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Avoiding Forbidden Sequences by Finding Suitable Initial Values

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Abstract

We present a short argument that for a wide class of recurrence relations of the form $x_{n+1} = h(x_n, \ldots, x_{n-k})$ and for an arbitrary fixed (forbidden) sequence (b_n) the sequence (x_n) avoids (b_n) (i.e., $\forall n \in \mathbb{N}\ x_n \neq b_n$) for almost all initial values $(x_{-k}, \ldots, x_0) \in \mathbb{R}^{k+1}$, excluding a set of Lebesgue measure 0. These sequences include algebraic recurrences and our result also generalizes the affirmative answer to Professor Ladas' [5, Conjecture 4.5.1]. We also consider some further generalizations on avoiding sequences of countable sets.

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1 Introduction

We start with Ladas' [5, Problem 4.5.1], which can be stated as follows:

Conjecture 1.1 *(see [5, 4.5.1]).* The recurrence equation

$$x_{n+1} = \frac{x_n + x_{n-1} + x_{n-2} \cdot x_{n-3}}{x_n \cdot x_{n-1} + x_{n-2} + x_{n-3}} \quad (n \geq 0) \tag{1.1}$$

has positive solutions which are not eventually equal to 1.

This problem requires finding initial values for $x_{-3}, \ldots, x_0 \in \mathbb{R}$ such that the recurrence equation (1.1) generates a sequence (x_n) for which the property

$$(\exists N \in \mathbb{N})\,(\forall n \geq N) \qquad x_n = 1 \tag{1.2}$$

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fails!

So far Conjecture 1.1 was confirmed by Stević in [7]. In the papers $[4, 6, 8, 9]$ the authors proved the global convergence of all positive solutions to the equilibrium of the equation (1.1). In [2] a close global convergence result was also obtained.

In this short note we show in Theorems 2.9 and 3.2 that for a wide class of recurrence relations of the form

$$x_{n+1} = h(x_n, \ldots, x_{n-k}) \quad (n \geq 0) \tag{1.3}$$

and for an arbitrary fixed (forbidden) sequence $(b_n) \subseteq \mathbb{R}$ the sequence (x_n) avoids (b_n), that is,

$$(\forall n \in \mathbb{N}) \qquad x_n \neq b_n$$

for almost all initial values $(x_{-k}, \ldots, x_0) \in \mathbb{R}^{k+1}$. "Almost all" means excluding a subset of $\mathbb{R}^{k+1}$ of Lebesgue measure 0. As a special case, this gives an affirmative answer to Conjecture 1.1 choosing $b_n = 1$ for all $n \in \mathbb{N}$. The class of recurrence relations (1.3) includes those of the form (1.1). An easy generalization is provided in Theorem 2.10 below: For the same class of recurrence relations in (1.3) and for any fixed forbidden sequence of countable sets $(B_n) \subseteq P(\mathbb{R})$ of real numbers, the sequence (x_n) avoids (B_n), i.e.,

$$\forall n \in \mathbb{N} \quad x_n \notin B_n$$

for almost all initial values $(x_{-k}, \ldots, x_0) \in \mathbb{R}^{k+1}$. This result also answers e.g., questions of the following nature: "Can one choose the initial values of the Fibonacci recurrence relation $f_{n+1} = f_n + f_{n-1}$ so that NO member of the old sequence

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

appears in the new sequence?", or "Is it possible to choose the initial values of a recurrence equation as in (1.3) such that NO member of the sequence would be integer (or rational, or even algebraic)?" Other directions of research in progress include trying to avoid neighbourhoods of sequences, i.e., to require

$$| \, x_n - b_n \, | > \varepsilon_n$$

for all $n \in \mathbb{N}$, or to reveal the set itself of good possible initial values for avoiding a given sequence. Complex numbers can be used instead of real numbers throughout.

2 Avoiding Sequences

In this section we prove for any fixed $N \in \mathbb{N}$ that all sequences $(x_n) \subseteq \mathbb{R}^N$ which satisfy recurrence relations having recursively weak set of roots (see Definition 2.4 below) avoid any fixed (forbidden) sequence $(b_n) \subseteq \mathbb{R}^N$ (i.e., $\forall n \; x_n \neq b_n$) for almost all initial values $x_{-k}, \ldots, x_0$ (excluding a set of Lebesgue measure 0). This is Theorem 2.9 below while Theorem 2.10 provides a generalization of this fact.

