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The maximum and minimum number of circuits and bases of matroids

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Abstract. In this paper we investigate the maximum and minimum number of bases and circuits and their structure in a matroid. This problem originates from the calculation of the number of minimal reactions and mechanisms in chemical stoichiometry.

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1 Introduction

The very first problem in our investigations was to determine the number of minimum reactions and mechanisms in chemical stoichiometry, which lead to the following linear algebraic (not geometric) notion [7]:

> *"A set of vectors $S \subset R^n$ is called a* ***simplex*** *iff S it is linearly dependent but all its proper subsets are independent."*

([6] investigates the relevant systems of linear equations while [9] generates all the simplexes in a given finite set of vectors in $\mathbf{R}^n$. The chemical motivation is shortly described e.g. in [3] and [9].)

In [3] and [2] the authors raised and partially solved the question *"How many simplexes (minimum or maximum) can be found in a given set of vectors in $\mathbf{R}^n$ of fixed cardinality, and what are the extreme constructions?"*

For an arbitrary matroid, the corresponding notion of a simplex is called a circuit, that is a dependent set all of whose proper subsets are independent. This paper concentrates on the following, more general problem:

"What is the minimum and maximum number of circuits and bases in matroids of given size and rank?"
We completely solve the maximum case and partially answer the minimum case, where the exact value of the *lower bound* remains open when neither parallel elements nor loops are allowed in the matroid.

The authors are grateful to Professor Á. Pethő for drawing their attention to this problem. Despite of ours and others' intensive search for relevant literature, only Murty's paper [4] on equicardinal matroids (where all circuits have the same size) was found.

The matroid terminology used mostly follows Oxley [5]. We denote a matroid M as a pair (S, F), where S is the ground set of M, and F is the set of independent subsets of S. We usually use m for the size of the matroid, i.e. the cardinality of S, and n for its rank n, i.e. the cardinality of any basis of M. We assume that $0 < n < m$, as the case $m = n$ is trivial. Through this note we use the following convenient notion:

DEFINITION. A circuit is called **small** if it consists of at most n elements. A circuit will be called **large** if it contains at least 3 elements.

Note, that in this sense a circuit can be small and large at the same time, as well.

2 On the maximum

In this section we count the maximum number of circuits and bases in matroids of size of m and rank n, exhibiting the structure of the resulting matroids.

2.1 Maximum number of circuits

Our results are summarized in the following theorem:

THEOREM 2.1 *If $m > n+1$, then only the uniform matroid $U_{m,n}$ contains the maximum number of circuits, $\binom{n+1}{m}$. If $m = n+1$, all matroids of size m and of rank n contain exactly 1 circuit.*

Proof. To prove Theorem 2.1 we first describe the Construction 1 to build a new matroid $M' = (S', F')$ from $M = (S, F)$.

CONSTRUCTION 1 Let u be an element of S. Then M' is obtained by freely adding a new element u' to $M\backslash u$. (See Oxley'x book, [5], Section 7.2.) Note that

$$S' := S\backslash\{u\} \cup \{u'\}$$

and

$$F' := \{f \in F : f \subseteq S\backslash\{u\}\} \cup \{f \cup \{u'\} : f \in F, f \subseteq S\backslash\{u\}, | f | \leq n-1\}.$$

It is straightforward to verify that the size and rank are preserved, and that the new element u' is not a member of any small circuits in M'. More importantly, if an element of $S\backslash\{u\}$ is not contained in a small circuit of M, then the same remains true in M'. Further the number of circuits in M' is at least that of M.

LEMMA 2.2 *The number of circuits in M is strictly less than $\binom{n+1}{m}$ whenever M contains a small circuit and $m > n+1$.*

Proof. Let $K \subseteq M$ be a fixed small circuit of ℓ elements, $\ell \leq n$. Using the above Construction 1 repeatedly $m - \ell$ times, we can replace each $u \in S\backslash K$ by a new element u' as described. The number of circuits was not decreased and in fact the circuits left are K itself and all other ones must be of size $n+1$. The number is therefore at most

$$1 + \sum_{i=0}^{\ell-1} \binom{\ell}{i}\binom{m-\ell}{n+1-i} = 1 + \binom{m}{n+1} - \binom{m-\ell}{n+1-\ell}$$

which is strictly less than $\binom{m}{n+1}$ iff $m > n+1$. □

For the case $m = n+1$, we have a base $\{u_1, u_2, \ldots, u_n\}$ and therefore $S = \{u_1, u_2, \ldots, u_n, v\}$ contains a unique circuit by the corollary of the weak axiom for circuits. This concludes the proof of Theorem 2.1. □

2.2 Bases

It turns out that the above Construction 1 does not decrease the number of bases either, and again only $U_{m,n}$ does have the maximum number of bases, namely $\binom{m}{n}$. We can assume that $m > n > 0$ and we consider a matroid $M = (S, F)$ of size m and rank n.

