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On Differentiability of Solutions with respect to Parameters in Neutral Differential Equations with State-Dependent Delays*

Ferenc Hartung
Department of Mathematics
University of Pannonia
8201 Veszprém, P.O. Box 158, Hungary
email: hartung.ferenc@uni-pannon.hu

Abstract

In this paper we consider a class of nonlinear neutral differential equations with state-dependent delays in both the neutral and the retarded terms. We study well-posedness and continuous dependence issues and differentiability of the parameter map with respect to the initial function and other possibly infinite dimensional parameters in a pointwise sense and also in the C-norm.

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1 Introduction

In this paper we consider state-dependent neutral functional differential equations (SD-NFDEs) of the form

$$\frac{d}{dt}\Big(x(t) - g(t, x_t, x(t - \rho(t, x_t, \chi)), \lambda)\Big) = f\Big(t, x_t, x(t - \tau(t, x_t, \xi)), \theta\Big) \qquad t \in [0, T], \tag{1.1}$$

with initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0]. \tag{1.2}$$

Here $\theta \in \Theta$, $\xi \in \Xi$, $\lambda \in \Lambda$ and $\chi \in X$ represent parameters in the functions f, τ, g and ρ, where Θ, Ξ, Λ and X are normed linear spaces with norms $|\cdot|_\Theta$, $|\cdot|_\Xi$, $|\cdot|_\Lambda$ and

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$|\cdot|_X$, respectively. The segment function x_t is defined by $x_t(s) = x(t+s)$, $s \in [-r, 0]$. See Section 3 below for the detailed assumptions on the initial value problem (IVP) (1.1)-(1.2). By a solution of the IVP (1.1)-(1.2) we mean a continuous function defined on an interval $[-r, \alpha]$, such that (i) $t \mapsto x(t) - g(t, x_t, x(t - \rho(t, x_t, \chi)), \lambda)$ is differentiable for $t \in [0, \alpha]$, (at the ends of the interval one sided derivatives exist); (ii) x satisfies (1.1) for $t \in [0, \alpha]$, and (iii) x satisfies the initial condition (1.2).

The study of state-dependent delay differential equations (SD-DDEs), i.e., the case when $g \equiv 0$ in (1.1) is an active research area (see [22] and its references). Much less work is devoted to SD-NFDEs, see, e.g., [1]–[6], [9], [10], [12], [18], [21], [24], [26], [36]–[38] and their references. Most of the above papers deal with SD-NFDEs of the form

$$x'(t) = h\Big(t, x(t), x(t - \tau(t, x(t))), x'(t - \eta(t, x(t)))\Big). \tag{1.3}$$

This equation is called in [29], [36], [37] as "explicit" SD-NFDE contrary to the "implicit" SD-NFDE (1.1). Well-posedness of such "explicit" SD-NFDEs was investigated in [11], [25].

Equation (1.1) can be considered as a natural "generalization" of NFDEs of the form

$$\frac{d}{dt}G(t, x_t) = f(t, x_t), \tag{1.4}$$

but (1.4) may also contain (1.1) depending on appropriate conditions on G and f, see assumptions on f in [22] for SD-DDEs, and [36] and [37] for similar conditions on "explicit" SD-NFDEs. Existence, uniqueness, stability and numerical approximation of special classes of (1.1) was studied in [3], [18], [20], [29]. Similar classes of abstract implicit SD-NFDEs were investigated in [5], [7], [30], [33].

Differentiability of solutions with respect to (wrt) parameters is an important qualitative question, but it also has natural application in the problem of identification of parameters [17]. But even for simple constant delay equations this problem leads to technical difficulties if the parameter is the delay [14], [28]. A similar difficulty arises in SD-DDEs. In the case when the initial function φ is continuously differentiable and satisfies the compatibility condition $\dot\varphi(0-) = f(0, \varphi, \varphi(-\tau(0, \varphi, \xi)), \theta)$, the corresponding solution $x(t, \varphi, \xi, \theta)$ of the IVP (1.1)-(1.2) with $g \equiv 0$ is differentiable wrt φ, ξ, θ for a fixed t [16]. Related is the work of Walther [34], [35], where the well-posedness of autonomous SD-DDEs is obtained using the space of continuously differentiable functions and restricting the parameters to those which generate continuously differentiable solutions. Walther also obtained differentiability of the solution with respect to the initial function in this space. Differentiability of solutions of SD-DDEs wrt parameters assuming the monotonicity of the time lag function along the solution instead of the above compatibility condition was investigated in [23], where the differentiability wrt the parameters was obtained in the $W^{1,p}$-norm. Recently, this result was improved in [19], where differentiability wrt parameters was proved for SD-DDEs for fixed t, and also using the C-norm.

In a recent paper [37] Walter studied continuous semiflows generated by "explicit" SD-NFDEs in the space of continuously differentiable functions, and differentiability and continuity of derivatives with respect to initial data. Differentiability wrt parameters of "implicit"

SD-NFDEs was proved in [18] for the case when the delay ρ in (1.1) is only time-dependent, and there are no parameters in the neutral term. The proof was based on the assumption that the parameters satisfy a compatibility condition similarly to the SD-DDE case above [16], [34], [35]. In this paper we extend this result for (1.1), where state-dependent delay and parameters are included in the neutral term, as well. In Theorem 3.2 below we discuss the well-posedness of the IVP (1.1)-(1.2), and in Theorem 4.4 and Corollary 4.5 below we show the differentiability of solutions of the IVP (1.1)-(1.2) wrt the parameters $(\varphi, \xi, \theta, \lambda, \chi)$ in a pointwise sense and also using the C-norm.

The organization of the paper is the following. In Section 2 we introduce some notations, and formulate some basic results will be used in the rest of the paper. In Section 3 we list our assumptions, and discuss well-posedness of the IVP (1.1)-(1.2), and then in Section 4, using and improving the method of [18], we study differentiability of solutions wrt parameters.

Note that for simplicity we present our results for the single state-dependent delay case, but all our results can be easily extended to the case when both g and f contain multiple state-dependent delays.

2 Notations and preliminaries

Throughout this paper a fixed norm on $\mathbb{R}^n$ and the corresponding matrix norm on $\mathbb{R}^{n\times n}$ are both denoted by $|\cdot|$. In a normed linear space $(X, |\cdot|_X)$ the open ball around a point x_0 with radius R is denoted by $\mathcal{B}_X(x_0; R)$, i.e., $\mathcal{B}_X(x_0; R) := \{x \in X \,:\, |x - x_0|_X < R\}$, and the corresponding closed ball by $\overline{\mathcal{B}}_X(x_0; R)$.

The space of continuous functions from $[-r, 0]$ to $\mathbb{R}^n$ is denoted by C, where the norm is the usual supremum norm $|\psi|_C = \max\{|\psi(\zeta)| : \zeta \in [-r, 0]\}$. The L^∞-norm of an essentially bounded Lebesgue measurable function $\psi\colon [-r, 0] \to \mathbb{R}^n$ is defined by $|\psi|_{L^\infty} := \operatorname{ess\,sup}\{|\psi(\zeta)| \,:\, \zeta \in [-r, 0]\}$. The space of absolutely continuous functions from $[-r, 0]$ to $\mathbb{R}^n$ with essentially bounded derivatives is denoted by $W^{1,\infty}$. The corresponding norm on $W^{1,\infty}$ is $|\psi|_{W^{1,\infty}} := \max\{|\psi|_C, |\dot\psi|_{L^\infty}\}$. We note that $\psi \in W^{1,\infty}$, if and only if ψ is Lipschitz continuous. The space of bounded linear operators between normed linear spaces X and Y is denoted by $\mathcal{L}(X, Y)$, and the norm on it is $|\cdot|_{\mathcal{L}(X,Y)}$.