In the next section we will reveal that a wide class of recurrence relations, including algebraic ones, have recursively weak set of roots – and so can avoid any forbidden sequence. These results give not only a quick and general solution to Professor Ladas' [5, Conjecture 4.5.1] but reveal a general behaviour of sequences satisfying a large class of recurrence relation in connection with their initial values. Some examples are mentioned at the end of the previous section. Our proof gives an "existence" results for the above mentioned problems.

In general we do not use special notation for vectors (elements of $\mathbb{R}^N$) but to avoid confusion we always try to indicate whether we are speaking about vectors or numbers. $\mathbb{N}_0$ stands for $\mathbb{N} \cup \{0\}$. To facilitate the forthcoming notions and notations, first we present a short look on the detailed effect of the recurrence equation (1.3) on the initial values. Starting with the fixed initial values $a_1, \ldots, a_{k+1} \in \mathbb{R}$ (where $0 \leq k$ is fixed), we have

$$\begin{cases} x_{-k} = a_1 \\ \quad \ldots \\ x_0 = a_{k+1} \end{cases}$$

Then, repeating (1.3) we can go further:

$$\begin{aligned} x_1 &= h\left(x_0, \ldots, x_{-k}\right) = h\left(a_{k+1}, \ldots, a_1\right), \\ x_2 &= h\left(x_1, a_{k+1}, \ldots, a_2\right) = h\left(h(a_{k+1}, \ldots, a_1), a_{k+1}, \ldots, a_2\right), \\ x_3 &= h(x_2, x_1, a_{k+1}, \ldots, a_3) = \\ &= h\left(\ h\left(h(a_{k+1}, \ldots, a_1), a_{k+1}, \ldots, a_2\right),\ h(a_{k+1}, \ldots, a_1),\ a_{k+1}, \ldots, a_3\ \right) \\ &\quad \ldots . \end{aligned}$$

To simplify this procedure we suggest the notations below in Definitions 2.1 through 2.4.[1] Moreover, we define these notions for vector-sequences $(\mathbf{x}_n) \subseteq \mathbb{R}^N$, that is, we allow functions

$$h : \mathbb{R}^{N \times K} \to \mathbb{R}^N, \qquad \text{for any } N, K \in \mathbb{N}.$$

Definition 2.1. For any function $h : \mathbb{R}^{N\times K} \to \mathbb{R}^N$ ($N, K \in \mathbb{N}$ are fixed), we define its **recurrence iteratives $\mathbf{h}^{[\mathbf{i}]}$** $: \mathbb{R}^{N\times K} \to \mathbb{R}^N$ for $i \in \mathbb{N}$ as follows: For any $a_K, \ldots, a_1 \in \mathbb{R}^N$, we let

$$\begin{aligned} h^{[0]}(a_K, \ldots, a_1) &:= a_K, \\ h^{[1]}(a_K, \ldots, a_1) &:= h(a_K, \ldots, a_1) \qquad \text{(i.e., } h \text{ itself)}, \\ h^{[2]}(a_K, \ldots, a_1) &:= h(h^{[1]}(\vec{\vec{a}}),\ a_K, \ldots, a_2), \\ h^{[3]}(a_K, \ldots, a_1) &:= h(h^{[2]}(\vec{\vec{a}}), h^{[1]}(\vec{\vec{a}}),\ a_K, \ldots, a_3), \\ &\quad \ldots \\ h^{[K]}(a_K, \ldots, a_1) &:= h(h^{[K-1]}(\vec{\vec{a}}), \ldots, h^{[1]}(\vec{\vec{a}}),\ a_K), \\ h^{[K+j]}(a_K, \ldots, a_1) &:= h(h^{[K+j-1]}(\vec{\vec{a}}), \ldots, h^{[j]}(\vec{\vec{a}})) \quad \text{for } j \in \mathbb{N}, \end{aligned} \tag{2.1}$$

[1] Since we want to extend our result to the most general case, the notation from the end of Definition 2.4 below will be a bit complicated.

where $\overrightarrow{\overrightarrow{a}}$ is short for $(a_K, \ldots, a_1)$ (sequence of vectors).