THEOREM 2.3 *Only the uniform matroid $U_{m,n}$ contains the maximum number of bases, namely $\binom{m}{n}$.*

Proof. We first verify that the number of bases does not decrease during the Construction 1, where an element $u \in S$ is replaced by an element u'. Let $B \subseteq S$ be a base in M. If $u \notin B$, then B remains a base in M'; otherwise, if $u \in B$, then $B\backslash\{u\} \cup \{u'\}$ is now a base in M'. This means that we have a one-one correspondence between the bases of M and some bases of M'. We now show that any matroid containing small circuits contains strictly less than $\binom{m}{n}$ bases. Let $M = (S, F)$ be a matroid containing a small circuit K of size ℓ where $\ell \leq n$. As before, replace all the elements u of $S\backslash K$ repeatedly by a corresponding u' as described in the Construction 1. In the final matroid the bases are exactly all the n-element subsets of S not containing K. The number of these subsets is

$$\sum_{i=0}^{\ell-1} \binom{\ell}{i}\binom{m-\ell}{n-i} = \binom{m}{n} - \binom{m-\ell}{n-\ell}$$

which is clearly strictly less than $\binom{m}{n}$ using $\ell \leq n \leq m$. □

3 On the minimum

In this section we give a lower bound for the number of circuits and bases being contained in a matroid of size m and of rank n. As opposed to the maximum case, the answer here for the minimum case depends on whether we allow loops or parallel elements; subsections 3.1 and 3.2 investigate separately these cases where we also describe the unique minimum configurations. As in [2], a third case excluding both loops and parallel elements remains *open*.

3.1 Allowing loops

In this subsection we analyze the minimum number of circuits and bases in matroids, allowing one element dependent sets, called loops. (These loops are necessarily circuits.) We shall assume that $m > n$, since the trivial case $m = n$ implies that such a matroid would have no circuit and only one base.

THEOREM 3.1 *For each m and n, there is a unique matroid M_0 of size m and of rank n containing the minimum number of bases, namely 1, when we allow loops in the matroid.*

Proof. Let $M_0 := (S_0, F_0)$ be the matroid of size m and rank n where $S_0 = \{u_1, \dots, u_n, v_1, \dots, v_{m-n}\}$, and $B = \{u_1, \dots, u_n\}$ is basis, and $v_1, \dots, v_{m-n}$ are loops, the only circuits of the matroid. (Note, that B is the the unique basis in the matroid M_0.) We show that any matroid but M_0 contains more than one base. Observe that such a matroid M contains a circuit, say K, of more than one element. Consider any element from $K \backslash B$. This element must be independent, by the definition of a circuit, and can therefore be extended it to a second base of M. □

THEOREM 3.2 *Any matroid M of size m and of rank n contains at least $n - m$ circuits. A matroid contains exactly $m - n$ circuits if and only if the circuits of the matroid are pairwise disjoint.*

Proof. Consider a base B of the matroid M. For any $u \in S \backslash B$, the corollary of the weak axiom for circuits implies that there is a (unique) circuit containing u included in $B \cup \{u\}$. We conclude that M has at least $m - n$ many circuits. Now suppose that M contains exactly $m - n$ circuits. Fix a base B of M, and let $S \backslash B = \{v_1, v_2, \dots, v_{m-n}\}$. For each $1 \le i \le m - n$, there is a circuit $K_i \subseteq B \cup \{v_i\}$. These circuits are different for $i \neq j$ since K_i must contain v_i, but K_j does not contain it. If there were two intersecting circuits K_i and K_j containing a common element u, then, by the strong axiom for circuits, the set $K_i \cup K_j \backslash \{u\}$ would contain a circuit K, necessarily distinct from all circuits K_α $(1 \le \alpha \le m - n)$, a contradiction. □

REMARK. The matroid M_0 mentioned above also contains exactly $m-n$ pairwise disjoint circuits, i.e. loops.

