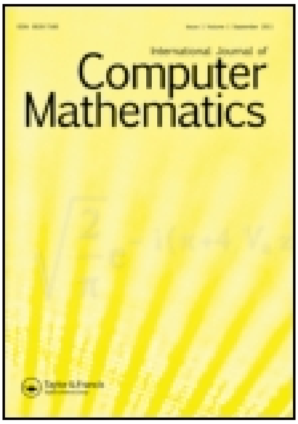

International Journal of Computer Mathematics
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gcom20

# Finding better weight functions for generalized Shepard's operator on infinite intervals

Biancamaria Della Vecchia [a] & István Szalkai [b]

[a] Dipartimento di Matematica, Istituto G. Castelnuovo, Università degli Studi di Roma 'La Sapienza', I-00185, Rome, Italy

[b] Department of Mathematics, Pannon University, Veszprém, H-8200, Hungary

Available online: 14 Jun 2011

To cite this article: Biancamaria Della Vecchia & István Szalkai (2011): Finding better weight functions for generalized Shepard's operator on infinite intervals, International Journal of Computer Mathematics, 88:13, 2838-2851

To link to this article: http://dx.doi.org/10.1080/00207160.2011.559542

*International Journal of Computer Mathematics*
Vol. 88, No. 13, September 2011, 2838–2851



# Finding better weight functions for generalized Shepard's operator on infinite intervals

Biancamaria Della Vecchia[a] and István Szalkai[b]*

[a]*Dipartimento di Matematica, Istituto G. Castelnuovo, Università degli Studi di Roma 'La Sapienza' I-00185 Rome, Italy;* [b]*Department of Mathematics, Pannon University, Veszprém, H-8200, Hungary*

(*Received 1 February 2007; resubmitted 17 November 2008; revised version received 22 April 2009; second revision received 11 July 2009; third revision received 17 November 2009; fourth revision received 11 July 2010; fifth revision received 20 July 2010; accepted 27 January 2011*)

In this paper, we investigate the difference of Shepard's generalized operators $S_\sigma$ from the approximated set of data for various weight functions $\sigma$. Bounds are given for the sizes of the 'bumps' shown on the graph of $S_\sigma$ for $\sigma(d) = 1/d$ in dimension $N = 1$, and the best weight function $\sigma$ for practical use is proposed.

**Keywords:** interpolation; approximation; generalized Shepard method

*2000 AMS Subject Classifications*: 41A05; 41A20; 41A36

## 1. Introduction

For any given set of datapoints $\{P_1, \ldots, P_M\} \subseteq \mathbb{R}^N$ in any dimension $N \geq 1$, real numbers $F_1, \ldots, F_M \in \mathbb{R}$ and fixed weight function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$, we investigate the generalized Shepard operator $S_\sigma : \mathbb{R}^N \to \mathbb{R}^+$ defined for any $P \in \mathbb{R}^N$ as

$$S_\sigma^{(M)}(P) := \frac{\sum_{i=1}^{M} F_i \sigma(d(P, P_i))}{\sum_{i=1}^{M} \sigma(d(P, P_i))},$$

where $d : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is any distance function on $\mathbb{R}^N$. (For simplicity we omit the superscript $M$ whenever it is clear from the context.)

The main advantage of the above simple formula is that it is applicable for any set of points $\{P_1, \ldots, P_M\} \subseteq \mathbb{R}^N$ (which is not our choice in general in practice). Let us highlight our main point of view: we consider $S_\sigma$ for constructing a surface matching any given set of data[1] $\{(P_1, F_1), \ldots, (P_M, F_M)\}$, i.e. we do not consider $S_\sigma$ for approximating any pre-given function $f : \mathbb{R}^N \to \mathbb{R}$.

This method is widely applied, e.g. in geography for dimension $N = 2$ (see, e.g. [10]).

*Corresponding author. Email: szalkai@almos.uni-pannon.hu

ISSN 0020-7160 print/ISSN 1029-0265 online

DOI: 10.1080/00207160.2011.559542
http://www.informaworld.com

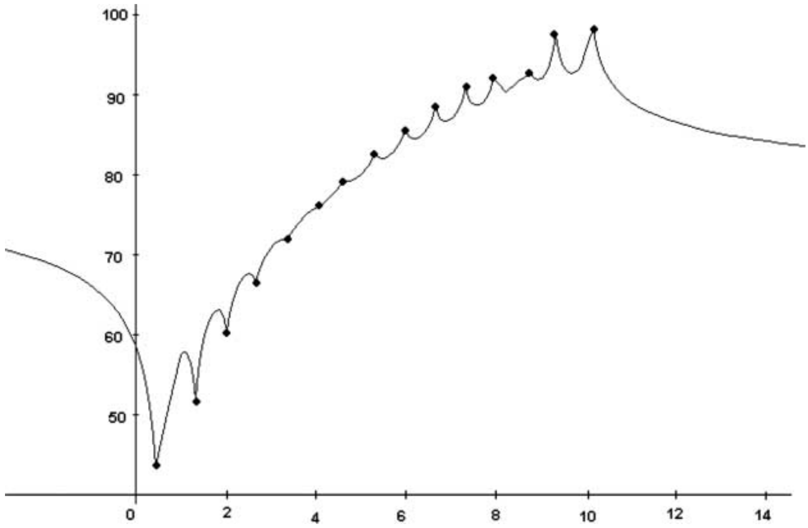


Figure 1. The graph of $S_\sigma$ for $\sigma(d) = 1/d$ in dimension $N = 1$.

For exact approximation (that is, $S_\sigma(P_i) = F_i$ for all $i \leq M$) $\sigma$ must satisfy

$$\lim_{d \to 0+} \sigma(d) = +\infty. \tag{1}$$

Further, we require

$$\lim_{d \to +\infty} \sigma(d) = 0 \tag{2}$$

since in our investigations $M \to \infty$ and so $d \to \infty$ (see [17]).