The derivative of a single variable function $v(t)$ wrt t is denoted by $\dot v$. Note that all derivatives we use in this paper are Fréchet derivatives. Suppose the function $F(x_1, \ldots, x_m)$ takes values in $\mathbb{R}^n$. The partial derivatives of F wrt its first, second, etc. arguments are denoted by D_1F, D_2F, etc. In the case when the argument x_1 of F is real, then we simply write $D_1F(x_1, \ldots, x_m)$ instead of the more precise notation $D_1F(x_1, \ldots, x_m)1$, i.e., here D_1F denotes the vector in $\mathbb{R}^n$ instead of the linear operator $\mathcal{L}(\mathbb{R}, \mathbb{R}^n)$. In the case when, let say, $x_2 \in \mathbb{R}^n$, then we identify the linear operator $D_2F(x_1, \ldots, x_m) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ by an $n \times n$ matrix.

The next lemma formalizes a method used frequently in functional inequalities (see, e.g., in [13]) and which will be used in the sequel, as well.

Lemma 2.1 ([18]) *Suppose* $h:\ [0,\alpha]\times[0,\infty)^3\to[0,\infty)$ *is monotone increasing in all variables, i.e., if* $0\le t_i\le s_i$ *for* $i=1,2,3,4$, *then* $h(t_1,t_2,t_3,t_4)\le h(s_1,s_2,s_3,s_4)$; $\eta:[0,\alpha]\to[0,r]$ *is such that* $a\le\eta(t)$ *for* $t\in[0,\alpha]$ *for some* $a>0$; $u\colon[-r,\alpha]\to[0,\infty)$ *is such that*

$$u(t)\le h(t,u(t),u(t-\eta(t)),|u_t|_C),\qquad t\in[0,\alpha],$$

and

$$|u_0|_C\le h(0,u(0),u(-\eta(0)),|u_0|_C).$$

Then

$$v(t)\le h(t,v(t),v(t-a),v(t)),\qquad t\in[0,\alpha],$$

where $v(t):=\sup\{u(s)\colon s\in[-r,t]\}$.

Finally, we recall the following two results which will be used later.

Lemma 2.2 ([13]) *Let* $a>0$, $b\ge 0$, $r_1>0$, $r_2\ge 0$, $r=\max\{r_1,r_2\}$, *and* $v:[0,\alpha]\to[0,\infty)$ *be continuous and non-decreasing. Let* $u\colon[-r,\alpha]\to[0,\infty)$ *be continuous and satisfy the inequality*

$$u(t)\le v(t)+bu(t-r_1)+a\int_0^t u(s-r_2)\,ds,\qquad t\in[0,\alpha].$$

Then $u(t)\le d(t)e^{ct}$ *for* $t\in[0,\alpha]$, *where* c *is the unique positive solution of* $cbe^{-cr_1}+ae^{-cr_2}=c$, *and*

$$d(t):=\max\left\{\frac{v(t)}{1-be^{-cr_1}},\ \max_{-r\le s\le 0}e^{-cs}u(s)\right\},\qquad t\in[0,\alpha].$$

Lemma 2.3 *Suppose* $\psi\in W^{1,\infty}$. *Then*

$$|\psi(b)-\psi(a)|\le|\dot\psi|_{L^\infty}|b-a|$$

for every $[a,b]\subset[-r,0]$.

3 Well-posedness and continuous dependence on parameters

Consider the SD-NFDE

$$\frac{d}{dt}\Big(x(t)-g(t,x_t,x(t-\rho(t,x_t,\chi)),\lambda)\Big)=f\Big(t,x_t,x(t-\tau(t,x_t,\xi)),\theta\Big)\qquad t\in[0,T],\quad(3.1)$$

and the initial condition

$$x(t)=\varphi(t),\qquad t\in[-r,0].\qquad(3.2)$$

Next we list our assumptions on the SD-NFDE (3.1) we will use throughout this paper. Let Θ, Ξ, Λ and X be normed linear spaces with norms $|\cdot|_\Theta$, $|\cdot|_\Xi$, $|\cdot|_\Lambda$ and $|\cdot|_X$, respectively, and let $\Omega_1 \subset C$, $\Omega_2 \subset \mathbb{R}^n$, $\Omega_3 \subset \Theta$, $\Omega_4 \subset \Xi$, $\Omega_5 \subset \mathbb{R}^n$, $\Omega_6 \subset \Lambda$ and $\Omega_7 \subset X$ be open subsets of the respective spaces. Let $0 < r_0 < r$ be fixed constants, and $T > 0$ be finite or $T = \infty$, in which case $[0, T]$ denotes the interval $[0, \infty)$. We assume:

(A1) (i) $f\colon \mathbb{R} \times C \times \mathbb{R}^n \times \Theta \supset [0, T] \times \Omega_1 \times \Omega_2 \times \Omega_3 \to \mathbb{R}^n$ is continuous;

(ii) $f(t, \psi, u, \theta)$ is locally Lipschitz continuous in ψ, u and θ in the following sense: for every finite $\alpha \in (0, T]$, for every closed subset $M_1 \subset \Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, compact subset $M_2 \subset \Omega_2$ of $\mathbb{R}^n$, and closed and bounded subset $M_3 \subset \Omega_3$ of Θ there exists a constant $L_1 = L_1(\alpha, M_1, M_2, M_3)$ such that

$$|f(t, \psi, u, \theta) - f(t, \bar\psi, \bar u, \bar\theta)| \le L_1\Big(|\psi - \bar\psi|_C + |u - \bar u| + |\theta - \bar\theta|_\Theta\Big),$$

for $t \in [0, \alpha]$, $\psi, \bar\psi \in M_1$, $u, \bar u \in M_2$ and $\theta, \bar\theta \in M_3$;

(iii) f is differentiable wrt its second, third and fourth variables, and the functions

$$\mathbb{R} \times C \times \mathbb{R}^n \times \Theta \supset [0, T] \times \Omega_1 \times \Omega_2 \times \Omega_3 \to \mathcal{L}(C, \mathbb{R}^n),\ (t, \psi, u, \theta) \mapsto D_2 f(t, \psi, u, \theta),$$
$$\mathbb{R} \times C \times \mathbb{R}^n \times \Theta \supset [0, T] \times \Omega_1 \times \Omega_2 \times \Omega_3 \to \mathbb{R}^{n\times n},\ (t, \psi, u, \theta) \mapsto D_3 f(t, \psi, u, \theta)$$

and

$$\mathbb{R} \times C \times \mathbb{R}^n \times \Theta \supset [0, T] \times \Omega_1 \times \Omega_2 \times \Omega_3 \to \mathcal{L}(\Theta, \mathbb{R}^n),\ (t, \psi, u, \theta) \mapsto D_4 f(t, \psi, u, \theta),$$

are continuous;

