \mathrm{PR}$. It is clear that the answer is no: By the definition of $\square$ we might have $0 = f^{\square}(0) = u(f(v(0)))$ for every $f \in \mathrm{PR}$. For every natural number n there exists a primitive recursive function f such that $f(v(0)) = n$ and so $0 = f^{\square}(0) = u(f(v(0))) = u(n)$, i.e. $u(n) = 0$ for every n, which leads to $f^{\square}(m) = u(f(v(m)))$ for every $f \in \mathrm{PR}$ and $m \in \mathsf{N}$. This is impossible.

Lemma 2. *Let f^{-1} be usual inverse function of f with respect to the operation $\circ$, i.e. $f^{-1} \circ f = f \circ f^{-1} = id$. Let $f \in \mathrm{PR}$ and $f(0) = 0$. Assume that f^{-1} exists and $f^{-1} \in \mathrm{PR}$. Then there exists exactly one $g \in \mathrm{PR}$ such that $f = g^{\square}$.*

Proof. If $f = g^{\square}$, then $f(0) = 0$ and $f(n+1) = g^{\square}(n+1) = g(g^{\square}(n)) = g(f(n))$, i.e. $f \circ \boldsymbol{s} = g \circ f$ and $g = f \circ \boldsymbol{s} \circ f^{-1}$, i.e. there is only one possible g and this g is suitable. $\square$

Corollary. *$id = f^{\square}$ iff $f = \boldsymbol{s}$.*

Theorem 1. *There are only two endomorphisms on $\langle \mathrm{PR}, \circ, \square \rangle$, namely $\boldsymbol{O}$ and $\boldsymbol{Id}$.*

Proof. There are two cases:

Case (a): $\boldsymbol{L}(id) = id$ where $\boldsymbol{L}$ is the considered endomorphism on $\langle \mathrm{PR}, \circ, \square \rangle$. Then $id = \boldsymbol{L}(id) = \boldsymbol{L}(\boldsymbol{s}^{\square}) = \boldsymbol{L}(\boldsymbol{s})^{\square}$ and so $\boldsymbol{L}(\boldsymbol{s}) = \boldsymbol{s}$, using the corollary of Lemma 2. For every $c \in \mathsf{N}$ and each $f \colon \mathsf{N} \to \mathsf{N}$ let $f^0 = id$ and $f^c = \underbrace{f \circ f \circ \ldots \circ f}_{c \text{ times}}$ for $c \neq 0$. Then for each constant function $\tilde{c}$ we have $\tilde{c} = \boldsymbol{s}^c \circ \boldsymbol{o}$ and

$$\boldsymbol{L}(\tilde{c}) = \boldsymbol{L}(\boldsymbol{s}^c \circ \boldsymbol{o}) = \boldsymbol{L}(\boldsymbol{s}^c \circ id^{\square}) = \boldsymbol{L}(\boldsymbol{s})^c \circ \boldsymbol{L}(id)^{\square} = \boldsymbol{s}^c \circ \boldsymbol{o} = \tilde{c},$$

i.e. $\boldsymbol{L}(\tilde{c}) = \tilde{c}$. Furthermore for every $f \in \mathrm{PR}$ and $c \in \mathsf{N}$ we have

$$(f(c))^{\sim} = \boldsymbol{L}((f(c))^{\sim}) = \boldsymbol{L}(f \circ \tilde{c}) = \boldsymbol{L}(f) \circ \tilde{c} = (\boldsymbol{L}(f)(c))^{\sim},$$

i.e. $f(c) = \boldsymbol{L}(f)(c)$, which implies $f = \boldsymbol{L}(f)$, i.e. $\boldsymbol{L} = \boldsymbol{Id}$.

Case (b): $\boldsymbol{L}(id) \neq id$. For short we set $\boldsymbol{L}(f) = f'$ for every $f \in \mathrm{PR}$, and $N' = \bigcup \{\mathrm{rg}(f') \mid f \in \mathrm{PR}\}$, where $\mathrm{rg}(f')$ denotes the range of f'. Firstly we examine whether N' equals to N or not. For every $f \in \mathrm{PR}$ we have $id \circ f = f$ and so $id' \circ f' = f'$, i.e. $id'(f'(c)) = f'(c)$ for every $c \in \mathsf{N}$ and for each $f \in \mathrm{PR}$. In other words: $id'|_{N'} = id|_{N'}$, and therefore $\mathrm{rg}(id') = N'$. From this follows $N' \neq \mathsf{N}$.