In the present paper, we restrict ourselves to dimension $N = 1$. However, the results we obtain can be used for any dimension, since any distortion of higher-dimensional surfaces (defined by $S_\sigma$) can be detected in a suitable one-dimensional intersection.

The starting point of our investigation was the surprising diagram of $S_{1/d}$, shown in Figure 1 (in dimension $N = 1$):

(A computer program for demonstrating and investigating different approximation methods is also in preparation in [13].)

Black dots in the above figure show the pairs $(P_i, F_i)$ for $i \leq M$. What disturbs us is that the approximating formula $S_\sigma(P)$ has big differences ('waves') in many places despite the almost linear dataset. (In other words: $S_\sigma$ tends to the average $\bar{F} := (F_1 + \cdots + F_M)/M$ not only when $P \to \infty$ but even when $P$ is inside the convex hull of the dataset $\{P_1, \ldots, P_M\}$.)

In this note, we show that these 'bumps' (big differences) are present almost in all cases. More precisely, we calculate the rate of these differences for several weight functions $\sigma$:

$$\sigma_1(d) := \frac{1}{d^\alpha} \quad (\alpha > 0) \text{ (Shepard's original formula)},$$

$$\sigma_2(d) := \frac{1}{d^\alpha} \exp(-\lambda d^\beta) \quad (\alpha, \beta, \lambda > 0),$$

$$\sigma_3(d) := \frac{1}{\ln^\beta(d+1)} \quad (\beta > 0),$$

$$\sigma_4(d) := \frac{1}{d^\alpha} \frac{1}{\ln^\beta(d+1)} \quad (\alpha, \beta > 0)$$

($\sigma_1$ is the original weight function of [7]. The others are our candidates for better approximation. We do not have so many choices since we have to ensure (1).)

For most of the investigated cases, the size of the differences goes to infinity when the number of the datapoints $M$ tends to infinity. This latter assumption requires infinite domain for the approximation. This is why we investigate $\lim_{M\to\infty}$ in Questions 1–3.

Though everyday approximations are done on finite intervals, in most cases we cannot choose as many datapoints $P_i$ as we like as, e.g. in the application Katona [10]. This could result in the unexpected waves as in Figure 1.

In the literature, numerous excellent properties of Shepard's original and generalized formulae are justified, see, e.g. in Allasia [1], Bojanic *et al.* [2], Della Vecchia *et al.* [3,4], Farwig [5], Gál and Szabados [6], Gordon and Wixom [7], Hoschek and Lasser [8], Mastroianni and Szabados [12], Szabados [14], Szalkai [16,17] or Zhou [19]. These good approximation properties are proved either assuming a special set of datapoints $\{P_1, \ldots, P_M\}$, or by investigating the limit-approximation in the case when the number of the datapoints $M$ tends to infinity on a fixed finite interval. Elimination of these restrictions is the main improvement of our analysis with respect to other investigations.

Katona [10], Láng-Lázi *et al.* [11] and Szalkai [15,18] tried to apply Shepard's original formula in practice. We suggest using the weight functions that we will select in Section 3.



### 1.1 *Preliminary definitions*

We are treating the $N = 1$-dimensional case[2]. In the present investigation, let us define the set of datapoints to be equidistant (they form an arithmetic progression), i.e. we fix $u, v > 0$ and we let

$$P_i := P_1 + (i-1)v \quad \text{and} \quad F_i := F_1 + (i-1)u, \ \text{for } i = 1, \ldots, M \tag{3}$$

(that is $P_i \in \mathbb{R}$ are in $N = 1$-dimension).

We investigate the difference of $S_\sigma$ from the straight line $\ell(x)$

$$\ell(x) = F_1 + \tau u \quad \text{for } x = P_1 + \tau v \ (\tau \in \mathbb{R})$$

($\ell$ connects all the points $(P_i, F_i)$) at the point $x_\tau \in (P_1, P_2)$

$$x_\tau = P_1 + \tau v \quad (\tau \in (0,1)),$$

i.e. we calculate

$$\Delta(x_\tau) = S_\sigma(x_\tau) - \ell(x_\tau)$$

for various weight functions $\sigma$. Our set of data $\{(P_i, F_i),\ i = 1, \ldots, M\}$ is equidistant, other sets of data are investigated in [ [17], Section 4.2].

The difference is

$$\begin{aligned}\Delta(x_\tau) &= S_\sigma(x_\tau) - \ell(x_\tau)\\ &= \frac{F_1\sigma(\tau v) + \sum_{i=2}^{M}(F_1 + (i-1)u)\sigma((i-1-\tau)v)}{\sigma(\tau v) + \sum_{i=2}^{M}\sigma((i-1-\tau)v)} - (F_1 + \tau u)\\ &= u\left(\frac{\sum_{j=1}^{M-1} j\sigma((j-\tau)v)}{\sigma(\tau v) + \sum_{j=1}^{M-1}\sigma((j-\tau)v)} - \tau\right).\end{aligned} \tag{4}$$

We investigate the following questions for fixed $\tau \in (0,1)$ (i.e. $x_\tau \in (P_1, P_2)$ is fixed[3]):

*Question 1* Is $\lim_{M\to\infty} \Delta(x_\tau) = \infty$ or $\lim_{M\to\infty} \Delta(x_\tau) < \infty$?

In the latter case: what is the value of $\lim_{M\to\infty} \Delta(x_\tau)/u$ approximatively?

*Question 2* For which weight functions $\sigma$ do we have $\lim_{M\to\infty} S_\sigma(x_\tau) > F_2$ for some $x_\tau \in (P_1, P_2)$?

(This inequality is equivalent to $\lim_{M\to\infty} \Delta(x_\tau)/u > 1-\tau$.)

*Question 3* For which point $x_\tau \in (P_1, P_2)$ is $\lim_{M\to\infty} \Delta(x_\tau)/u$ maximal?

Similar questions might be investigated for the approximation in the finite interval $[0, a]$ ($S_\sigma$ is invariant for vertical translation but not for vertical zooming).

