DEPARTMENT OF MATHEMATICS
UNIVERSITY OF VESZPRÉM

PREPRINT

István Szalkai:

AN OPEN PROBLEM CONCERNING SPANNED SUBGRAPHS OF INFINITE GRAPHS

Volume 2/5 (1991)

VESZPRÉM
EGYETEM U. 10.
P.O.B. 158.
H-8201

AN OPEN PROBLEM CONCERNING SPANNED SUBGRAPHS OF INFINITE GRAPHS

István Szalkai

Let the graph $K = (V,E)$ be defined as $V:=\mathbb{N}$ (the set of natural numbers) and let $(m,n)\in E$ iff m and n are relative prime. Then we easily have:

STATEMENT 0 *A graph G is a spanned (=induced) subgraph of K iff*

$$(*) \qquad \mathrm{chr}(G\setminus\Gamma(P)) < \omega_o \qquad \text{for every } P\in V(G)$$

where

$$\Gamma(P) := \{\, Q\in V : (Q,P)\in E(G) \,\}$$

is the set of neighbours of P. □

This statement is an easy exercise, but is the starting point of the following general problem:

DEFINITION *For (finite or infinite) cardinals $\kappa\geq\lambda$ denote, by $K_{\kappa,\lambda}=(V_{\kappa,\lambda},E_{\kappa,\lambda})$ the following graph defined as $V_{\kappa,\lambda}:=[\kappa]^{\lambda}=\{X\subseteq\kappa : |X|=\lambda\}$ and $(X,Y)\in E_{\kappa,\lambda}$ iff $X\cap Y=\emptyset$. The graphs $K_{\kappa,<\lambda}=(V_{\kappa,<\lambda},E_{\kappa,<\lambda})$ and $K_{\kappa,\leq\lambda}=(V_{\kappa,\leq\lambda},E_{\kappa,\leq\lambda})$ are defined analogously: $V_{\kappa,<\lambda}:=[\kappa]^{<\lambda}=\{X\subseteq\kappa : |X|<\lambda\}$ and $V_{\kappa,\leq\lambda}:=[\kappa]^{\leq\lambda}=\{X\subseteq\kappa : |X|\leq\lambda\}$, resp.* □

The problem now is: to characterize all the spanned subgraphs of $K_{\kappa,\lambda}$, $K_{\kappa,<\lambda}$ or $K_{\kappa,\leq\lambda}$. This problem was raised by the author in

1987 ([C,p.585]) . The following results are derived easily:

STATEMENT 1 *A countable graph G is a spanned subgraph of* $K_{\kappa,<\omega_0}$ *iff* (*) *holds.* □

STATEMENT 2 *A graph G of size at most λ vertices is a spanned subgraph of* $K_{\kappa,<\lambda}$ *(κ is arbitrary) iff*

(*λ) $\mathrm{chr}(G\setminus\Gamma(P)) < \lambda$ for every $P\in V(G)$ □

However, for graphs of size larger than λ (*λ) is not sufficient as Komjáth Péter pointed out:

STATEMENT 3 *The bipartite graph on* λ^+ *(ie.* λ^+ *many disjoint edges) is* not *a spanned subgraph of* $K_{\kappa,<\lambda}$ □

Supposing $\Gamma(P_1)\neq\Gamma(P_2)$ for $P_1,P_2\in V(G)$, and $\lambda^{<\lambda}:=\sum_{\mu<\lambda}\lambda^{\mu}=\lambda$ the following conditions are neccessary for any graph G of size larger than λ^+ to be a spanned subgraph of $K_{\kappa,<\lambda}$:

(⁑) *There is no* $[\lambda:\lambda^+]$ *complete bipartite graph in* $\bar{G}$ *(= the complement of G)* □

Supposing even *chr(G)=2* , that is $V(G)=A\cup B$, $A\cap B=\emptyset$ we have:

(⁝) *There is an almost disjoint partition* $\{X_i\in[A]^{\lambda^+}: i\in I\}$ *of* A *(that is* $|X_i\cap X_j|<\lambda$ *for* $i\neq j\in I$*) such that*
$(\forall y\in B)\ (\exists\vartheta_y<\lambda)\ (\forall i\in I)$ either $\Gamma(y)\cap X_i=\emptyset$
or $|X_i\setminus\Gamma(y)|\le\vartheta_y$ □

Unfortunately enough I do not have any sufficient condition for graphs G of size larger than λ to be a spanned subgraph of $K_{\kappa,<\lambda}$ or $K_{\kappa,\lambda}$ or $K_{\kappa,\le\lambda}$ even in case $|G|=\lambda^+$ and $\mathrm{chr}(G)=2$.

David Wagner (Canada) [W] has general results of this problem for countable hypergraphs.

Wilson Castrellon (Colombia) posed the following related problem: Define $K_{\mathbb{R}}$ as $V(K_{\mathbb{R}})=\mathbb{R}$ (ℝ is the set of real numbers) and $(r,s)\in E(K_{\mathbb{R}})$ iff r/s is irrational. Then characterize all the spanned subgraphs of $K_{\mathbb{R}}$. Only the following partial result is known:

STATEMENT 4 *A countable (or finite) graph is a spanned subgraph of* $K_{\mathbb{R}}$ *iff*

(+) $(p,q)\notin E(G)$ ⇒ either $(p,r)\notin E(G)$ or $(q,r)\in E(G)$
for $p,q,r\in V(G)$ □

REFERENCES

[C] Szalkai, I.: *Problem* , in: Colloquia Math.Soc.J. Bolyai, vol 52 , Conference on Combinatorics , Eger, Hungary , 1987

[W] Wagner, D.: *Representing hypergraphs by Coprimality* , handwritten, 1985

University of Veszprém
Department of Mathematics and
Computer Science
H-8201 Veszprém, P.O.Box 158.
H u n g a r y