(A2) (i) $\tau\colon \mathbb{R} \times C \times \Xi \supset [0, T] \times \Omega_1 \times \Omega_4 \to \mathbb{R}$ is continuous, and

$$0 \le \tau(t, \psi, \xi) \le r, \qquad \text{for } t \in [0, T],\ \psi \in \Omega_1 \text{ and } \xi \in \Omega_4;$$

(ii) $\tau(t, \psi, \xi)$ is locally Lipschitz continuous in ψ and ξ in the following sense: for every finite $\alpha \in (0, T]$, closed subset $M_1 \subset \Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, and closed and bounded subset $M_4 \subset \Omega_4$ of Ξ there exists a constant $L_2 = L_2(\alpha, M_1, M_4)$ such that

$$|\tau(t, \psi, \xi) - \tau(t, \bar\psi, \bar\xi)| \le L_2\Big(|\psi - \bar\psi|_C + |\xi - \bar\xi|_\Xi\Big)$$

for $t \in [0, \alpha]$, $\psi, \bar\psi \in M_1$ and $\xi, \bar\xi \in M_4$;

(iii) τ is differentiable wrt its second and third variables, and the maps

$$\mathbb{R} \times C \times \Xi \supset [0, T] \times \Omega_1 \times \Omega_4 \to \mathcal{L}(C, \mathbb{R}), \quad (t, \psi, \xi) \mapsto D_2\tau(t, \psi, \xi)$$

and

$$\mathbb{R} \times C \times \Xi \supset [0, T] \times \Omega_1 \times \Omega_4 \to \mathcal{L}(\Xi, \mathbb{R}), \quad (t, \psi, \xi) \mapsto D_3\tau(t, \psi, \xi)$$

are continuous;

(A3) (i) $g\colon \mathbb{R}\times C\times\mathbb{R}^n\times\Lambda\supset[0,T]\times\Omega_1\times\Omega_5\times\Omega_6\to\mathbb{R}^n$ is continuous;

(ii) g is locally Lipschitz continuous in the following sense: for every $\alpha\in(0,T]$, closed subset $M_1\subset\Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, compact subset $M_5\subset\Omega_5$ of $\mathbb{R}^n$ and closed and bounded subset $M_6\subset\Omega_6$ of Λ there exists $L_3=L_3(\alpha,M_1,M_5,M_6)$ such that

$$\begin{aligned}&|g(t,\psi,u,\lambda)-g(\bar t,\bar\psi,\bar u,\bar\lambda)|\\&\qquad\le\ L_3\Big(|t-\bar t|+\max_{\zeta\in[-r,-r_0]}|\psi(\zeta)-\bar\psi(\zeta)|+|u-\bar u|+|\lambda-\bar\lambda|_\Lambda\Big),\end{aligned}$$

for $t,\bar t\in[0,\alpha],\ \psi,\bar\psi\in M_1,\ u,\bar u\in M_5,\ \lambda,\bar\lambda\in M_6$;

(iii) g is differentiable wrt its second, third and fourth arguments, and the maps

$$\mathbb{R}\times C\times\mathbb{R}^n\times\Lambda\supset[0,T]\times\Omega_1\times\Omega_5\times\Omega_6\to\mathcal{L}(C,\mathbb{R}^n),\ (t,\psi,u,\lambda)\mapsto D_2g(t,\psi,u,\lambda),$$
$$\mathbb{R}\times C\times\mathbb{R}^n\times\Lambda\supset[0,T]\times\Omega_1\times\Omega_5\times\Omega_6\to\mathbb{R}^{n\times n},\ (t,\psi,u,\lambda)\mapsto D_3g(t,\psi,u,\lambda)$$

and

$$\mathbb{R}\times C\times\mathbb{R}^n\times\Lambda\supset[0,T]\times\Omega_1\times\Omega_5\times\Omega_6\to\mathcal{L}(\Lambda,\mathbb{R}^n),\ (t,\psi,u,\lambda)\mapsto D_4g(t,\psi,u,\lambda)$$

are continuous;

(iv) D_2g, D_3g and D_4g are locally Lipschitz continuous wrt its first three variables in the following sense: for every $\alpha\in(0,T]$, closed subset $M_1\subset\Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, compact subset $M_5\subset\Omega_5$ of $\mathbb{R}^n$ and closed and bounded subset $M_6\subset\Omega_6$ of Λ there exist $L_4=L_4(\alpha,M_1,M_5,M_6)$ and $L_5=L_5(\alpha,M_1,M_5,M_6)$ such that

$$\begin{aligned}&|D_2g(t,\psi,u,\lambda)h-D_2g(\bar t,\bar\psi,\bar u,\lambda)h|\\&\qquad\le\ L_4\Big(|t-\bar t|+\max_{\zeta\in[-r,-r_0]}|\psi(\zeta)-\bar\psi(\zeta)|+|u-\bar u|\Big)\max_{\zeta\in[-r,-r_0]}|h(\zeta)|,\\&\qquad\quad+L_4\max\Big\{|h(\zeta)-h(\bar\zeta)|\colon\ \zeta,\bar\zeta\in[-r,-r_0],\ |\zeta-\bar\zeta|\le L_5|t-\bar t|\Big\},\\&|D_3g(t,\psi,u,\lambda)-D_3g(\bar t,\bar\psi,\bar u,\lambda)|\\&\qquad\le\ L_4\Big(|t-\bar t|+\max_{\zeta\in[-r,-r_0]}|\psi(\zeta)-\bar\psi(\zeta)|+|u-\bar u|\Big),\\&|D_4g(t,\psi,u,\lambda)-D_4g(\bar t,\bar\psi,\bar u,\lambda)|_{\mathcal{L}(\Lambda,\mathbb{R}^n)}\\&\qquad\le\ L_4\Big(|t-\bar t|+\max_{\zeta\in[-r,-r_0]}|\psi(\zeta)-\bar\psi(\zeta)|+|u-\bar u|\Big),\end{aligned}$$

for $t,\bar t\in[0,\alpha],\ \psi,\bar\psi\in M_1,\ u,\bar u\in M_5,\ \lambda\in M_6,\ h\in C$;

(A4) (i) $\rho\colon \mathbb{R}\times C\times X\supset[0,T]\times\Omega_1\times\Omega_7\to\mathbb{R}$ is continuous, and

$$0<r_0\le\rho(t,\psi,\chi)\le r,\qquad t\in[0,T],\quad\psi\in\Omega_1,\quad\chi\in\Omega_7;$$

(ii) ρ is locally Lipschitz continuous in the following sense: for every $\alpha \in (0,T]$, closed subset $M_1 \subset \Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$, and bounded and closed subset $M_7 \subset \Omega_7$ of X there exists $L_6 = L_6(\alpha, M_1, M_7)$ such that

$$|\rho(t,\psi,\chi) - \rho(\bar{t},\bar{\psi},\bar{\chi})| \leq L_6\Big(|t-\bar{t}| + \max_{\zeta\in[-r,-r_0]} |\psi(\zeta) - \bar{\psi}(\zeta)| + |\chi - \bar{\chi}|_X\Big)$$

for $t, \bar{t} \in [0,\alpha]$, $\psi, \bar{\psi} \in M_1$, and $\chi, \bar{\chi} \in M_7$;