According to Definition 2.1, we suggest to write $f(a_n, \ldots, a_1)$ for functions f instead of $f(a_1, \ldots, a_n)$ in many parts of the present paper. Our method will also handle recurrence equations with cases – using logical formulas. In order to speak easily about these equations we prefer the generalization with multiindices of recurrence iteratives. This means, that in each step we take care of all possible former branchings, that is taking accounts of which former steps were taken.[2]

Definition 2.2. Let $h_0, \ldots, h_{L+1} : \mathbb{R}^{N\times K} \to \mathbb{R}^N$ be functions and $\Phi_0, \ldots, \Phi_L$ be properties (formulae) on $\mathbb{R}^{N\times K}$. Now the function $h : \mathbb{R}^{N\times K} \to \mathbb{R}^N$ is defined from the functions $h_0, \ldots, h_{L+1}$ with the cases $\Phi_0, \ldots, \Phi_L$ if and only if

$$h(a_K, \ldots, a_1) = \begin{cases} h_0(a_K, \ldots, a_1) & \text{if} \quad \Phi_0(a_K, \ldots, a_1) \\ \quad \ldots & \\ h_L(a_K, \ldots, a_1) & \text{if} \quad \Phi_L(a_K, \ldots, a_1) \\ h_{L+1}(a_K, \ldots, a_1) & \text{otherwise.} \end{cases} \tag{2.2}$$

Definition 2.3. If the function $h : \mathbb{R}^{N\times K} \to \mathbb{R}^N$ is defined with cases in (2.2), then we define its recurrence multi-iteratives $h^{[\vec{i}]}$ for any multiindex $\vec{i} = < i_1, \ldots, i_s > \in \mathbb{N}_0^s$ (vector of length s of indices) for any $s \in \mathbb{N}$, supposing $0 \le i_t \le L+1$ for $1 \le t \le s$, by induction on s as follows: For any $a_1, \ldots, a_K \in \mathbb{R}^N$, we let

$$\begin{aligned} h^{<i_1>}(a_K, \ldots, a_1) \; &: = h_{i_1}(a_K, \ldots, a_1), \\ h^{<i_1,i_2>}(a_K, \ldots, a_1) \; &: = h_{i_2}\left(h^{<i_1>}(\overrightarrow{\overrightarrow{a}}),\; a_K, \ldots, a_2\right), \\ h^{<i_1,i_2,i_3>}(a_K, \ldots, a_1) \; &: = h_{i_3}\left(h^{<i_1,i_2>}(\overrightarrow{\overrightarrow{a}}), h^{<i_1>}(\overrightarrow{\overrightarrow{a}}),\; a_K, \ldots, a_3\right), \\ &\quad \ldots \\ h^{<i_1,\ldots,i_K>}(a_K, \ldots, a_1) \; &: = h_{i_K}\left(h^{<i_1,\ldots,i_{K-1}>}(\overrightarrow{\overrightarrow{a}}), \ldots, h^{<i_1>}(\overrightarrow{\overrightarrow{a}}),\; a_K\right), \\ h^{<i_1,\ldots,i_{j+K}>}(a_K, \ldots, a_1) \; &: = h_{i_{j+K}}\left(h^{<i_1,\ldots,i_{j+K-1}>}(\overrightarrow{\overrightarrow{a}}), \ldots, h^{<i_1,\ldots,i_j>}(\overrightarrow{\overrightarrow{a}})\right), \end{aligned}$$

where $\overrightarrow{\overrightarrow{a}} = (a_K, \ldots, a_1)$.

In what follows we always assume that $0 \le i_t \le L+1$ for $1 \le t \le s$ in any multiindex $\vec{i} =< i_1, \ldots, i_s >\in \mathbb{N}^s$ ($s \in \mathbb{N}$).

Definition 2.4. (i) A function $h : \mathbb{R}^{N\times K} \to \mathbb{R}^N$ **has a recursively weak set of roots** if and only if for each $i \in \mathbb{N}$ and for each $\mathbf{b} \in \mathbb{R}^N$, the set (of solutions)

$$\left\{ (\mathbf{x}_1, \ldots, \mathbf{x}_K) \in \mathbb{R}^{N\times K} \;:\; h^{[i]}(\mathbf{x}_1, \ldots, \mathbf{x}_K) = \mathbf{b} \right\} \tag{2.3}$$

has Lebesgue measure 0.

[2]The forthcoming Definition 2.3 is similar to the previous one but rather complicated – we beg the reader's pardon.