Now we remark the following simple fact: $f'|_{N'} = g'|_{N'}$ implies $f' = g'$, for every function f and g. (Since for every $y \in \mathsf{N}$: $f'(y) = (f' \circ id')(y) = f'(id'(y)) = g'(id'(y)) = (g' \circ id'(y) = g'(y).)$

Obviously $\boldsymbol{o}' = (id^{\Box\Box})' = (id')^{\Box\Box} = \boldsymbol{o}$. Furthermore for every $a \in \mathsf{N}$ we have:

$$\tilde{a}' = (\boldsymbol{s}^a \circ \boldsymbol{o})' = \boldsymbol{s}'^a \circ \boldsymbol{o} = (\boldsymbol{s}'^{\Box}(a))^{\sim} = \boldsymbol{s}'^{\Box} \circ \tilde{a} = id' \circ \tilde{a},$$

i.e.

$$(1) \qquad id' \circ \tilde{a}' = id' \circ \tilde{a}, \quad \text{for every } a \in \mathsf{N}.$$

Therefore $id'' \circ \tilde{a} = id'' \circ \tilde{a}' = (id' \circ \tilde{a})' = \tilde{a}' = \tilde{a}$ if $a \in N'$, and so $id''|_{N'} = id|_{N'} = id'|_{N'}$ and $id'' = id'$ from the previous remark. Furthermore for every $f \in \mathrm{PR}$ and $y \in \mathsf{N}$ we have $(id' \circ f \circ id')(y) \in N'$ and $f' = id' \circ f' \circ id'$ and therefore

$$\begin{aligned}((id' \circ f \circ id')(y))^{\sim} &= [((id' \circ f \circ id')(y))^{\sim}]' = [id' \circ f \circ id' \circ \tilde{y}]' \\ &= id'' \circ f' \circ id'' \circ \tilde{y}' = id' \circ f' \circ id' \circ \tilde{y}',\end{aligned}$$

and using (1) we get

$$id' \circ f' \circ id' \circ \tilde{y} = f' \circ y = (f'(y))^{\sim},$$

i.e. $id' \circ f \circ id' = f'$. We know that $\mathrm{rg}(id') = N' \neq \mathsf{N}$, and so $id' \circ id' \neq id$. Moreover $id'(0) = \boldsymbol{s}'^{\Box}(0) = 0$, i.e. $0 \in \mathrm{rg}(id')$. If id' is a constant function then $id' = \boldsymbol{o}$ and $\boldsymbol{L} = \boldsymbol{O}$. Now suppose that id' were not constant. Then we could apply Lemma 1 choosing $u = v = id'$, and by this Lemma we obtain a contradiction. $\Box$

The following corollary shows the importance of this theorem.

Corollary. *Let $g_1, \ldots, g_k \in \mathrm{PR}$ and $\omega_1, \ldots, \omega_r$ be operators on* PR. *Suppose that there is a finite procedure to calculate $f^{\Box}$ and $f \circ g$ from the functions $f, g \in \mathrm{PR}$ and $g_1, \ldots, g_k$ by the help of the above operators. If $\boldsymbol{L} \in \mathrm{End}(\langle \mathrm{PR}, \omega_1, \ldots, \omega_r \rangle)$ and $\boldsymbol{L}(g_i) = g_i$ for $i = 1, 2, \ldots, k$, then $\boldsymbol{L} \in \mathrm{End}(\langle \mathrm{PR}, \circ, \Box \rangle)$ and so $\boldsymbol{L} = \boldsymbol{Id}$.*

The corollary says that the theorem is true in many usual structures of primitive recursive functions. For example:

(a) Let $\omega_1 = \circ$ and $\omega_2 = \Box(m)$, where m is a fixed natural number. For every function f we have $f^{\Box} = \boldsymbol{p}^m \circ (\boldsymbol{s}^m \circ f \circ \boldsymbol{p}^m)^{\Box(m)}$. So we can put $g_1 = \boldsymbol{p}$ and $g_2 = \boldsymbol{s}$. By our corollary, if $\boldsymbol{L} \in \mathrm{End}(\langle \mathrm{PR}, \circ, \Box(m) \rangle)$ and $\boldsymbol{L}(\boldsymbol{s}) = \boldsymbol{s}$, $\boldsymbol{L}(\boldsymbol{p}) = \boldsymbol{p}$ then $\boldsymbol{L} = \boldsymbol{Id}$.

(b) At this point let f^{-1} be defined only for surjective functions as $f^{-1}(x) = \min\{y: f(y) = x\}$ for every x. J. Robinson showed in [2] how to calculate $f^{\Box}$ from the functions f, $\boldsymbol{s}$ and q by the help of the operators $\circ$, $^{-1}$, and $+$. (The proof can be found in [9], Theorems 3.49 and 3.50 in Part I, too.) She also showed how to calculate $f^{\Box}$ from the function f and two certain complicated functions u and v (which are independent of f) by the help of the operators $\circ$ and $^{-1}$. So we got the following statements:

If $\boldsymbol{L} \in \mathrm{End}(\langle \mathrm{PR}, \circ, +, ^{-1} \rangle)$ and $\boldsymbol{L}(\boldsymbol{s}) = \boldsymbol{s}$, $\boldsymbol{L}(\boldsymbol{q}) = \boldsymbol{q}$, then $\boldsymbol{L} = \boldsymbol{Id}$.