(iii) ρ is differentiable wrt its second and third arguments, and the maps

$$\mathbb{R}\times C\times X \supset [0,T]\times\Omega_1\times\Omega_7 \to \mathcal{L}(C,\mathbb{R}), \quad (t,\psi,\chi) \mapsto D_2\rho(t,\psi,\chi)$$

and

$$\mathbb{R}\times C\times X \supset [0,T]\times\Omega_1\times\Omega_4 \to \mathcal{L}(X,\mathbb{R}), \quad (t,\psi,\chi) \mapsto D_3\rho(t,\psi,\chi)$$

are continuous;

(iv) $D_2\rho$ and $D_3\rho$ are locally Lipschitz continuous wrt its first and second variables in the following sense: for every $\alpha \in (0,T]$, closed subset $M_1 \subset \Omega_1$ of C which is also a bounded subset of $W^{1,\infty}$ and bounded and closed subset $M_7 \subset \Omega_7$ of X there exist $L_7 = L_7(\alpha, M_1, M_7)$ and $L_8 = L_8(\alpha, M_1, M_7)$ such that

$$\begin{aligned}
&|D_2\rho(t,\psi,\chi)h - D_2\rho(\bar{t},\bar{\psi},\chi)h| \\
&\quad\leq\; L_7\Big(|t-\bar{t}| + \max_{\zeta\in[-r,-r_0]} |\psi(\zeta)-\bar{\psi}(\zeta)|\Big) \max_{\zeta\in[-r,-r_0]} |h(\zeta)| \\
&\qquad + L_7 \max\{|h(\zeta)-h(\bar{\zeta})| \colon\ \zeta,\bar{\zeta} \in [-r,-r_0],\ |\zeta-\bar{\zeta}| \leq L_8|t-\bar{t}|\},
\end{aligned}$$

and

$$|D_3\rho(t,\psi,\chi) - D_3\rho(\bar{t},\bar{\psi},\chi)|_{\mathcal{L}(X,\mathbb{R})} \leq L_7\Big(|t-\bar{t}| + \max_{\zeta\in[-r,-r_0]} |\psi(\zeta)-\bar{\psi}(\zeta)|\Big)$$

for $t, \bar{t} \in [0,\alpha]$, $\psi, \bar{\psi} \in M_1$, $\chi \in M_7$, $h \in C$.

It is easy to see that (A3) (ii) and (A4) (ii) yield that $g(t,\psi,u,\lambda)$ and $\rho(t,\psi,\chi)$ depend only on the restriction of ψ to the interval $[-r,-r_0]$, since if $\psi(\zeta) = \bar{\psi}(\zeta)$ for $\zeta \in [-r,-r_0]$, then $g(t,\psi,u,\lambda) = g(t,\bar{\psi},u,\lambda)$ and $\rho(t,\psi,\chi) = \rho(t,\bar{\psi},\chi)$. It also follows from (A3) (ii), (iii) and (A4) (ii), (iii) that

$$|D_2 g(t,\psi,u,\lambda)h| \leq |D_2 g(t,\psi,u,\lambda)|_{\mathcal{L}(C,\mathbb{R}^n)} \max_{\zeta\in[-r,-r_0]} |h(\zeta)|$$

and

$$|D_2\rho(t,\psi,\chi)h| \leq |D_2\rho(t,\psi,\chi)|_{\mathcal{L}(C,\mathbb{R})} \max_{\zeta\in[-r,-r_0]} |h(\zeta)|$$

hold for $t \in [0,T]$, $\psi \in \Omega_1$, $u \in \Omega_5$, $\lambda \in \Omega_6$, $\chi \in \Omega_7$ and $h \in C$.

It follows from the assumptions on M_1 in (A1) (ii), (A2) (ii), (A3) (ii), (iv) and (A4) (ii), (iv) that it has no interior in C. Note that assumptions (A1) and (A2) are practically identical to those used in [23] for SD-DDEs, i.e., for the case when $g \equiv 0$. (See also [8] or [23] for well-posedness of SD-DDEs.) The key assumptions in this paper are that ρ is bounded below by $r_0 > 0$ (see (A4) (i)), and $g(t, \psi, u, \lambda)$ and $\rho(t, \psi, \chi)$ depend only on the restriction of ψ to the interval $[-r, -r_0]$. Similar assumption is used for SD-NFDEs in [18], see condition (g1) in [36], [37], and for PDEs with state-dependent delays in [32]. The particular form of the Lipschitz continuity assumed in (A3) (ii), (iv) and (A4) (ii), (iv) is motivated by the specific form (3.3) and (3.4) of the functions g and ρ, respectively (see Lemma 3.1 below). We comment that the Arzelà-Ascoli theorem yields that closed subsets of C which are bounded subsets of $W^{1,\infty}$ are compact in C.

Assumptions (A3) and (A4) are naturally satisfied, e.g., in the case when $\Lambda = X = W^{1,\infty}([0, T], \mathbb{R})$, and g and ρ have the form

$$g(t, \psi, u, \lambda) = \bar{g}\Big(t, \psi(-\eta^1(t)), \dots, \psi(-\eta^k(t)), \int_{-r}^{-r_0} A(t, \zeta)\psi(\zeta)\, d\zeta, u, \lambda(t)\Big) \tag{3.3}$$

and

$$\rho(t, \psi, \chi) = \bar{\rho}\Big(t, \psi(-\nu^1(t)), \dots, \psi(-\nu^\ell(t)), \int_{-r}^{-r_0} B(t, \zeta)\psi(\zeta)\, d\zeta, \chi(t)\Big), \tag{3.4}$$

where $t \in [0, T]$, $\psi \in C$, $u \in \mathbb{R}^n$, $\lambda \in \Lambda$, $\chi \in X$ and $0 < r_0 < r$. The next lemma shows that assumption (A4) is satisfied under natural assumptions on $\bar{\rho}$. Clearly, (A3) can be also satisfied under similar assumptions on $\bar{g}$.

Lemma 3.1 *Assume $X = W^{1,\infty}([0, T], \mathbb{R})$, and ρ has the form (3.4), where*

(i) $\bar{\rho}\colon [0, T] \times \mathbb{R}^{n\times(\ell+1)} \times \mathbb{R} \to \mathbb{R}$ *is continuous,* $\nu^1, \dots, \nu^\ell\colon [0, T] \to \mathbb{R}$ *are continuous,* $B\colon [0, T] \times [-r, -r_0] \to \mathbb{R}^{n\times n}$ *is continuous, and*

$$0 < r_0 \le \bar{\rho}(t, u_1, \dots, u_{\ell+1}, v) \le r, \qquad t \in [0, T], \quad u_1, \dots, u_{\ell+1} \in \mathbb{R}^n, \quad v \in \mathbb{R},$$

and

$$0 < r_0 \le \nu^i(t) \le r, \qquad i = 1, \dots, \ell, \quad t \in [0, T];$$

(ii) $\bar{\rho}$ *is twice continuously differentiable;*

(iii) $\nu^1, \dots, \nu^\ell\colon [0, T] \to \mathbb{R}$ *and* $B\colon [0, T] \times [-r, -r_0] \to \mathbb{R}^{n\times n}$ *are locally Lipschitz continuous wrt t, i.e., for every $\alpha \in (0, T]$ there exist $L_9 = L_9(\alpha)$ and $L_{10} = L_{10}(\alpha)$ such that*

$$|\nu^i(t) - \nu^i(\bar{t})| \le L_9|t - \bar{t}|, \qquad t, \bar{t} \in [0, \alpha], \quad i = 1, \dots, \ell,$$

and

$$|B(t, \zeta) - B(\bar{t}, \zeta)| \le L_{10}|t - \bar{t}|, \qquad t, \bar{t} \in [0, \alpha], \quad \zeta \in [-r, -r_0].$$

Then ρ satisfies assumptions (A4) (i)–(iv).

