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On the Periodicity of the Sequence $x_{n+1} = \max\left\{\frac{A_0}{x_n}, \frac{A_1}{x_{n-1}}, \dots, \frac{A_k}{x_{n-k}}\right\}$

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We give a complete characterization of the behaviour of the sequence cited in the title with negative coefficients, i.e. we settle *Conjecture 2.3.2 and Problem 2.3.1*, and give a partial answer for *Problem 2.3.2* of Ladas (*J. Difference Equ. and Appl.* **2** (1996) 339–341).

We also give a similar argument for an already known result for the case when all the coefficients have *the same* fixed positive value.

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1 THE PROBLEM

In [1] the following sequence was highlighted for detailed investigation:

For any fixed real numbers $A_0, A_1, \dots, A_k$ and $a_0, a_1, \dots, a_k$ (k is any fixed natural number) such that $A_k \neq 0$, define

$$x_{n+1} = \max\left\{\frac{A_0}{x_n}, \frac{A_1}{x_{n-1}}, \dots, \frac{A_k}{x_{n-k}}\right\} \quad (n \geq k) \tag{1}$$

and

$$x_0 = a_0, \quad x_1 = a_1, \quad \dots, \quad x_k = a_k. \tag{1a}$$

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It is useful to rewrite the recursion (1) as

$$x_{n+1} = \min \frac{1}{\{x_n/A_0, ((x_{n-1})/A_1), \ldots, ((x_{n-k})/A_k)\}} \quad (n \geq k). \qquad (2)$$

Some properties of this sequence are listed and conjectured in [1]. The case when all the coefficients A_i are negative is totally open. The main conjecture is the following. For *any* coefficients $A_i \in \mathbf{R}$ and for *any* initial values $a_i \in \mathbf{R}$ $(0 \leq i \leq k)$ and $k \in \mathbf{N}$ this sequence is eventually periodic if and only if it is bounded, and moreover it is always bounded for *positive numbers* $A_i \in \mathbf{R}$ and $a_i \in \mathbf{R}$ $(0 \leq i \leq k)$. The case $k = 1$ is handled in [2].

In the present note we give a *complete* characterization of the behaviour of the sequences satisfying (1) and (1a) in the case when all the coefficients A_i $(0 \leq i \leq k)$ are *negative* (see Theorem A). This settles Conjecture 2.3.2 and Problem 2.3.1 of [1], and gives a partial answer for Problem 2.3.2.

In Theorem A we also give a simple argument for an already stated result for the case when all the coefficients have the same fixed positive value.

2 THE NEGATIVE COEFFICIENTS CASE

In this section we *completely* describe the behaviour of the sequence when all the coefficients are negative: $A_i < 0$, $A_k \neq 0$, but $a_i \in \mathbf{R}$ are *arbitrary* real numbers for $i \leq k$.

THEOREM A *For any* $k \geq 0$ *and* $A_i < 0$, $A_k \neq 0$, $a_i \in \mathbf{R}$ $(i \leq k)$ *the following statements are equivalent*:

(i) *the sequence* (x_n) *is periodic;*
(ii) *the sequence* (x_n) *is periodic with period* $k+2$*;*
(iii) $A_i = A_{k-i}$ *for* $0 \leq i \leq k$*;*
(iv) *the sequence* (x_n) *is bounded.*

Proof Observe first that A_i/x_{n-i} and hence x_{n+1} are positive if and only if x_{n-i} is negative for some $i < k$. This implies that enlarging n step by step we leave behind all the negative elements of the sequence. That is, we reach to an n_0 such that

$$x_{n_0-i} > 0 \quad \text{for } 0 \le i \le k. \tag{$*1$}$$

Let

$$z_0 = x_{n_0+1}.$$

Clearly z_0 is negative, and by ($*1$), the previous $k+1$ elements of the sequence are positive. This by ($*1$) implies that the *next* $k+1$ elements of the sequence are

$$x_{n_0+2+i} = \frac{A_i}{z_0} \quad \text{for } 0 \le i \le k \tag{$*2$}$$

and all of them are positive.

Now one can easily see that

$$z_1 = x_{n_0+k+3} = z_0 \cdot \max\left\{\frac{A_i}{A_{k-i}} : 0 \le i \le k\right\}.$$

A repeated argument shows that for every natural number $t \in N$

$$z_t = x_{n_0+1+t\cdot(k+2)} = z_0 \cdot K^t,$$

where

$$K = \max\left\{\frac{A_i}{A_{k-i}} : i \le k\right\} \quad .$$

This clearly shows (iii) $\Leftrightarrow$ (iv).

Checking now the terms between z_t and z_{t+1} we get (iii) $\Rightarrow$ (i) + (ii).

Since (i) $\Rightarrow$ (iv) is obvious, Theorem A is proved.

Observe also that $A_i \neq 0$ must hold for $i \le k$ if there is no positive term among these coefficients.

The argument given above shows that for $n \ge k+2$ the solution consists of positive semicycles of length $k+1$, followed by negative semicycles of length 1, etc., or the other way around (i.e. replace positive by negative), hence there exists an $N \in \{1, 2, \ldots, k+2\}$ such that $x_N < 0$.

The above result confirms Conjecture 2.3.2 and answers Problem 2.3.1 of [1], and moreover gives a partial answer for Problem 2.3.2 of [1].

3 THE SAME POSITIVE COEFFICIENTS CASE

Suppose now that all the coefficients $A_i \in \mathbf{R}$ have the same fixed positive value $A_i = A$. We now prove that the sequence is periodic also in this case, using an argument similar to the previous proof.

THEOREM B *In the case $A_i = A > 0$ $(i \leq k)$ where A is any fixed real number, the sequence (x_n) is periodic with period $k+2$.*

Proof Let $\alpha = \sqrt{A}$. Observe first that $A_i/x_{n-i} > \alpha$ and $x_{n+1} > \alpha$ hold exactly in the case if $x_{n-i} < \alpha$ for some $i < k$. This implies that step by step enlarging n we reach an n_0 such that

$$x_{n_0-i} > \alpha \quad \text{for } i \leq k. \tag{$*3$}$$

The above inequality clearly implies

$$x_{n_0+1} < \alpha$$

and so the next $k+1$ elements of the sequence are

$$x_{n_0+2+i} = \frac{1}{x_{n_0}+1} \quad \text{for } i \leq k. \tag{$*4$}$$

In other words, all they have the same value which is greater than α. Then one can see that

$$x_{n_0+k+3} = \frac{1}{x_{n_0}+2} = x_{n_0+1}$$

and also that the sequence is periodic with period $k+2$.

So Theorem B is proved.

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References

[1] Ladas, G.: On the recursive sequence $x_{n+1} = \max\{A_0/x_n, (A_1/(x_{n-1})), \ldots, (A_k/(x_{n-k}))\}$ *J. Difference Equ. and Appl.* **2** (1996) 339–341.

[2] Amleh, A.M., Hoag, J. and Ladas, G.: A difference equation with eventually periodic solutions, *Computers and Math. with Appl.*, to appear.