(ii) Let the function h be defined by cases from the functions $h_0, \ldots, h_{L+1}$. Now h **has a multiple recursively weak set of roots** if and only if for each multiindex $\vec{i} =< i_1, \ldots, i_s >\in \mathbb{N}^s$ and for each $\mathbf{b} \in \mathbb{R}^N$, the set

$$\left\{ (\mathbf{x}_1, \ldots, \mathbf{x}_K) \in \mathbb{R}^{N \times K} \ : \ h^{\vec{i}}(\mathbf{x}_1, \ldots, \mathbf{x}_K) = \mathbf{b} \right\}$$

has Lebesgue measure 0.

In other words, we require that the inverse image (or preimage) $\left(h^{\vec{i}}\right)^{-1}(b)$ of b for any multiindex $\vec{i}$ and for any $b \in \mathbb{R}^N$ has Lebesgue measure 0. In what follows, we talk about functions h which are defined by cases, multiindices and multiple recursively weak sets, but at the same time we allow, of course simple functions, indices and sets, too. When talking about Lebesgue measure of sets of different dimensions the following notations will be useful:

Notation 2.5. (i) for any vector $\mathbf{c} \in \mathbb{R}^u$, we define the set $[(\mathbf{c}, \ .\ )] \subseteq \mathbb{R}^v$ as

$$[(\mathbf{c}, \ .\ )] := \left\{ \mathbf{x} \in \mathbb{R}^v : (\mathbf{c}, \mathbf{x}) \in \mathbb{R}^{u \times v} \right\},$$

(ii) $L_d(X)$ stands for the d-dimensional Lebesgue measure of the set $X \subseteq \mathbb{R}^d$.

We will use also the following well-known lemma which is an easy consequence of Fubini's theorem (see e.g., [3, Ch. 36, Thm. A]):

Lemma 2.6. *Let $H \subseteq \mathbb{R}^{u \times v}$ be given. H has Lebesgue measure 0:*

$$L_{u \times v}(H) = 0$$

if and only if all (except a set of Lebesgue measure 0) v-dimensional intersections of H have Lebesgue measure 0:

$$L_v(H \cap [(\mathbf{c}, \cdot)]) = 0$$

for all (but a set of Lebesgue measure 0) vectors $\mathbf{c} \in \mathbb{R}^u$.

Now we proceed towards Theorem 2.9 below.

Lemma 2.7. *Let $f : \mathbb{R}^{N \times K} \to \mathbb{R}^N$ be any function and $\mathbf{b} \in \mathbb{R}^N$ be any vector. If for all (but a set of Lebesgue measure 0) of $(K-1)$-tuples $(\mathbf{c}_1, \ldots, \mathbf{c}_{K-1}) \in \mathbb{R}^{N \times (K-1)}$ the set (of solutions)*

$$\left\{ \mathbf{x} \in \mathbb{R}^N \ : \ f(\mathbf{c}_1, \ldots, \mathbf{c}_{K-1}, \mathbf{x}) = \mathbf{b} \right\} \tag{2.4}$$

has Lebesgue measure 0, then the inverse image $f^{-1}(\mathbf{b}) \subseteq \mathbb{R}^{N \times K}$ has Lebesgue measure 0, too.

Proof. Denote $(\mathbf{c}_1, \ldots, \mathbf{c}_{K-1})$ by $\vec{\mathbf{c}}$. Since

$$f^{-1}(\mathbf{b}) = \bigcup \left\{ f^{-1}(\mathbf{b}) \cap \left[\left(\vec{\mathbf{c}}, \cdot\right)\right] \ : \ \vec{\mathbf{c}} \in \mathbb{R}^{N \times (K-1)} \right\}$$

and $f^{-1}(\mathbf{b}) \cap \left[\left(\vec{\mathbf{c}}, \cdot\right)\right]$ is exactly the set in (2.4), we can use Lemma 2.6. $\square$

To state our main results, we also need the following notion.

Definition 2.8. (i) Any two sequences $(x_n),(y_n) \subseteq \mathbb{R}^N$ **avoid each other** if and only if

$$(\forall n \in \mathbb{N}) \qquad x_n \neq y_n.$$

(ii) The sequence (x_n) **avoids** the sequence of sets $(B_n) \subseteq P(\mathbb{R}^N)$ if and only if

$$(\forall n \in \mathbb{N}\,) \quad x_n \notin B_n.$$

Now we can state our first main result.