Moreover, if in addition $\bar\chi, \nu^1, \dots, \nu^\ell \in C^1([0,T],\mathbb{R})$ and B is continuously differentiable wrt its first argument, then $\rho(t,\psi,\bar\chi)$ is differentiable wrt t for $t\in[0,T]$ and $\psi\in C^1$, and the map $[0,T]\times C^1\to\mathbb{R}$, $(t,\psi)\mapsto D_1\rho(t,\psi,\bar\chi)$ is continuous.

Proof (A4) (i) is clearly satisfied under the assumptions of the lemma with $\Omega_1 = C$ and $\Omega_7 = X$. Suppose $\alpha\in(0,T]$, M_1 is a closed subset of C which is also a bounded subset of $W^{1,\infty}$, and $M_7\subset X$ is closed and bounded. Then there exists $R_1>0$ and $R_2>0$ such that $M_1\subset\overline{\mathcal{B}}_{W^{1,\infty}}(0;R_1)$ and $M_7\subset\overline{\mathcal{B}}_X(0;R_2)$. We have

$$\Big|\int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta\Big| \le b_{max}R_1 r, \qquad t\in[0,\alpha],\ \psi\in M_1,$$

where

$$b_{max} = b_{max}(\alpha) := \max\{|B(t,\zeta)|\colon\ t\in[0,\alpha],\ \zeta\in[-r,-r_0]\}. \tag{3.5}$$

Let

$$\begin{aligned} L_{11} \ :=\ & \max_{i=1,\dots,\ell+3}\max\Big\{|D_i\bar\rho(t,u_1,\dots,u_{\ell+1},v)|\colon\ t\in[0,\alpha],\ u_1,\dots,u_\ell\in\overline{\mathcal{B}}_{\mathbb{R}^n}(0;R_1),\\ & \qquad u_{\ell+1}\in\overline{\mathcal{B}}_{\mathbb{R}^n}(0;b_{max}R_1r),\ v\in\overline{\mathcal{B}}_{\mathbb{R}}(0;R_2)\Big\}. \end{aligned}$$

Then Lemma 2.3 yields for $t\in[0,\alpha]$, $\psi,\bar\psi\in M_1$, and $\chi,\bar\chi\in M_7$

$$\begin{aligned} &|\rho(t,\psi,\chi)-\rho(t,\bar\psi,\bar\chi)|\\ &\quad= \Big|\bar\rho\Big(t,\psi(-\nu^1(t)),\dots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0}B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\\ &\qquad -\bar\rho\Big(t,\bar\psi(-\nu^1(t)),\dots,\bar\psi(-\nu^\ell(t)),\int_{-r}^{-r_0}B(t,\zeta)\bar\psi(\zeta)\,d\zeta,\bar\chi(t)\Big)\Big|\\ &\quad\le L_{11}\Big(\sum_{i=1}^{\ell}|\psi(-\nu^i(t))-\bar\psi(-\nu^i(t))| + \int_{-r}^{-r_0}|B(t,\zeta)||\psi(\zeta)-\bar\psi(\zeta)|\,d\zeta + |\chi(t)-\bar\chi(t)|\Big)\\ &\quad\le L_{11}(\ell+rb_{max})\Big(\max_{\zeta\in[-r,-r_0]}|\psi(\zeta)-\bar\psi(\zeta)| + |\chi-\bar\chi|_X\Big). \end{aligned}$$

To show the Lipschitz continuity of ρ wrt t consider for $t,\bar t\in[0,\alpha]$, $\psi\in M_1$, $\chi\in M_7$

$$\begin{aligned} &|\rho(t,\psi,\chi)-\rho(\bar t,\psi,\chi)|\\ &\quad\le \Big|\bar\rho\Big(t,\psi(-\nu^1(t)),\dots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0}B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\\ &\qquad -\bar\rho\Big(\bar t,\psi(-\nu^1(\bar t)),\dots,\psi(-\nu^\ell(\bar t)),\int_{-r}^{-r_0}B(\bar t,\zeta)\psi(\zeta)\,d\zeta,\chi(\bar t)\Big)\Big| \end{aligned}$$

$$\leq L_{11}\Big(|t-\bar{t}|+\sum_{i=1}^{\ell}|\psi(-\nu^i(t))-\psi(-\nu^i(\bar{t}))|+\int_{-r}^{-r_0}|B(t,\zeta)-B(\bar{t},\zeta)||\psi(\zeta)|\,d\zeta$$
$$+|\chi(t)-\chi(\bar{t})|\Big)$$
$$\leq L_{11}\Big(|t-\bar{t}|+\sum_{i=1}^{\ell}|\dot{\psi}|_{L^\infty}|\nu^i(t)-\nu^i(\bar{t})|+L_{10}r|\psi|_C|t-\bar{t}|+\operatorname*{ess\,sup}_{s\in[0,\alpha]}|\dot{\chi}(s)||t-\bar{t}|\Big).$$

Therefore (A4) (ii) holds with $L_6:=\max\{L_{11}(\ell+rb_{max}),L_{11}(1+\ell R_1L_9+L_{10}rR_1+R_2)\}$.

The differentiability of $\bar{\rho}$ yields for $t\in[0,T]$, $\psi\in C$, $\chi\in X$, $h\in C$ and $\eta\in X$

$$D_2\rho(t,\psi,\chi)h$$
$$=\sum_{i=1}^{\ell}D_{i+1}\bar{\rho}\Big(t,\psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0}B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)h(-\nu^i(t))$$
$$+D_{\ell+2}\bar{\rho}\Big(t,\psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0}B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\int_{-r}^{-r_0}B(t,\zeta)h(\zeta)\,d\zeta$$

and

$$D_3\rho(t,\psi,\chi)\eta=D_{\ell+3}\bar{\rho}\Big(t,\psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0}B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\eta(t),$$

and clearly, $D_2\rho(t,\psi,\chi)\in\mathcal{L}(C,\mathbb{R})$ and $D_3\rho(t,\psi,\chi)\in\mathcal{L}(X,\mathbb{R})$ are continuous in t, ψ and χ.