The following well-known results will be useful to our work:

LEMMA 1 *Let $a_0, a_1, \ldots \in \mathbb{R}_+$ with $a_j \to 0$. Then the fractions*

$$\frac{\sum_{j=0}^{M} j a_j}{\sum_{j=0}^{M} a_j}$$

*have a finite limit for $M \to \infty$ if and only if $\sum_{j=0}^{\infty} j a_j < \infty$.*

## 2. Investigating the weight functions

Now we investigate the weight functions $\sigma_1$–$\sigma_4$ in detail.

### 2.1 *The weight function $\sigma_1(d) = 1/d^\alpha$*

Since $\sigma$ now is homogeneous, we have

$$\frac{\Delta(x_\tau)}{u} = \frac{\sum_{j=1}^{M-1} j\sigma(j-\tau)}{\sigma(\tau) + \sum_{j=1}^{M-1} \sigma(j-\tau)} - \tau = \frac{\sum_{j=1}^{M-1} j/(j-\tau)^\alpha}{1/\tau^\alpha + \sum_{j=1}^{M-1} 1/(j-\tau)^\alpha} - \tau.$$

It is well known that the denominator is convergent iff $\alpha > 1$, while the numerator is convergent iff $\alpha > 2$.

This means that Shepard's original formula

$$S_\alpha(P) := \frac{\sum_{i=1}^{M} F_i 1/(d(P, P_i))^\alpha}{\sum_{i=1}^{M} 1/(d(P, P_i))^\alpha}$$

must have as large bumps as one likes for all $1 < \alpha \le 2$, while the size of bumps is bounded for $2 < \alpha$:

THEOREM 1 *For all $1 < \alpha \le 2$ the limit $\lim_{M\to\infty} \Delta(x_\tau)/u = \infty$ diverges, while for $\alpha > 2$ we have*

$$\frac{1/(1-\tau)^\alpha + \zeta(\alpha-1) - 1}{1/\tau^\alpha + 1/(1-\tau)^\alpha + \zeta(\alpha)} - \tau \le \lim_{M\to\infty} \frac{\Delta(x_\tau)}{u} \le \frac{1/(1-\tau)^\alpha + \zeta(\alpha-1) + \zeta(\alpha)}{1/\tau^\alpha + 1/(1-\tau)^\alpha + \zeta(\alpha) - 1} - \tau, \quad (5)$$

*where $\zeta$ is Riemann's zeta function.*

*Proof* In the case of $\alpha > 2$ in order to approximate the value of $\lim_{M\to\infty} \Delta(x_\tau)/u$ we write for the denominator

$$\frac{1}{\tau^\alpha} + \frac{1}{(1-\tau)^\alpha} + \sum_{j=2}^{M-1} \frac{1}{j^\alpha} < \frac{1}{\tau^\alpha} + \sum_{j=1}^{M-1} \frac{1}{(j-\tau)^\alpha} < \frac{1}{\tau^\alpha} + \frac{1}{(1-\tau)^\alpha} + \sum_{j=2}^{M-1} \frac{1}{(j-1)^\alpha},$$

i.e.

$$\frac{1}{\tau^\alpha} + \frac{1}{(1-\tau)^\alpha} + \zeta(\alpha) - 1 \le \lim_{M\to\infty}(\text{den}) \le \frac{1}{\tau^\alpha} + \frac{1}{(1-\tau)^\alpha} + \zeta(\alpha)$$

and for the numerator

$$\frac{1}{(1-\tau)^\alpha} + \sum_{j=2}^{M-1} \frac{j}{j^\alpha} < \sum_{j=1}^{M-1} \frac{j}{(j-\tau)^\alpha} < \frac{1}{(1-\tau)^\alpha} + \sum_{j=2}^{M-1} \frac{j-1+1}{(j-1)^\alpha},$$

i.e.

$$\frac{1}{(1-\tau)^\alpha} + \zeta(\alpha-1) - 1 \le \lim_{M\to\infty}(\text{num}) \le \frac{1}{(1-\tau)^\alpha} + \zeta(\alpha-1) + \zeta(\alpha),$$

which implies the estimation (5), answering Question 1. ■

Question 2 could be answered by the inequality

$$1 - \tau \le \frac{1/(1-\tau)^\alpha + \zeta(\alpha-1) - 1}{1/\tau^\alpha + 1/(1-\tau)^\alpha + \zeta(\alpha)} - \tau,$$

i.e.

$$\frac{1}{\tau^\alpha} + \frac{1}{(1-\tau)^\alpha} + \zeta(\alpha) \le \frac{1}{(1-\tau)^\alpha} + \zeta(\alpha-1) - 1$$

or by the much simpler one

$$\frac{1}{\tau^\alpha} + 1 \le \zeta(\alpha-1) - \zeta(\alpha). \tag{6}$$

For each fixed $\alpha$ the left-hand side has a minimal value for $\tau = 1$, so Equation (6) admits a solution for $\tau$ iff

$$2 \le \zeta(\alpha-1) - \zeta(\alpha). \tag{7}$$

From our computational experiments we learned that Equation (7) holds for

$$2 < \alpha < 2.3617$$

and does not hold for $1 < \alpha < 2$ or $\alpha > 2.3617$.

For Question 3, we should find the maximal value(s) of

$$\frac{\Delta(x_\tau)}{u} := \frac{1/(1-\tau)^\alpha + \zeta(\alpha-1) - 1}{1/\tau^\alpha + 1/(1-\tau)^\alpha + \zeta(\alpha)} - \tau,$$

where $\tau \in (0, 1)$ for each fixed $\alpha > 2$.