3.2 Allowing parallel elements, no loops

As before, $M = (S, F)$ denotes an arbitrary fixed matroid of size m and of rank n. Recall that two elements are called parallel if together they form a circuit. In this subsection we determine the minimum number of circuits and bases in the case where M may not contain loops, but where parallel elements are allowed.

We describe a second construction to modify the matroid in order to reduce the number of bases and circuits. Using this Construction 2 we will describe the unique structures of matroids having the minimum number of circuits and bases.

CONSTRUCTION 2 Let $u_1 \in S$ be any fixed element such that, when deleting it from S, the rank does not decrease (i.e. $r(M) = r(M \backslash \{u_1\})$). For example, any element which is a member of a circuit has this property. Fix further a second arbitrary element $u_2 \in S$ and a new element $u' \notin S$. We now define the matroid $M' := (S', F')$ by $S' := S \backslash \{u_1\} \cup \{u'\}$, and

$$F' := \{f \in F : f \subseteq S \backslash \{u_1\}\} \cup \{f \cup \{u'\} : f \cup \{u_2\} \in F, f \subseteq S \backslash \{u_1, u_2\}\}.$$

In practice, this Construction 2 will be used when u_1 and u_2 are members of a common circuit. The effect is essentially that we delete u_1 from the matroid and add a new u' parallel to u_2.

LEMMA 3.3 *$M' = (S', F')$ is again a matroid of size m and of rank n.*

Proof. The size and rank of M' have not changed since $|S'| = |S| = m$, and since by Construction 2, u_1 was chosen so that its removal does not decrease the rank of M. What must be verified carefully is that M' is a matroid, although only the so-called independence augmentation axiom requires a proof; that is we must show that if f_1 and f_2 are members of F' with $|f_1| < |f_2|$, then there is an element $e \in f_2 \setminus f_1$ such that $f_1 \cup e \in F'$. There are four cases, depending whether $u' \in f_i$.
The only interesting case is when $u' \notin f_1$ and $u' \in f_2$. This means that $f_1 \in F$, $u_1 \notin f_1$; and if $f_2' = f_2 \setminus \{u'\}$ then $f_2' \cup \{u_2\} \in F$, $f_2' \subseteq S \setminus \{u_1, u_2\}$. But $|f_1| < |f_2' \cup \{u_2\}|$ and there is therefore an $e \in f_2' \cup \{u_2\} \backslash f_1$ such that $f_1 \cup \{e\} \in F$. If $e = u_2$, then $f_1 \cup \{u'\}$ is as desired; if otherwise $e \neq u_2$, then $f_1 \cup \{e\}$ is good enough. □

We are now ready to investigate the effect of this Construction 2 on the number of circuits and of bases.

3.2.1 Circuits

In order to find the structure of the extreme *minimum* matroid, we investigate the effect of the above Construction 2 with a careful choice of the element u_1.

LEMMA 3.4 *Suppose that $u_1, u_2 \in S$ are contained in a same large circuit, and denote k_i the number of circuits containing u_i but not u_j. If $k_1 \ge k_2$, then* *deleting u_1 from M and adding a new element parallel to u_2 into M as in Construction 2 the number of circuits does not increase.*

Proof. Denote by k_{12} the number of circuits containing both u_1 and u_2, and as usual let u' be the new element we just added to M. Notice that exactly that circuits what contain u_1 were deleted during the Construction 2, i.e. the number of the circuits is $k_1 + k_{12}$. We constructed new circuits, namely the two-element circuit $\{u_2, u'\}$ and the circuits now containing u' instead of u_2 in M (but not u_1), we have k_2 many of them. Thus the number of circuits is changed by $k_2 + 1 - k_1 - k_{12}$, which is not positive since $k_1 \geq k_2$ and $k_{12} \geq 1$. Moreover, the number of circuits remains unchanged iff $k_1 = k_2$ and $k_{12} = 1$. □

Using Construction 2 repeatedly we eventually reach a matroid not containing any large circuits, only circuits consisting of two parallel elements. Therefore a matroid having the minimum number of circuits must be among this kind. The following theorem says that all matroids having the minimum number of circuits are among this kind.

THEOREM 3.5 *Suppose that there are no large circuits and no loops in the matroid M, and let $\{a_1, a_2, \ldots, a_n\}$ be any fixed base. If ϑ_i denotes the number of elements in M parallel to a_i (including a_i itself) for $i = 1, 2, \ldots, n$, then M contains the minimum number of circuits iff $|\vartheta_i - \vartheta_j| \leq 1$ for $i \neq j$.*

Proof. It is not difficult to verify that the assumptions on M together with the weak axiom for circuits imply that every element of S is parallel to one and exactly one of the a_i's, and therefore $\sum_{i=1}^{n} \vartheta_i = m$.