Similarly, if $\psi\in C^1$, $\nu^i\in C^1$ $(i=1,\ldots,\ell)$, B is continuously differentiable wrt t, and $\chi\in C^1([0,T],\mathbb{R})$, then for $t\in[0,T]$

$$D_1\rho(t,\psi,\chi)$$
$$=D_1\bar{\rho}\Big(t,\psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0}B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)$$
$$-\sum_{i=1}^{\ell}D_{i+1}\bar{\rho}\Big(t,\psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0}B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\dot{\psi}(-\nu^i(t))\dot{\nu}^i(t)$$
$$+D_{\ell+2}\bar{\rho}\Big(t,\psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0}B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\int_{-r}^{-r_0}D_1B(t,\zeta)\psi(\zeta)\,d\zeta$$
$$+D_{\ell+3}\bar{\rho}\Big(t,\psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0}B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\dot{\chi}(t).$$

Moreover, it is easy to see that the function $[0,T]\times C^1\to\mathbb{R}$, $(t,\psi)\mapsto D_1\rho(t,\psi,\chi)$ is continuous.

Let

$$L_{12}:=\max_{i,j=1,\ldots,\ell+3}\max\Big\{|D_jD_i\bar{\rho}(t,u_1,\ldots,u_{\ell+1},v)|:\ t\in[0,\alpha],$$
$$u_1,\ldots,u_\ell\in\overline{\mathcal{B}}_{\mathbb{R}^n}(0;R_1),\quad u_{\ell+1}\in\overline{\mathcal{B}}_{\mathbb{R}^n}(0;b_{max}R_1r),\ v\in\overline{\mathcal{B}}_{\mathbb{R}}(0;R_2)\Big\}.$$

Then for $t \in [0,\alpha]$, $\psi, \bar{\psi} \in M_1$, $\chi, \bar{\chi} \in M_7$ and $h \in C$ we get

$$
\begin{aligned}
&|D_2\rho(t,\psi,\chi)h - D_2\rho(t,\bar{\psi},\bar{\chi})h| \\
&= \Big|\sum_{i=1}^{\ell} D_{i+1}\bar{\rho}\Big(t, \psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)), \int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta, \chi(t)\Big) h(-\nu^i(t)) \\
&\quad + D_{\ell+2}\bar{\rho}\Big(t, \psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)), \int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta, \chi(t)\Big) \int_{-r}^{-r_0} B(t,\zeta)h(\zeta)\,d\zeta \\
&\quad - \sum_{i=1}^{\ell} D_{i+1}\bar{\rho}\Big(t, \bar{\psi}(-\nu^1(t)),\ldots,\bar{\psi}(-\nu^\ell(t)), \int_{-r}^{-r_0} B(t,\zeta)\bar{\psi}(\zeta)\,d\zeta, \bar{\chi}(t)\Big) h(-\nu^i(t)) \\
&\quad - D_{\ell+2}\bar{\rho}\Big(t, \bar{\psi}(-\nu^1(t)),\ldots,\bar{\psi}(-\nu^\ell(t)), \int_{-r}^{-r_0} B(t,\zeta)\bar{\psi}(\zeta)\,d\zeta, \bar{\chi}(t)\Big) \int_{-r}^{-r_0} B(t,\zeta)h(\zeta)\,d\zeta\Big| \\
&\le L_{12}\Big(\sum_{j=1}^{\ell} |\psi(-\nu^j(t)) - \bar{\psi}(-\nu^j(t))| + \int_{-r}^{-r_0} |B(t,\zeta)||\psi(\zeta)-\bar{\psi}(\zeta)|\,d\zeta + |\chi(t)-\bar{\chi}(t)|\Big) \\
&\quad \times \Big(\sum_{i=1}^{\ell} |h(-\nu^i(t))| + \int_{-r}^{-r_0} |B(t,\zeta)||h(\zeta)|\,d\zeta\Big) \\
&\le L_7^*(\max_{\zeta\in[-r,-r_0]} |\psi(\zeta)-\bar{\psi}(\zeta)| + |\chi-\bar{\chi}|_X) \max_{\zeta\in[-r,-r_0]} |h(\zeta)|
\end{aligned}
$$

with $L_7^* := L_{12}(\ell + rb_{max})^2$.

Similarly, for $t \in [0,\alpha]$, $\psi, \bar{\psi} \in M_1$, $\chi, \bar{\chi} \in M_7$, $\eta \in X$ we have

$$
\begin{aligned}
&|D_3\rho(t,\psi,\chi)\eta - D_3\rho(t,\bar{\psi},\bar{\chi})\eta| \\
&= \Big|D_{\ell+3}\bar{\rho}\Big(t, \psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)), \int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta, \chi(t)\Big)\eta(t) \\
&\quad - D_{\ell+3}\bar{\rho}\Big(t, \bar{\psi}(-\nu^1(t)),\ldots,\bar{\psi}(-\nu^\ell(t)), \int_{-r}^{-r_0} B(t,\zeta)\bar{\psi}(\zeta)\,d\zeta, \bar{\chi}(t)\Big)\eta(t)\Big| \\
&\le L_{12}\Big(\sum_{i=1}^{\ell} |\psi(-\nu^i(t)) - \bar{\psi}(-\nu^i(t))| + \int_{-r}^{-r_0} |B(t,\zeta)||\psi(\zeta)-\bar{\psi}(\zeta)|\,d\zeta \\
&\quad + |\chi(t)-\bar{\chi}(t)|\Big)|\eta|_X \\
&\le L_7^*(\max_{\zeta\in[-r,-r_0]} |\psi(\zeta)-\bar{\psi}(\zeta)| + |\chi-\bar{\chi}|_X)|\eta|_X.
\end{aligned}
$$

For $t, \bar{t} \in [0,\alpha]$, $\psi \in M_1$, $\chi \in M_7$ and $h \in C$ we have

$$
\begin{aligned}
&|D_2\rho(t,\psi,\chi)h - D_2\rho(\bar{t},\psi,\chi)h| \\
&\le \sum_{i=1}^{\ell} \Big| D_{i+1}\bar{\rho}\Big(t, \psi(-\nu^1(t)),\ldots,\psi(-\nu^\ell(t)), \int_{-r}^{-r_0} B(t,\zeta)\psi(\zeta)\,d\zeta, \chi(t)\Big) \\
&\quad - D_{i+1}\bar{\rho}\Big(\bar{t}, \psi(-\nu^1(\bar{t})),\ldots,\psi(-\nu^\ell(\bar{t})), \int_{-r}^{-r_0} B(\bar{t},\zeta)\psi(\zeta)\,d\zeta, \chi(\bar{t})\Big)\Big| |h(-\nu^i(t))|
\end{aligned}
$$