### 2.2 *The weight function* $\sigma_2(d) = 1/d^\alpha \exp(-\lambda d^\beta)$

In this case, $\Delta(x_\tau)/u$ reads as

$$\begin{aligned}\frac{\Delta(x_\tau)}{u} &= \frac{\sum_{j=1}^{M-1} j\exp(-\lambda((j-\tau)v)^\beta)/((j-\tau)v)^\alpha}{\exp(-\lambda(\tau v)^\beta)/(\tau v)^\alpha + \sum_{j=1}^{M-1}\exp(-\lambda((j-\tau)v)^\beta)/((j-\tau)v)^\alpha} - \tau \\ &= \frac{\sum_{j=1}^{M-1} jE^{(j-\tau)^\beta}/(j-\tau)^\alpha}{E^{\tau^\beta}/\tau^\alpha + \sum_{j=1}^{M-1} E^{(j-\tau)^\beta}/(j-\tau)^\alpha} - \tau \end{aligned} \tag{8}$$

where

$$E := \exp(-\lambda v^\beta)$$

($v$ was defined in Equation (3)).

Since $0 < E$ and $\tau < 1$, we can easily prove

THEOREM 2 $\lim_{M\to\infty} \Delta(x_\tau)/u$ *is convergent for all* $\alpha, \beta, \lambda > 0$, $\tau \in (0, 1)$.

*Proof* We use Lemma 0 for the sequence $a_j = E^{(j-\tau)^\beta}/(j-\tau)^\alpha$ $(1 \le j)$, $a_0 = E^{\tau^\beta}/\tau^\alpha$. The assumptions $a_j > 0$ and $a_j \to 0$ clearly hold since $|E| < 1$ and $1 \le j$. The numerator can be estimated as

$$\sum_{j=1}^{\infty} j \frac{E^{(j-\tau)^\beta}}{(j-\tau)^\alpha} \le a_1 + \sum_{j=2}^{\infty} j E^{(j-1)^\beta} = a_1 + \sum_{i=1}^{\infty} (i+1) E^{i^\beta}. \tag{9}$$

Using the fact that

$$\lim_{i\to\infty} \frac{i^\beta}{\log_{1/E}(i)} = \infty,$$

we can find $i_0 \in \mathbb{N}$ such that $i^\beta > 3\log_{1/E}(i)$ for $i > i_0$ . This proves Equation (9) since

$$\begin{aligned}\sum_{i=1}^{\infty} (i+1) E^{i^\beta} &\le \sum_{i=1}^{i_0} (i+1) E^{i^\beta} + \sum_{i=i_0}^{\infty} (i+1) E^{3\log_{1/E}(i)} \\ &\le c + \sum_{i=i_0}^{\infty} \frac{i+1}{i^3},\end{aligned}$$

which clearly converges. The denominator does not exceed the numerator so it converges as well. ■

Now we present detailed calculations for the case $\alpha = \beta = 1$ (calculations for the general case of $\alpha$ and $\beta$ are lengthy). In this case, the numerator of Equation (8) is

$$\begin{aligned}\sum_{j=1}^{M-1} j \frac{E^{(j-\tau)}}{(j-\tau)} &= \sum_{j=1}^{M-1} \left(1 + \frac{\tau}{j-\tau}\right) E^{(j-\tau)} = E^{-\tau} \sum_{j=1}^{M-1} E^j + \tau \sum_{j=1}^{M-1} \frac{E^{(j-\tau)}}{j-\tau} \\ &= \frac{E}{E^\tau} \frac{E^{M-1}-1}{E-1} + \tau \mathcal{I}_M(E,\tau),\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_M(E,\tau) := \sum_{j=1}^{M-1} \frac{E^{(j-\tau)}}{(j-\tau)} &= \frac{E^{1-\tau}}{1-\tau} + \sum_{j=2}^{M-1} \int_0 E^{(j-\tau-1)}\,\mathrm{d}E \\ &= \frac{E^{1-\tau}}{1-\tau} + \int_0 E^{1-\tau} \sum_{J=0}^{M-3} E^J\,\mathrm{d}E = \frac{E^{1-\tau}}{1-\tau} + \int_0 E^{1-\tau} \frac{E^{M-2}-1}{E-1}\,\mathrm{d}E \\ &= \frac{E^{1-\tau}}{1-\tau} + \int_0^E x^{1-\tau} \frac{x^{M-2}-1}{x-1}\,\mathrm{d}x\end{aligned}$$

which has a limit $(M \to \infty)$

$$\mathcal{I}_\infty(E,\tau) := \frac{E^{1-\tau}}{1-\tau} + \int_0^E x^{1-\tau} \frac{1}{1-x}\,\mathrm{d}x.$$

So we finally obtain:

THEOREM 3

$$\mathcal{L}(E,\tau) := \lim_{M\to\infty} \frac{\Delta(x_\tau)}{u} = \frac{E^{1-\tau}/(1-E) + \tau\mathcal{I}_\infty(E,\tau)}{E^\tau/\tau + \mathcal{I}_\infty(E,\tau)} - \tau \quad \textit{for}\ \alpha=\beta=1. \tag{10}$$

*This answers Question* 1.

It is easy to see that

$$\int x^{1-\tau}\frac{1}{1-x}\,\mathrm{d}x = \frac{x_2^{1-\tau}F_1(1-\tau;1;2-\tau;x)-1}{1-\tau},$$

with ${}_2F_1(w,z,y,x)$ being the hypergeometric function, so that

$$\mathcal{I}_\infty(E,\tau) = \frac{E^{1-\tau}}{1-\tau}\cdot{}_2F_1(1-\tau;1;2-\tau;E). \tag{11}$$

Figures 2–4 show 3D views and intersections of $\mathcal{L}$ vs. $E$ and $\tau$ in different scaling. The hypergeometric function on the right-hand-side of Equation (11) was computed by means of a routine included in the package of special functions by Jin and Zhang [9]. Points 0 and 1 are excluded from the plots.

Since we are looking for the best approximating function $S_\sigma$ including $E=\exp(-\lambda v^\beta)$, we can conclude in the case $\alpha=\beta=1$ the following:

After estimating the largest or most common values of $v$ we must choose $\lambda$ such that

$$E=\exp(-\lambda v) < 0.6$$

which will make $\lim_{M\to\infty}\Delta(x_\tau)/u$ very small!