Theorem 2.9. *We consider the N-dimensional recurrence equation (i.e., $(\mathbf{x}_n) \subseteq \mathbb{R}^N$)*

$$\mathbf{x}_{n+1} = h\left(\mathbf{x}_n, \ldots, \mathbf{x}_{n-k}\right), \tag{2.5}$$

where the function $h : \mathbb{R}^{N\times(k+1)} \to \mathbb{R}^N$ may be defined with cases and has (multiple) recursively weak set of roots. Let $(\mathbf{b}_n) \subseteq \mathbb{R}^N$ be any fixed (N-dimensional) sequence. Then for almost all initial values $\mathbf{x}_{-k}, \ldots, \mathbf{x}_0 \in \mathbb{R}^N$ (excluding a set of Lebesgue measure 0 of $\mathbb{R}^{N\times(k+1)}$) the corresponding solution $(\mathbf{x}_n)$ of Eq. (2.5) *avoids $(\mathbf{b}_n)$.*

Proof. Let us denote the initial values for $(\mathbf{x}_{-k}, \ldots, \mathbf{x}_0)$ by $(\mathbf{a}_0, \ldots, \mathbf{a}_k) \in \mathbb{R}^{N\times(k+1)}$, i.e., let us prescribe

$$\mathbf{x}_{-k} := \mathbf{a}_0, \ldots, \mathbf{x}_0 := \mathbf{a}_k. \tag{2.6}$$

For any fixed $n \in \mathbb{N}$ the requirement $\mathbf{x}_n \neq \mathbf{b}_n$ is ensured by the inequalities

$$h^{\vec{i}}\left(\mathbf{a}_0, \ldots, \mathbf{a}_k\right) \neq \mathbf{b}_n$$

for any multiindex $\vec{i} \in \mathbb{N}^n$ of length n. But these inequalities (for each $n \in \mathbb{N}$ countable many) hold for all but a set of Lebesgue measure 0 of the initial value vectors $(\mathbf{a}_0, \ldots, \mathbf{a}_k) \in \mathbb{R}^{N\times(k+1)}$. The union of these countable times countable many sets has still Lebesgue measure 0, which proves our statement. $\square$

In the next theorem we give a generalization of Theorem 2.9.

Theorem 2.10. *Let us consider the N-dimensional recurrence equation* (2.5), *where h has a (multiple) recursively weak set of roots. Let $(\mathbf{B}_n) \subseteq P(\mathbb{R}^N)$ be a fixed sequence of countable sets of N-dimensional vectors. Then for almost all initial values $x_{-k}, \ldots, x_0 \in \mathbb{R}^N$ (excluding a set of Lebesgue measure 0 of $\mathbb{R}^{N\times(k+1)}$), the corresponding solution $(\mathbf{x}_n)$ of Eq.* (2.5) *avoids (B_n).*

Proof. We introduce the sets $B_n \subseteq \mathbb{R}^N$ for $n \in \mathbb{N}$ by

$$B_n = \{b_n^{(i)} : i \in \mathbb{N}\}.$$

Now, defining the sequences

$$\mathbf{b}^{(i)} := \left(b_n^{(i)}\right)_{n\in\mathbb{N}}$$

for $i \in \mathbb{N}$, we have, by Theorem 2.9, for all $i \in \mathbb{N}$ a set $\mathcal{I}_i \subseteq \mathbb{R}^{N\times(k+1)}$ of Lebesgue measure $L(\mathcal{I}_i) = 0$ of initial value-vectors

$$(x_{-k}, \ldots, x_0) \in \mathbb{R}^{N\times(k+1)}$$

which generates sequences (x_n) only which does not avoid the sequence $\mathbf{b}^{(i)}$. The union of these sets $\mathcal{I}_i$ is still of Lebesgue measure 0. $\square$

3 Functions which have Recursively Weak Set of Roots

In this section we examine which types of recurrence relations have recursively weak set of roots. Unfortunately we do not have a general result for this question, but let us mention at the beginning that not every continuous function $f : \mathbb{R} \to \mathbb{R}$, which is not constant in any interval, has the property that the inverse image $f^{-1}(b)$ has Lebesgue measure 0 for any $b \in \mathbb{R}$.[3] Furthermore one can easily find even continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that for any $b \in \mathbb{R}$, the inverse images $f^{-1}(b)$ and $g^{-1}(b)$ both have Lebesgue measure 0 but for the composition $f \circ g$ this is NOT true.