$$
\begin{aligned}
&+\Big|D_{\ell+2}\bar\rho\Big(t,\psi(-\nu^1(t)),\dots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0}B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\\
&\quad -D_{\ell+2}\bar\rho\Big(\bar t,\psi(-\nu^1(\bar t)),\dots,\psi(-\nu^\ell(\bar t)),\int_{-r}^{-r_0}B(\bar t,\zeta)\psi(\zeta)\,d\zeta,\chi(\bar t)\Big)\Big|\\
&\quad\quad\times\int_{-r}^{-r_0}|B(t,\zeta)||h(\zeta)|d\zeta\\
&+\sum_{i=1}^{\ell}\Big|D_{i+1}\bar\rho\Big(\bar t,\psi(-\nu^1(\bar t)),\dots,\psi(-\nu^\ell(\bar t)),\int_{-r}^{-r_0}B(\bar t,\zeta)\psi(\zeta)\,d\zeta,\chi(\bar t)\Big)\Big|\\
&\quad\quad\times|h(-\nu^i(t))-h(-\nu^i(\bar t))|\\
&+\Big|D_{\ell+2}\bar\rho\Big(\bar t,\psi(-\nu^1(\bar t)),\dots,\psi(-\nu^\ell(\bar t)),\int_{-r}^{-r_0}B(\bar t,\zeta)\psi(\zeta)\,d\zeta,\chi(\bar t)\Big)\Big|\\
&\quad\quad\times\int_{-r}^{-r_0}|B(t,\zeta)-B(\bar t,\zeta)||h(\zeta)|\,d\zeta\\
\le\ & L_{12}\Big(|t-\bar t|+\sum_{j=1}^{\ell}|\psi(-\nu^j(t))-\psi(-\nu^j(\bar t))|+\int_{-r}^{-r_0}|B(t,\zeta)-B(\bar t,\zeta)||\psi(\zeta)|\,d\zeta\\
&\quad\quad+|\chi(t)-\chi(\bar t)|\Big)\Big(\sum_{i=1}^{\ell}|h(-\nu^i(t))|+\int_{-r}^{-r_0}|B(t,\zeta)||h(\zeta)|\,d\zeta\Big)\\
&+L_{11}\Big(\sum_{i=1}^{\ell}|h(-\nu^i(t))-h(-\nu^i(\bar t))|+\int_{-r}^{-r_0}|B(t,\zeta)-B(\bar t,\zeta)||h(\zeta)|\,d\zeta\Big)\\
\le\ & (L_{12}(1+\ell R_1L_9+rL_{10}R_1+R_2)(\ell+rb_{max})+rL_{11}L_{10})|t-\bar t|\max_{\zeta\in[-r,-r_0]}|h(\zeta)|\\
&+L_{11}l\max\{|h(\zeta)-h(\bar\zeta)|:\ \zeta,\bar\zeta\in[-r,-r_0],\ |\zeta-\bar\zeta|\le L_9|t-\bar t|\}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&|D_3\rho(t,\psi,\chi)\eta-D_3\rho(\bar t,\psi,\chi)\eta|\\
&\le\ \Big|D_{\ell+3}\bar\rho\Big(t,\psi(-\nu^1(t)),\dots,\psi(-\nu^\ell(t)),\int_{-r}^{-r_0}B(t,\zeta)\psi(\zeta)\,d\zeta,\chi(t)\Big)\eta(t)\\
&\quad\quad -D_{\ell+3}\bar\rho\Big(\bar t,\psi(-\nu^1(\bar t)),\dots,\psi(-\nu^\ell(\bar t)),\int_{-r}^{-r_0}B(\bar t,\zeta)\psi(\zeta)\,d\zeta,\chi(\bar t)\Big)\eta(t)\Big|\\
&\quad +\Big|D_{\ell+3}\bar\rho\Big(\bar t,\psi(-\nu^1(\bar t)),\dots,\psi(-\nu^\ell(\bar t)),\int_{-r}^{-r_0}B(\bar t,\zeta)\psi(\zeta)\,d\zeta,\chi(\bar t)\Big)\Big||\eta(t)-\eta(\bar t)|\\
&\le\ L_{12}\Big(|t-\bar t|+\sum_{i=1}^{\ell}|\psi(-\nu^i(t))-\psi(-\nu^i(\bar t))|+\int_{-r}^{-r_0}|B(t,\zeta)-B(\bar t,\zeta)||\psi(\zeta)|\,d\zeta\\
&\quad\quad +|\chi(t)-\chi(\bar t)|\Big)|\eta|_X+L_{11}|\eta|_X|t-\bar t|,
\end{aligned}
$$

so (A4) (iv) holds with $L_7:=\max\{L_7^*,L_{12}(1+\ell R_1L_9+rL_{10}R_1+R_2)(\ell+rb_{max})+rL_{11}L_{10}+L_{11},L_{11}\ell\}$ and $L_8=L_9$. □

We define the parameter space $\Gamma := W^{1,\infty} \times \Xi \times \Theta \times \Lambda \times X$, and use the notation $\gamma = (\varphi, \xi, \theta, \lambda, \chi)$ or $\gamma = (\gamma^\varphi, \gamma^\xi, \gamma^\theta, \gamma^\lambda, \gamma^\chi)$ for the components of $\gamma \in \Gamma$, and $|\gamma|_\Gamma := |\varphi|_{W^{1,\infty}} + |\xi|_\Xi + |\theta|_\Theta + |\lambda|_\Lambda + |\chi|_X$ for the norm on Γ. We introduce the set of feasible parameters

$$\begin{aligned}\Pi \;:=\; \Big\{&(\varphi, \xi, \theta, \lambda, \chi) \in \Gamma\colon\ \varphi \in \Omega_1,\ \ \varphi(-\tau(0,\varphi,\xi)) \in \Omega_2,\ \ \theta \in \Omega_3,\ \ \xi \in \Omega_4,\\ &\qquad\qquad \varphi(-\rho(0,\varphi,\chi)) \in \Omega_5,\ \ \lambda \in \Omega_6,\ \ \chi \in \Omega_7, \Big\}.\end{aligned}$$

We will show in Theorem 3.2 below that Π is an open subset of Γ. Next define the special parameter set

$$\begin{aligned}\mathcal{M} := \Big\{&(\varphi, \xi, \theta, \lambda, \chi) \in \Pi\ :\ g(t,\psi,u,\lambda) \text{ and } \rho(t,\psi,\chi) \text{ are differentiable wrt } t,\\ &\text{and the maps } (t,\psi,u) \mapsto D_1 g(t,\psi,u,\lambda) \text{ and } (t,\psi) \mapsto D_1\rho(t,\psi,\chi)\\ &\text{are continuous for } t \in [0,T],\ \psi \in \Omega_1,\ u \in \Omega_2; \qquad \varphi \in C^1;\\ &\quad \dot\varphi(0-) = D_1 g(0,\varphi,\varphi(-\rho(0,\varphi,\chi)),\lambda) + D_2 g(0,\varphi,\varphi(-\rho(0,\varphi,\chi)),\lambda)\dot\varphi\\ &\quad +D_3 g(0,\varphi,\varphi(-\rho(0,\varphi,\chi)),\lambda)\dot\varphi(-\rho(0,\varphi,\chi))\\ &\quad \times(1 - D_1\rho(0,\varphi,\chi) - D_2\rho(0,\varphi,\chi)\dot\varphi) + f(0,\varphi,\varphi(-\tau(0,\varphi,\xi)),\theta)\Big\}.\end{aligned}$$

Note that an analogous set was used for neutral FDEs in order to guarantee the existence of a continuous semiflow on a subset of C^1 in [27].

Next we show that under the assumptions listed in the beginning of this section the IVP (3.1)-(3.2) has a unique solution which depends continuously on the parameter $\gamma = (\varphi, \xi, \theta, \lambda, \chi)$ in the C-norm. The solution of the IVP (3.1)-(3.2) corresponding to a parameter γ and its segment function at t are denoted by $x(t,\gamma)$ and $x_t(\cdot,\gamma)$, respectively.