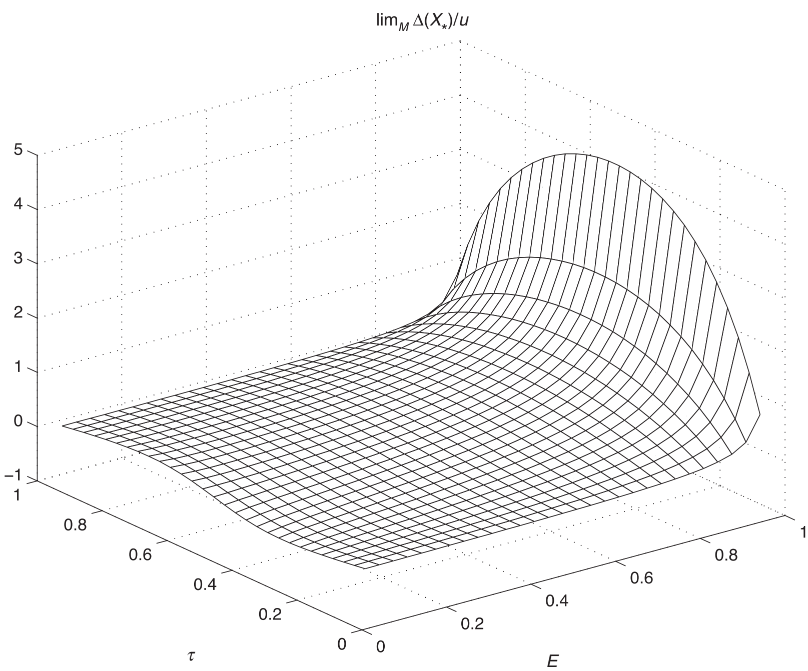


Figure 2. Plot $\lim_\infty \Delta(x_\tau)/u$ vs. $\tau$ and $E$, scale $[-1, 5]$.

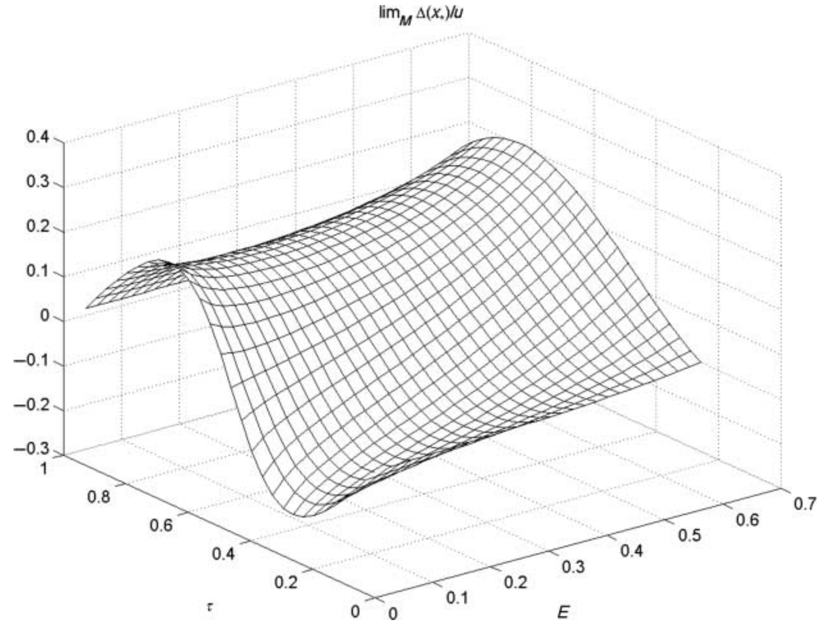


Figure 3. Plot $\lim_{\infty} \Delta(x_\tau)/u$ vs. $\tau$ and $E$, scale $[-0.3, 0.4]$.

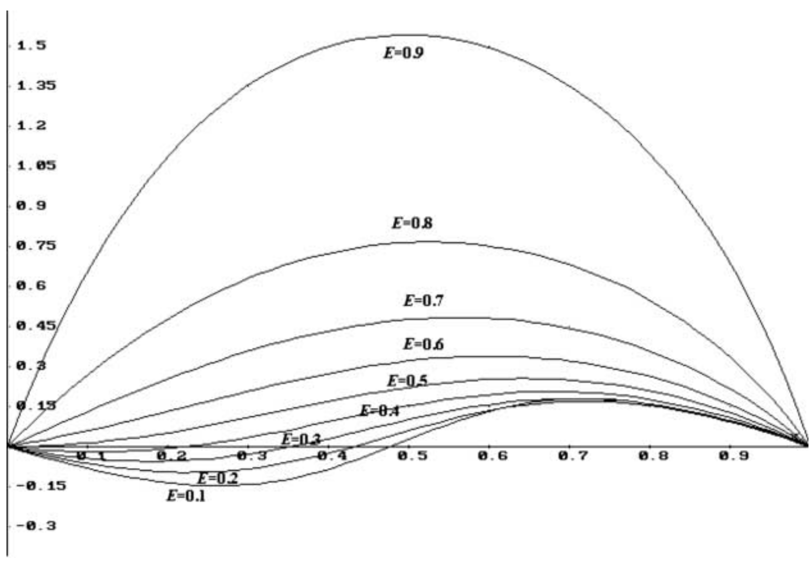


Figure 4. Plot $\lim_{\infty} \Delta(x_\tau)/u$ vs. $\tau$ for $E = 0.1, 0.2, \ldots, 0.9$.