Suppose on the contrary that $\vartheta_j > \vartheta_\ell + 1$ for some $j, \ell \leq n$. Delete a_j and add an element parallel to a_ℓ, as in Construction 2. Since there are no large circuits in our matroid, the number of circuits in M is

$$\binom{\vartheta_j}{2} + \binom{\vartheta_l}{2} + \sum_{i \neq j, \ell} \binom{\vartheta_i}{2}$$

which becomes in M' to be

$$\binom{\vartheta_j - 1}{2} + \binom{\vartheta_\ell + 1}{2} + \sum_{i \neq j, \ell} \binom{\vartheta_j}{2}.$$

These expressions clearly show that the number of circuits did strictly decrease. Defining a relation on S by $b \sim c$ if they are parallel to the same a_i is an equivalence relation, we obtain that a matroid as above contains the minimum number of circuits exactly in the case when the equivalence classes of parallel vectors have almost all the same size, i.e. differing by at most one. □

COROLLARY 3.6 *The minimum number of circuits in a matroid of size m and of rank n, where $m = an + b$, $(0 \leq b < n)$, is*

$$b \cdot \binom{a+1}{2} + (n - b) \cdot \binom{a}{2}$$

and in particular, if m is a multiple of n,

$$n \cdot \binom{\frac{m}{n}}{2}.$$ □

Now we turn to the exact structure of the matroids containing the minimum number of circuits. We will see that for small matroids there are more possibilities while the structure of *large* matroids are unique.

THEOREM 3.7 **a)** *For $m < 2n$, a matroid of size m and rank n contains the minimum number of circuits iff all its circuits are disjoint.*
b) *For $m \geq 2n$, a matroid contains the minimum number of circuits iff it contains only 2-element circuits (i.e. parallel elements), and the sizes of the equivalence classes of parallel elements differ by at most* 1.

NOTE. There are many matroids satisfying a) while the matroids described in b) are, in fact, isomorphic.

The proof of the above theorem is based upon the following lemmas.

LEMMA 3.8 *If M contains two large circuits K and L, then $|K \cap L| \leq 1$.*

Proof. If $K \cap L$ contains two distinct elements $u_1 \neq u_2$, then the proof of Lemma 3.4 using $k_{12} \geq 2$ shows that M does not contain the minimum number of circuits. □

LEMMA 3.9 *Let K be a large circuit and let $u \notin K$ be arbitrary. Then either u is parallel to some element of K, or else u is not contained in any large circuits intersecting K.*

Proof. Suppose that L is a large circuit containing u and intersecting K. Using Lemma 3.8, we must have exactly one element in $K \cap L$, say v. Then, by the strong axiom of circuits, there is a circuit $H \subseteq K \cup L \setminus \{v\}$ containing u. If H is large, then at least one of the two sets $H \cap L$ and $H \cap K$ has at least two elements, contradicting the previous lemma. If H is small, then u must be parallel to an element of K. This completes the proof. □

LEMMA 3.10 *No element of a large circuit can be parallel to any element of the matroid.*

Proof. Consider a large circuit $K = \{u_1, u_2, \ldots, u_p\}$ (i.e. $p \geq 3$). Suppose to the contrary that an element of K, say u_1, is parallel to some other element $u_1' \neq u_1$. We claim that $K' = \{u_1', u_2, \ldots, u_p\}$ is again a large circuit, contradicting Lemma 3.8. If $K' \in F$, then we could extend it to a base B, but now $B \cup \{u_1\}$ would contain the distinct circuits K and $\{u_1, u_1'\}$, contradicting the weak axiom for circuits. However, every proper subset of K' does belong to F; indeed otherwise such a subset must contain a circuit L, which cannot be large by Lemma 3.8 again. But L cannot be a two-element circuit since u_1' is parallel to u_1. □

The above lemmas imply that each large circuit must be disjoint from every (large or small) circuit in M. Now it remains to consider small circuits.