Definition 3.1. A function $h : \mathbb{R}^m \to \mathbb{R}^N$ is called **algebraic** if and only if it can be expressed with the four basic operators + , - , * , / (of course h may contain any real numbers as constants, too).

In this section we prove the following result.

Theorem 3.2. *Any nonconstant*[4] *algebraic function $h : \mathbb{R}^n \to \mathbb{R}$ has a multiple recursively weak set of roots.*

The proof of Theorem 3.2 uses the following easy corollary of the fundamental theorem of algebra:

Lemma 3.3. *For each polynomial $p(x_1, \ldots, x_n)$ not identically 0 there are* **finite** *sets $F_1, \ldots, F_n \subset \mathbb{R}$ with the property such that the equation*

$$p(x_1, \ldots, x_n) = 0 \tag{3.1}$$

has only finitely many roots $(x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfying $x_j \notin F_j$ for all $j \leq n$. In other words, the equation (3.1) *may have infinitely many roots only if $x_j \in F_j$ for some $j \leq n$.*

[3]For example take any closed set $C \subseteq \mathbb{R}$ of positive measure. Since the complement of C is a union of disjoint open sets, we can define $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = 0$ for $x \in C$ and the reversed parabola $f(x) := (x-a)(b-x)$ for $a \leq x \leq b$ for any interval (a, b) of C's complement. Many more variations are possible, too.

[4]The term "nonconstant" (not identically constant) means that there is no real number $c \in \mathbb{R}$ such that $h(x_n, \ldots, x_1) = c$ for all $(x_n, \ldots, x_1) \in \mathbb{R}^n$.

Though Lemma 3.3 may be well known, we include its short proof in the Appendix. Lemma 3.3 implies easily that the set of roots of each polynomial has Lebesgue measure 0. Theorem 3.2 states that for each $i \in \mathbb{N}$ and for each $b \in \mathbb{R}$, the set (of solutions)

$$\left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ h^{[i]}(x_1, \ldots, x_n) = b \right\}$$

has Lebesgue measure 0. Though all iterates $h^{[i]}$ are also algebraic, we cannot use Lemma 3.3 directly: We have to ensure that all iterates $h^{[i]}$ are not equal identically to b, too! The following result proves Theorem 3.2.

Theorem 3.4. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a nonconstant algebraic function. Then for every $b \in \mathbb{R}$ there exist finite (or empty) sets $E^b_{j,i} \subset \mathbb{R}$ ($j \leq n$, $i \in \mathbb{N}$) such that the equality*

$$h^{[i]}(x_n, \ldots, x_1) = b \tag{3.2}$$

has only finitely many roots $(x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfying $x_j \notin E^b_{j,i}$ for all $j \leq n$. In other words, the equality (3.2) *may have infinitely many roots only if $x_j \in E^b_{j,i}$ for some $j \leq n$.*

Proof. It is easy to see that all algebraic functions have the form

$$h(x_n, \ldots, x_1) = \frac{k_1(x_n, \ldots, x_1)}{k_2(x_n, \ldots, x_1)}$$

for some polynomials k_1 and k_2. So, all equalities

$$h(x_n, \ldots, x_1) = \frac{k_1(x_n, \ldots, x_1)}{k_2(x_n, \ldots, x_1)} = b$$

are equivalent to

$$k_1(x_n, \ldots, x_1) - b \cdot k_2(x_n, \ldots, x_1) = 0. \tag{3.3}$$

We now use induction on $i \in \mathbb{N}$, $i \geq 1$. If $i = 1$, then, by (3.3), we can let

$$E^{(b)}_{j,1} := F^{(n)}_j \left(k_1(x_n, \ldots, x_1) - b \cdot k_2(x_n, \ldots, x_1) \right).$$

Let $1 \lneqq i \leq n$. Using the notation $\overset{\Rightarrow}{x} := (x_n, \ldots, x_1)$, we have to investigate the equality

$$h^{[i]}\left(\overset{\Rightarrow}{x}\right) = h\left(h^{[i-1]}\left(\overset{\Rightarrow}{x}\right), \ldots, h^{[1]}\left(\overset{\Rightarrow}{x}\right), x_n, \ldots, x_i\right) = b. \tag{3.4}$$