Let us note that formula (10) for $\mathcal{L}(E, \tau)$ can also be written as

$$\begin{aligned}\mathcal{L}(E, \tau) &= \frac{\tau[E^\tau/\tau + \mathcal{I}(E, \tau)] + E^{1-\tau}/(1-E) - E^\tau}{E^\tau/\tau + \mathcal{I}(E, \tau)} - \tau \\ &= \frac{E^{1-\tau}/(1-E) - E^\tau}{E^\tau/\tau + E^{1-\tau}/(1-\tau) + \int_0^E x^{1-\tau}/(1-x)\,\mathrm{d}x} \\ &= \frac{\tau(1-\tau)[E^{1-\tau} - (1-E)E^\tau]}{(1-E)[(1-\tau)E^\tau + \tau E^{1-\tau} + \tau(1-\tau)\int_0^E x^{1-\tau}/(1-x)\,\mathrm{d}x]}.\end{aligned}$$

It is easy to see that, for fixed $E \in (0, 1)$

$$\lim_{\tau\to 0} \mathcal{L}(E, \tau) = 0 \quad \text{and} \quad \lim_{\tau\to 1} \mathcal{L}(E, \tau) = 0,$$

which correspond to the fact that $S_\sigma$ is exact (that is, $S_\sigma(P_i) = F_i$).

For Question 2, we should solve the inequality

$$\frac{E^{1-\tau}/(1-E) + \tau\mathcal{I}(E, \tau)}{E^\tau/\tau + \mathcal{I}(E, \tau)} - \tau > 1 - \tau,$$

that is

$$\frac{E^{1-\tau}}{1-E} + \tau\left(\frac{E^{1-\tau}}{1-\tau} + \int_0^E \frac{x^{1-\tau}}{1-x}\,\mathrm{d}x\right) > \frac{E^\tau}{\tau} + \frac{E^{1-\tau}}{1-\tau} + \int_0^E \frac{x^{1-\tau}}{1-x}\,\mathrm{d}x,$$

i.e.

$$\frac{E^{1-\tau}}{1-E}E - \frac{E^\tau}{\tau} > (1-\tau)\int_0^E \frac{x^{1-\tau}}{1-x}\,\mathrm{d}x,$$

or, using the hypergeometric function ${}_2F_1$,

$$\frac{E}{1-E} - \frac{E^{2\tau-1}}{\tau} >_2 F_1(1-\tau; 1; 2-\tau; E) - 1.$$

Some more computer experiments are necessary for solving this inequality, we do not include them here.

### 2.3 *The weight function* $\sigma_3(d) = 1/\ln^\beta(d+1)$

Now $\Delta(x_\tau)/u$ reads as

$$\frac{\Delta(x_\tau)}{u} = \frac{\sum_{j=1}^{M-1} j/\ln^\beta((j-\tau)v+1)}{1/\ln^\beta(\tau v+1) + \sum_{j=1}^{M-1} 1/\ln^\beta((j-\tau)v+1)} - \tau.$$

Since

$$\sum_{j=1}^{\infty} \frac{1}{\ln^\beta((j-\tau)v+1)} \geq \sum_{j=2}^{\infty} \frac{1}{\ln^\beta(j^2)} - c = \infty$$

we see that:

THEOREM 4 *For the weight function* $\sigma(d) = 1/\ln^\beta(d+1)$ *we have for all* $\beta > 0$

$$\lim_{M\to\infty} \frac{\Delta(x_\tau)}{u} = \infty,$$

*answering Question* 1.

### 2.4 *The weight function $\sigma_4(d) = 1/d^\alpha 1/\ln^\beta(d+1)$*

In this case, we have

$$\frac{\Delta(x_\tau)}{u} = \frac{\sum_{j=1}^{M-1} j/(j-\tau)^\alpha v^\alpha \ln^\beta((j-\tau)v+1)}{1/\tau^\alpha v^\alpha \ln^\beta(\tau v+1) + \sum_{j=1}^{M-1} 1/(j-\tau)^\alpha v^\alpha \ln^\beta((j-\tau)v+1)} - \tau$$
$$= \frac{\sum_{j=1}^{M-1} j/(j-\tau)^\alpha \ln^\beta((j-\tau)v+1)}{1/\tau^\alpha \ln^\beta(\tau v+1) + \sum_{j=1}^{M-1} 1/(j-\tau)^\alpha \ln^\beta((j-\tau)v+1)} - \tau.$$

We will use the following fact from elementary calculus:

LEMMA 2 *The sum*

$$\mathcal{L}(\alpha,\beta,v) := \sum_{\substack{j=1\\ jv\neq 1}}^{\infty} \frac{1}{j^\alpha \ln^\beta(jv)} \quad (v > 0 \textit{ fixed})$$

*is convergent iff either $\alpha = 1$ and $\beta > 1$ or $\alpha > 1$ and $\beta > 0$.*

Now we can start answering Question 1:

$$\sum_{j=1}^{\infty} \frac{1}{(j-\tau)^\alpha \ln^\beta((j-\tau)v+1)} \geq \sum_{j=1}^{\infty} \frac{1}{j^\alpha \ln^\beta(j(v+1))} = \mathcal{L}(\alpha,\beta,v+1),$$

$$\sum_{j=1}^{\infty} \frac{1}{(j-\tau)^\alpha \ln^\beta((j-\tau)v+1)} \leq \frac{1}{(1-\tau)^\alpha \ln^\beta((1-\tau)v+1)} + \sum_{\substack{j=2\\ (j-1)v\neq 1}}^{\infty} \frac{1}{(j-1)^\alpha \ln^\beta((j-1)v)}$$
$$= \frac{1}{(1-\tau)^\alpha \ln^\beta((1-\tau)v+1)} + \mathcal{L}(\alpha,\beta,v)$$

and

$$\sum_{j=1}^{M-1} \frac{j}{(j-\tau)^\alpha \ln^\beta((j-\tau)v+1)} \geq \sum_{j=1}^{\infty} \frac{1}{j^{\alpha-1} \ln^\beta(j(v+1))} = \mathcal{L}(\alpha-1,\beta,v+1),$$

$$\sum_{j=1}^{\infty} \frac{j}{(j-\tau)^\alpha \ln^\beta((j-\tau)v+1)} \leq \frac{1}{(1-\tau)^\alpha \ln^\beta((1-\tau)v+1)} + \sum_{\substack{j=2\\ (j-1)v\neq 1}}^{\infty} \frac{j-1}{(j-1)^\alpha \ln^\beta((j-1)v)} + \sum_{\substack{j=2\\ (j-1)v\neq 1}}^{\infty} \frac{1}{(j-1)^\alpha \ln^\beta((j-1)v)}$$
$$= \frac{1}{(1-\tau)^\alpha \ln^\beta((1-\tau)v+1)} + \mathcal{L}(\alpha-1,\beta,v) + \mathcal{L}(\alpha,\beta,v),$$