If $\boldsymbol{L} \in \mathrm{End}(\langle \mathrm{PR}, \circ, ^{-1} \rangle)$ and $\boldsymbol{L}(u) = u$ and $\boldsymbol{L}(v) = v$, then $\boldsymbol{L} = \boldsymbol{Id}$.

The proof of Theorem 1 shows that we used a few properties of our structure $\langle \mathrm{PR}, \circ, \Box \rangle$ only. This implies the following generalizations:

Theorem 1A. *Let $\langle P, \circ, \square \rangle$ be an arbitrary algebraic structure on which the following axioms hold:*

(a) *$\langle P, \circ \rangle$ is a semigroup with unit element id.*

(b) *There exists exactly one s in P such that $s^{\square} = id$. We denote by PS the set of the left-hand singular elements of $\langle P, \circ \rangle$, i.e. for every $c \in PS$ and $f \in P$ let $c \circ f = c$.*

(c) $(\forall f, g \in P)\ ((\forall c \in PS)\ f \circ c = g \circ c) \Rightarrow f = g)$.

(d) $(\exists c_0 \in PS)\ (\forall f \in P)\ f^{\square\square} = c_0$.

(e) $(\forall c \in PS)\ (\exists k_c \in \mathsf{N})\ c = \underbrace{s \circ s \circ \ldots \circ s}_{k_c \text{ times}} \circ c_0$.

If $\boldsymbol{L} \in \operatorname{End}(\langle P, \circ, \square \rangle)$ and $\boldsymbol{L}(id) = id$ then $\boldsymbol{L} = \boldsymbol{Id}$.

Theorem 1B. *Let $\langle P, \circ, \square \rangle$ be an arbitrary algebraic structure. Suppose that all the above axioms* (a)–(e) *and the following axiom hold:*

(f) $(\forall x_1, x_2, y, z \in PS)\ (\exists f \in P)\ (f \circ z = x_1\ \&\ f \circ y = x_2)$.

Then there are only two endomorphisms on $\langle P, \circ, \square \rangle$, namely $\boldsymbol{Id}$ and $\boldsymbol{O}$.

§ 2. In this section we examine the generations of PR. Except from Theorem 3, we consider arbitrary functions $f\colon \mathsf{N} \to \mathsf{N}$.

Lemma 3. *Let f be an arbitrary function. If $f^{\square}$ is not injective, then $\operatorname{rg}(f^{\square})$ is a finite set.*

Proof. By the definition of $f^{\square}$ from $f^{\square}(n) = f^{\square}(m)$ for any $m > n$ it follows that $\operatorname{rg}(f^{\square}) = \{f^{\square}(0), \ldots, f^{\square}(m-1)\}$. $\square$

Note that in the case above $f^{\square}$ is a periodic function and its period is $m - n$. L. Lovász asked whether for every periodic function f there is a function g such that $f = g^{\square}$. The answer is the following: Let the sequence $f(i)$ be periodic with the period $m - n$, then there exists such a g iff $f(0) = 0$ and the numbers $f(0), f(1), \ldots, f(m-1)$ are all distinct.

Lemma 4. *If f is not injective, then $f^{\square}$ is not surjective.*

Proof. *Let* $i = f(k_1) = f(k_2)$, where $k_1 \neq k_2$. If $f^{\square}$ is surjective, then there exist natural numbers h_1, h_2 such that $k_1 = f^{\square}(h_1)$ and $k_2 = f^{\square}(h_2)$. Then $i = f(k_1) = f(f^{\square}(h_1)) = f^{\square}(h_1 + 1)$ and in similar way we get $i = f^{\square}(h_2 + 1)$. We know that $h_1 + 1 \neq h_2 + 1$ and because of Lemma 3, $f^{\square}$ is not surjective. This contradiction proves the lemma. $\square$

From this point on for an arbitrary function a we denote by $\langle a \rangle$ the closure of $\{a\}$ with respect to the operators $\circ$ and $\square$.

Lemma 5. *If a is an arbitrary injective function, then for every member f of $\langle a \rangle$ either f is injective or $rg(f)$ is finite.*

Proof. The *order* of an element f in $\langle a \rangle$ is defined as the minimal number of operations $\circ$ and $\square$ which are necessary to generate f from a. Now the lemma is proved by induction on the order of f. As the assertion is true for a, the lemma holds for order 0. Assume the assertion is true for order $k \leqq n$, and let $\operatorname{ord}(f) = n + 1$.