By the induction hypothesis, this equality might have infinitely many roots only if

$$x_j \in E^{(b)}_{j-i+1,\,1} \qquad (i \leq j \leq n)$$

and

$$h^{[\ell]}\left(\overset{\Rightarrow}{x}\right) = \beta \in E^{(b)}_{n-i+\ell+1,\,1} \quad (1 \leq \ell \leq i-1),$$

i.e.,

$$x_j \in E_{j,\ell}^{(\beta)}, \quad \text{where} \quad \beta \in E_{n-i+\ell+1,\,1}^{(b)} \qquad (1 \le \ell \le i-1, 1 \le j \le n)\,.$$

So, for $1 \le j < i$, we may let

$$E_{j,i}^{(b)} := \bigcup_{\ell=1}^{i-1} \left(\cup \left\{ E_{j,\ell}^{(\beta)} : \beta \in E_{n-i+\ell+1,\,1}^{(b)} \right\} \right),$$

while for $i \le j \le n$,

$$E_{j,i}^{(b)} := \bigcup_{\ell=1}^{i-1} \left(\cup \left\{ E_{j,\ell}^{(\beta)} : \beta \in E_{n-i+\ell+1,\,1}^{(b)} \right\} \right) \cup E_{j-i+1,\,1}^{(b)}.$$

If $i > n$, then (3.4) changes to

$$h^{[i]} \left(\overset{\Rightarrow}{x} \right) = h \left(h^{[i-1]} \left(\overset{\Rightarrow}{x} \right), \ldots, h^{[i-n]} \left(\overset{\Rightarrow}{x} \right) \right) = b. \tag{3.5}$$

Similarly to the case $1 \lneqq i \le n$, we can let

$$E_{j,i}^{(b)} := \bigcup_{\ell=1}^{i-1} \left(\cup \left\{ E_{j,\ell}^{(\beta)} : \beta \in E_{n-i+\ell+1,\,1}^{(b)} \right\} \right)$$

for $1 \le j \le n$. □

Theorem 3.4 clearly implies Theorem 3.2 (using Lemmas 2.6, 2.7 and Theorem 2.9). Our further research will include higher dimensional, not only algebraic functions h with cases.

4 Appendix

Though Lemma 3.3 may be well known, we include its short proof here.

Proof of Lemma 3.3. We use induction on $n \in \mathbb{N}$. The sets F_j depend on p, and we must handle many polynomials so we use the notation $F_j^{(n)}(p)$. If $n = 1$, then we let $F_1 = F_1^{(1)} := \emptyset$ since p is nonconstant. Now let $n > 1$. Write p for $1 \le j \le n$ as

$$p(x_1, \ldots, x_n) = \sum_{t=0}^{\Delta_j} (x_j)^t \cdot p_{j,t} \left(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n \right),$$

where $\Delta_j \in \mathbb{N}$ and

$$p_{j,t} : \mathbb{R}^{n-1} \to \mathbb{R} \qquad (1 \le j \le n,\ 0 \le t \le \Delta_j)$$

are suitable polynomials. (Some of $p_{j,t}$ may be constant but not all of them.) By the induction hypothesis, we have finite sets $F_{\ell,j}^{(n-1)}(p_{j,t}) \subset \mathbb{R}$ such that if $x_\ell \notin F_{\ell,j}^{(n-1)}(p_{j,t})$, $\ell \neq j$, then the polynomials $p_{j,t}$ may have finitely many roots[5]

$$(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in \mathbb{R}^{n-1}.$$

Let $R_{\ell,j}^{(n-1)}(p_{j,t})$ bet the set of the x_ℓ-components of these roots:

$$R_{\ell,j}^{(n-1)}(p_{j,t}) := \{x_\ell \in \mathbb{R} \mid (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \text{ is a root of } p_{j,t}\} \quad (\ell \neq j)$$

and further

$$F_{\ell,j}^{(n)} := \left(\bigcup_{t=0}^{\Delta_j} F_{\ell,j}^{(n-1)}(p_{j,t}) \right) \cup \left(\bigcap_{t=0}^{\Delta_j} R_{\ell,j}^{(n-1)}(p_{j,t}) \right)$$

and finally

$$F_\ell^{(n)} := \bigcup_{j=1, j\neq\ell}^{n} F_{\ell,j}^{(n)}.$$

The sets $F_\ell^{(n)}$ for $\ell \leq n$ are clearly finite and satisfy the requirement of the lemma. $\square$

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