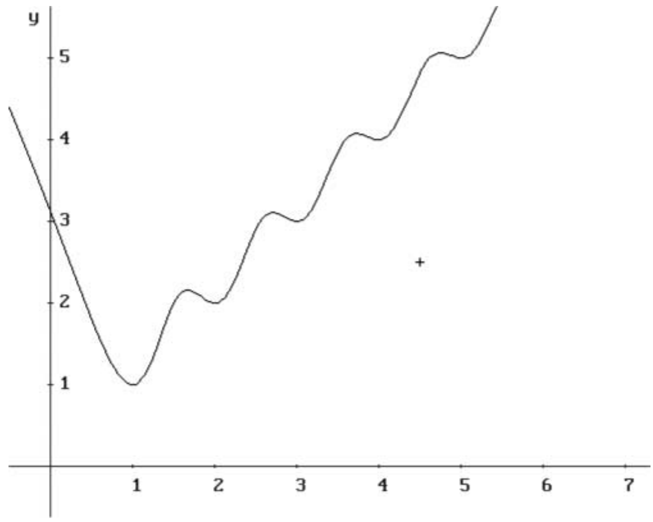


Figure 5. The graph of $S_\sigma$ for $\sigma(d) = 1/d^{2.01}$.



which implies

THEOREM 5 *For the weight function* $\sigma(d) = 1/d^\alpha 1/\ln^\beta(d+1)$ *the limit* $\lim_{M\to\infty} \Delta(x_\tau)/u$ *is convergent if and only if either* $\alpha = 2$ *and* $\beta > 1$ *or* $\alpha > 2$ *and* $\beta > 0$.

*In the above cases we have*

$$\frac{\mathcal{L}(\alpha-1,\beta,v+1)}{1/\tau^\alpha \ln^\beta(\tau v+1) + 1/(1-\tau)^\alpha \ln^\beta((1-\tau)v+1) + \mathcal{L}(\alpha,\beta,v)} - \tau \le \lim_{M\to\infty} \frac{\Delta(x_\tau)}{u}$$

*and*

$$\lim_{M\to\infty} \frac{\Delta(x_\tau)}{u} \le \frac{1/(1-\tau)^\alpha \ln^\beta((1-\tau)v+1) + \mathcal{L}(\alpha-1,\beta,v) + \mathcal{L}(\alpha,\beta,v)}{1/\tau^\alpha \ln^\beta(\tau v+1) + \mathcal{L}(\alpha-1,\beta,v+1)} - \tau.$$

## 3. Conclusions

In the previous sections, we have seen that for most of the weight functions $\sigma$ the relative size $\Delta(x_\tau)/u$ of the bumps may be convergent or divergent depending on its parameters. In general, the quicker $\sigma(d)$ tends to 0 as $d \to \infty$, the smaller $\Delta(x_\tau)/u$. In other words we have that:

Among the investigated weight functions $\sigma_1$ through $\sigma_4$ we found

$$\sigma_2(d) := \frac{1}{d}\exp(-\lambda d)$$

to be 'smoothest', i.e. $\lim_{M\to\infty} \Delta(x_\tau)/u$ could be acceptably small for suitable $\lambda$.

For practical applications we recommend first to estimate the largest, or the most common values of $v$ (the distances of the measuring datapoints, see Equation (3)), then to choose $\lambda$ as

$$\exp(-\lambda v) < 0.6.$$

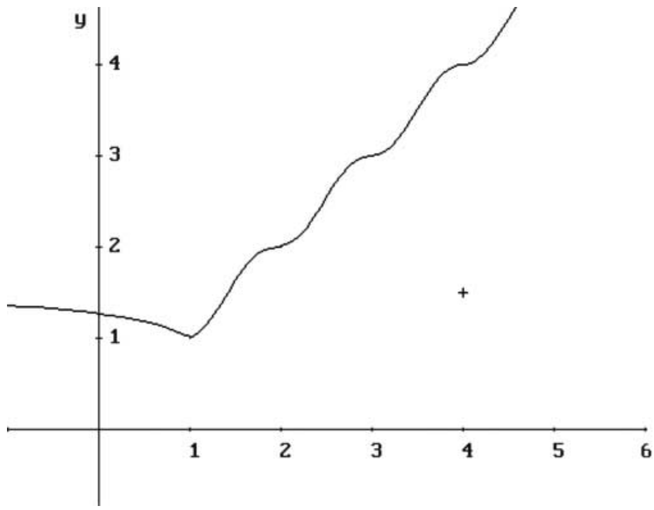


Figure 6. The graph of $S_\sigma$ for $\sigma(d) = \mathrm{e}^{-d}/d$.

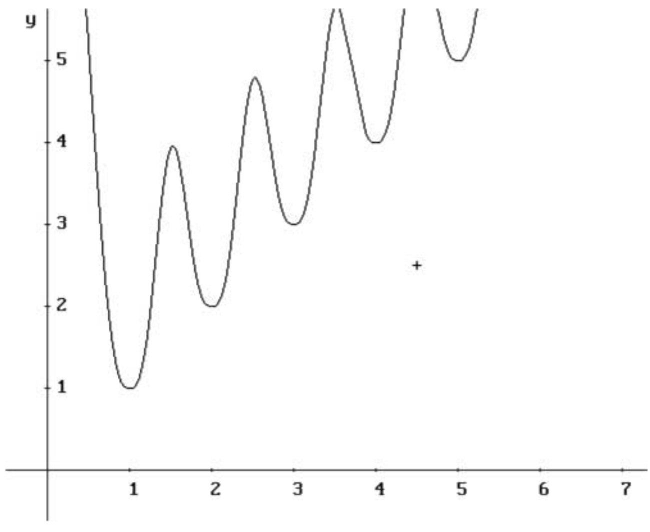


Figure 7. The graph of $S_\sigma$ for $\sigma(d) = 1/\ln^3(d+1)$.