Case 1: $f = g^{\square}$ and $\operatorname{ord}(g) = n$. If $\operatorname{rg}(g)$ is finite, then $\operatorname{rg}(g^{\square})$ is also finite. If g is injective and $g^{\square}$ is not injective, then by Lemma 3 $\operatorname{rg}(g^{\square})$ is finite.

Case 2: $f = g \circ h$ and $\mathrm{ord}(g)$, $\mathrm{ord}(h) \leqq n$. If g and h are injective, then $g \circ h$ is injective. If $\mathrm{rg}(g)$ or $\mathrm{rg}(h)$ is finite, then $\mathrm{rg}(g \circ h)$ is finite. □

Lemma 6. *For every element f of $\langle a \rangle$ either there exists a suitable natural number k such that $f = a^k$ or* $(\mathrm{rg}(f) \subseteqq \mathrm{rg}(a^\square)$.

Proof. For every natural number m and each function f we have $f^m \circ f^\square = f^\square \circ \boldsymbol{s}^m$ and $(f^m)^\square = f^\square \circ (\boldsymbol{s}^m)^\square$. Taking these identities into account we get the following scheme for the construction of $\langle a \rangle$ on the strength of the definition of the order of the elements in $\langle a \rangle$:

$$a, a^2, a^\square, a^3, (a^2)^\square = a^\square \circ (\boldsymbol{s}^2)^\square,\ a^{\square\square} = \boldsymbol{o},$$
$$a^4, (a^3)^\square = a^\square \circ (\boldsymbol{s}^3)^\square, a^\square \circ a^\square, \ldots$$
$$\ldots$$
$$a^m, (a^m)^\square = a^\square \circ (\boldsymbol{s}^m)^\square,\ (a^\square)^m = a^\square \circ (a^\square)^{m-1},\ a^m \circ a^\square = a^\square \circ \boldsymbol{s}^m, a^\square \circ a^m.$$

A short look of this scheme yields the proof. □

Theorem 2. *Let a be an arbitrary function from* N *to* N. *Then either there exists no bijection in $\langle a \rangle$ or for every member f of $\langle a \rangle$ it holds that f is injective or* $\mathrm{rg}(f)$ *is finite.*

Proof. If a is injective, the assertion follows by Lemma 5. If a is not injective, then a^m is not injective, too. In this case we prove that there is no bijective function in $\langle a \rangle$. Assume on the contrary that there exists a bijective member f in $\langle a \rangle$. Then $f \neq a^m$ because f is injective. But f is surjective and by Lemma 6 then $a^\square$ must be a surjective function. By Lemma 4 this is a contradiction which proves the theorem. □

Theorem 3. *There is no primitive recursive function which generates all monoton increasing primitive recursive functions. In particular, there is no primitive recursive function a such that* $\langle a \rangle = \mathrm{PR}$.

Proof. Because of Theorem 2 id and $\boldsymbol{p}$ can not be at the same time in $\langle a \rangle$. □

We now give a more general algebraic form of Theorem 2 similar to Theorem 1A. Let (g) be the following axiom:

(g) $PS = \{c_0, c_1, \ldots\}$ (i.e. PS is countable) and, for every $f \in P$ and each natural number i, $f^\square \circ c_0 = c_0$ and $f^\square \circ c_{i+1} = f^\square \circ f \circ c_i$ hold.

Really it is a very strong axiom: From (c) and (g) one can easily prove the axioms (d), (e) and half of (b). If we identify the elements f of P with functions f mapping from PS to PS with $f(c) = f \circ c$, then we can easily prove Lemma 3 – Lemma 6 and so we get

Theorem 2A. *Let $\langle P, \circ, \square \rangle$ an arbitrary algebraic structure on which axiom* (g) *holds. Then for every element a of P either there exists no bijection in $\langle a \rangle$ or for every $f \in \langle a \rangle$ it holds that f is injective or* $\mathrm{rg}(f)$ *is finite.*

Note that we can show by the help of Theorem 2 that several subspaces of $\langle \mathsf{N}^\mathsf{N}, \circ, \square \rangle$ can not be generated from only one function, e.g. $\{f\colon f(0) = 0$ and f is strictly monoton$\} \cup \{\boldsymbol{o}\}$, etc. Till now I have not found a monotone increasing primitive recursive function which is not in $\langle \boldsymbol{s} \rangle$. This is not an important question but I am interested in it. The results of this paper seem to be the first ones concerning the algebraic properties of $\langle \mathrm{PR}, \circ, \square \rangle$. I think it is interesting and useful to investigate similar problems, for example to study other properties of the operators $\circ$ and $\square$, to investigate other operators on PR (e.g. $\Sigma(f)(n) = f(0) + \ldots + f(n)$ or f^{-1}) or to raise usual or unusual algebraic questions about $\langle \mathrm{PR}, \circ, \square \rangle$.

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