LEMMA 3.11 *If M contains a large circuit, then there are no three pairwise parallel elements.*

Proof. Let K be a large circuit and suppose that three pairwise parallel elements exist in M. By Lemma 3.10, we may assume that none of these three elements belong to K. Let u_1 be one of these three elements and let u_2 be any element of K. Use again Construction 2 to delete u_1 and add a new element u_2' parallel to u_2; this is possible as M is assumed to contain no loops and therefore u_1 satisfies the hypothesis for Construction 2. Using a calculation similar to Theorem 3.5 and the same notation, we have $k_2 = 1$ since K is the only circuit containing u_2, $k_1 \geq 2$ since we have at least two elements parallel to u_1, and finally $k_{12} = 0$ since u_1 is not contained in any large circuit intersecting K by Lemma 3.9 and is not parallel to any element of K by Lemma 3.10. Therefore, during the Construction 2, we deleted k_1 circuits; we added the circuit $\{u_2, u_2'\}$ of course as well as the circuit $K \setminus \{u_2\} \cup \{u_2'\}$. Thus the number of circuits has changed by $1 + 1 - k_1 \leq 0$, so this number certainly did not increase. But now we can use the procedure described in Lemma 3.10 to decrease the number of circuits. This contradiction shows that M was not minimum. □

Proof of Theorem 3.7. The above results show that all circuits in M must be disjoint in the presence of a large circuit. In this case fix any base B of M. For each element u of $S \setminus B$, $B \cup \{u\}$ must contain a circuit, which in turn must contain at least one element of B. Since all circuits are assumed to be pairwise disjoint, $S \setminus B$ can contain at most n elements. Therefore for $m \geq 2n$, a matroid with the minimum number of circuits cannot contain any large circuit. In the lack of large circuit, using Theorem 3.5, the equivalence classes of parallel elements must have almost the same size. This implies the statement of Theorem 3.7. □

REMARK. The last part of the above proof describes uniquely the structures of matroids containing the minimum number of circuits when $m \geq 2n$.

3.2.2 Bases

The structure of matroids containing the minimum number of bases is always unique, as described in the following Theorem.

THEOREM 3.12 *A matroid M of size m and rank n contains the minimum number of bases iff it has a base $\{a_1, a_2, \ldots, a_n\}$ such that all other elements in M are parallel to a_1.*

As in the previous subsection, we use Construction 2 to achieve the minimum number of bases. The following result describes the effect of this construction on the number of bases.

LEMMA 3.13 *Let K be a large circuit in M and let $u_1, u_2 \in K$. Denote by ℓ_1 the number of bases containing u_1 but not u_2, and similarly for ℓ_2. Then deleting u_1 and adding a new element parallel to u_2 (as in Construction 2), the number of bases strictly decreases whenever $\ell_1 \geq \ell_2$.*

Proof. Denote by ℓ_{12} the number of bases containing both u_1 and u_2. By deleting u_1, we loose exactly the bases containing u_1, that is $\ell_1 + \ell_{12}$ many of them. By adding a new element to M parallel to u_2, we gain ℓ_2 many new bases. Clearly the set $\{u_1, u_2\}$ is independent since K is assumed to be a large circuit, so it can be extended to a base, which implies $\ell_{12} \geq 1$. This, together with $\ell_1 \geq \ell_2$ means that the number of bases strictly decreases. □

Using the above result, we can remove each large circuit of the matroid while decreasing the number of bases. In other words, the matroids containing the minimum number of bases do not contain any large circuit.

Proof of Theorem 3.12. Suppose that M does not contain any large circuit, and let $B = \{a_1, a_2, \ldots, a_n\}$ be any fixed base of M. By adding any other element u to this set, we obtain the collection $B \cup \{u\}$ which must contain a circuit, and therefore u must be parallel to one of the base elements a_i, since M does not contain large circuits.
Denote k_i the number of elements from B parallel to a_i (including a_i itself); clearly $\sum_{i=1}^{n} k_i = m$. Now the number of bases (picking an element from each equivalence class) is

$$\prod_{i=1}^{n} k_i.$$

But in the case $k_\ell \geq k_j \geq 2$, we can delete an element parallel to a_j and add a new element to M parallel to a_ℓ; the number of bases changes to

$$\prod_{i \neq j, \ell}^{n} k_i \cdot (k_j - 1) \cdot (k_\ell + 1)$$

which is strictly less. This implies that all but one k_i is 1. □

COROLLARY 3.14 *The minimum number of bases is $m - n + 1$, and the minimum configuration is unique.*

PROBLEM. Characterize the matroids with the minimum number of circuits and bases, when neither parallel elements or loops are allowed.

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