(In the present paper, we could make detailed computations only in the case $\alpha = \beta = 1$ for the function $\sigma_2$.)

Though we used the data set (3) for our computations, we think that our conclusions above are valid also for any other data set, since the 'smoothness' of $S_\sigma$ depends on the rate of Equations (1) and (2) which is influenced by $\lambda$ and $\upsilon$ above rather than by the data set.

In conclusion, we present some graphs of $S_\sigma$ for some $\sigma$. In all examples in Figures 5–8, $M = 100$, $P_i = i$, $F_i = i$ $(1 \leq i \leq M)$.

Computational experiments were made by Derive 4.0 and Maple (Scientific Workplace 3.0).

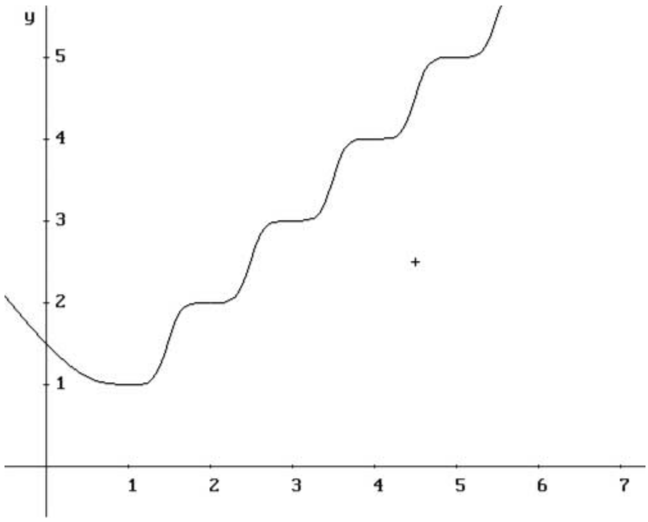


Figure 8. The graph of $S_\sigma$ for $\sigma(d) = 1/d^2 \ln^{1.5}(d+1)$.

## Acknowledgements

The authors thank *Balázs Szalkai* for a lot of computer help and the referees for their valuable remarks.

## Notes

1. In practice, these data are obtained by measuring and not by using a formula.
2. Which can be embedded in some higher dimensional space.
3. The assumption $0 < \tau < 1$ is not a restriction in fact, since the limit $\lim_{M\to\infty} \Delta(x_*)$ we are discussing in this paper is the same for any fixed point $x_* \in (P_0, \infty)$. This is why we may restrict ourselves to the interval $(P_0, P_1)$.

## References

[1] Allasia, G., *A class of interpolating positive linear operators: Theoretical and computational aspects*, in *Approximation Theory, Wavelets and Applications*, S.P. Singh, ed., Kluwer, Dordrecht, 1995, pp. 1–36.
[2] R. Bojanic, B. Della Vecchia, and G. Mastroianni, *On the approximation of bounded functions with discontinuities of the first kind by generalized shepard operators*, Acta Math. Hung. 85 (1999), pp. 29–57.
[3] B. Della Vecchia, G. Mastroianni, and J. Szabados, *Balázs–Shepard operators on infinite I. intervals*, Ann. Univ. Sci. Budapest, Sect. Comp. 16 (1996), pp. 93–102.
[4] B. Della Vecchia, G. Mastroianni, and J. Szabados: Weighted approximation of functions with endpoint or inner singularities by iterated exponential weights, *East J. Approx.* 9 (2003), 215–227.
[5] R. Farwig, *Rate of convergence of shepard's global interpolation formula*, Math. Comput. (1986), pp. 577–590.
[6] S.G. Gál and J. Szabados, *On the preservation of global smoothness by some interpolation operators*, Studia Sci.M. Hung. 35 (1999), pp. 397–414.
[7] W.J. Gordon and J.A. Wixom, *Shepard's method of 'Metric interpolation' to bivariate and multivariate interpolation*, Math. Comput. 32 (1978), pp. 253–264.
[8] J. Hoschek and D. Lasser, *Scattered data interpolation und approximation*, Computer Aided Geometric Design, Chapter 9, A.K.Peters Ltd, Wellesley, MA, 1993.
[9] J. Jin and S. Zhang, *Computation of special functions*, Wiley-Interscience, New York, 1996.
[10] E. Katona, *Constructing digital elevation models (in Hungarian)*, Polygon XI (2002), pp. 35–55.
[11] M. Láng-Lázi, J. Heszberger, M. Molnárné-Jobbágy, and G. Viczián, *Spline functions in chemistry: Approximation of surfaces over triangle domains*. Int. J. Comput. Math. 86(9) (2009), pp. 1634–1644.
[12] G. Mastroianni and J. Szabados, *Balázs–Shepard operators on infinite intervals II*, J. Approx. Theory 90 (1997), pp. 1–8.
[13] Á. Nagy, *A computer-help for demonstrating and investigating approximating methods (in Hungarian)*, Dissertation, Pannon University, 2010.

[14] J. Szabados, *Direct and converse approximation theorems for the shepard operator*, J. Approx. Theory Appl. 7 (1991), pp. 63–76.
[15] I. Szalkai, *SALT3DIM.exe – A Program for Handling 4 Component Mixtures*, University of Veszprém, Preprint, no. 047, 1996.
[16] I. Szalkai, *Handling multicomponent systems in $R^n$, I: Theoretical results*, J. Math. Chem. 25 (1999), pp. 31–46.
[17] I. Szalkai, *Scattered data interpolation via improved Shepard's method*, University of Veszprém, Preprint, no. 076, 1999.
[18] I. Szalkai, *Handling multicomponent systems in $R^n$ II: Computational results*, Submitted for publication, 2000.
[19] X.L. Zhou, *The saturation class of shepard operators*, Acta Math. Hung. 80 (1998), pp. 293–310.