Theorem 3.2 *Assume (A1) (i), (ii), (A2) (i), (ii), (A3) (i), (ii) and (A4) (i)–(ii), and let* $\widehat\gamma \in \Pi$. *Then there exist* $\delta > 0$ *and* $0 < \alpha \le T$ *finite numbers such that*

(i) $P := \mathcal{B}_\Gamma(\widehat\gamma;\delta) \subset \Pi$*;*

(ii) the IVP (3.1)-(3.2) *has a unique solution* $x(t,\gamma)$ *on* $[-r,\alpha]$ *for all* $\gamma \in P$*;*

(iii) there exist a closed subset $M_1 \subset C$ *which is also a bounded and convex subset of* $W^{1,\infty}$, $M_2 \subset \Omega_2$ *and* $M_5 \subset \Omega_5$ *compact and convex subsets of* $\mathbb{R}^n$*, such that* $x(t) := x(t,\gamma)$ *satisfies*

$$x_t \in M_1, \qquad x(t-\tau(t,x_t,\xi)) \in M_2, \quad \text{and} \quad x(t-\rho(t,x_t,\chi)) \in M_5 \tag{3.6}$$

for $t \in [0,\alpha]$ *and* $\gamma = (\varphi,\xi,\theta,\lambda,\chi) \in P$*;*

(iv) $x_t(\cdot,\gamma) \in W^{1,\infty}$ *for* $t \in [0,\alpha]$, $\gamma \in P$*, and there exist* $N = N(\alpha,\delta)$ *and* $L = L(\alpha,\delta)$ *such that*

$$|x_t(\cdot,\gamma)|_{W^{1,\infty}} \le N, \qquad t \in [0,\alpha], \quad \gamma \in P, \tag{3.7}$$

and

$$|x_t(\cdot,\gamma) - x_t(\cdot,\bar\gamma)|_C \le L|\gamma - \bar\gamma|_\Gamma, \qquad t \in [0,\alpha], \quad \gamma,\bar\gamma \in P. \tag{3.8}$$

(v) Moreover, if (A3) (iii) and (A4) (iii) are also hold, then the function $x(\cdot,\gamma)\colon [-r,\alpha]\to \mathbb{R}^n$ is continuously differentiable for $\gamma\in\mathcal{M}\cap P$.

Proof (i) Let $\widehat\gamma := (\widehat\varphi,\widehat\xi,\widehat\theta,\widehat\lambda,\widehat\chi)\in\Pi$. Since $\Omega_1,\dots,\Omega_7$ are open subsets of their respective spaces, there exists $\delta_1>0$ such that $\overline{\mathcal{B}}_C(\widehat\varphi;\delta_1)\subset\Omega_1$, $\overline{\mathcal{B}}_\Theta(\widehat\theta;\delta_1)\subset\Omega_3$, $\overline{\mathcal{B}}_\Xi(\widehat\xi;\delta_1)\subset\Omega_4$, $\overline{\mathcal{B}}_\Lambda(\widehat\lambda;\delta_1)\subset\Omega_6$ and $\overline{\mathcal{B}}_X(\widehat\chi;\delta_1)\subset\Omega_7$. Introduce the vectors $w_1 := \widehat\varphi(-\tau(0,\widehat\varphi,\widehat\xi))$ and $w_2 := \widehat\varphi(-\rho(0,\widehat\varphi,\widehat\chi))$. Let $\varepsilon_1>0$ be such that $\overline{\mathcal{B}}_{\mathbb{R}^n}(w_1;\varepsilon_1)\subset\Omega_2$ and $\overline{\mathcal{B}}_{\mathbb{R}^n}(w_2;\varepsilon_1)\subset\Omega_5$. The map

$$\mathbb{R}\times C\times\Xi\supset[0,T]\times\Omega_1\times\Omega_4\to\mathbb{R}^n,\quad (t,\psi,\xi)\mapsto\psi(-\tau(t,\psi,\xi))$$

is continuous, since

$$\begin{aligned}
&|\psi(-\tau(t,\psi,\xi))-\bar\psi(-\tau(\bar t,\bar\psi,\bar\xi))|\\
&\quad\le |\psi(-\tau(t,\psi,\xi))-\bar\psi(-\tau(t,\psi,\xi))|+|\bar\psi(-\tau(t,\psi,\xi))-\bar\psi(-\tau(\bar t,\bar\psi,\bar\xi))|\\
&\quad\le |\psi-\bar\psi|_C+|\bar\psi(-\tau(t,\psi,\xi))-\bar\psi(-\tau(\bar t,\bar\psi,\bar\xi))|\\
&\quad\to 0,\qquad \text{as } t\to\bar t,\ \psi\to\bar\psi,\ \xi\to\bar\xi.
\end{aligned}$$

Similarly, the map $\mathbb{R}\times C\times\Xi\supset[0,T]\times\Omega_1\times\Omega_7\to\mathbb{R}^n$, $(t,\psi,\chi)\mapsto\psi(-\rho(t,\psi,\chi))$ is also continuous, therefore there exist $\delta_2\in(0,\delta_1]$ and $T_1\in(0,T]$ such that

$$\begin{aligned}
&|\psi(-\tau(t,\psi,\xi))-w_1|<\varepsilon_1,\qquad |\psi(-\rho(t,\psi,\chi))-w_2|<\varepsilon_1\\
&\qquad\text{for } t\in[0,T_1],\ \psi\in\mathcal{B}_C(\widehat\varphi;\delta_2),\ \xi\in\mathcal{B}_\Xi(\widehat\xi;\delta_2)\text{ and }\chi\in\mathcal{B}_X(\widehat\chi;\delta_2).
\end{aligned}\tag{3.9}$$

In particular, we get that for $\gamma:=(\varphi,\theta,\lambda,\xi)\in\mathcal{B}_\Gamma(\widehat\gamma;\delta_2)$ it follows $\varphi\in\Omega_1$, $\varphi(-\tau(0,\varphi,\xi))\in\Omega_2$, $\theta\in\Omega_3$, $\xi\in\Omega_4$, $\varphi(-\rho(0,\varphi,\xi))\in\Omega_5$, $\lambda\in\Omega_6$ and $\xi\in\Omega_7$. Therefore, part (i) of the theorem holds for any $0<\delta\le\delta_2$.

Fix $\varepsilon_0>0$. The continuity of the map $(t,\psi,\xi,\theta)\mapsto f(t,\psi,\psi(-\tau(t,\psi,\xi)),\theta)$ yields that there exist $\delta_3\in(0,\delta_2]$ and $T_2\in(0,T_1]$ such that

$$|f(t,\psi,\psi(-\tau(t,\psi,\xi)),\theta)-f(0,\widehat\varphi,\widehat\varphi(-\tau(0,\widehat\varphi,\widehat\xi)),\widehat\theta)|<\varepsilon_0$$

for $t\in[0,T_2]$, $\psi\in\mathcal{B}_C(\widehat\varphi;\delta_3)$, $\xi\in\mathcal{B}_\Xi(\widehat\xi;\delta_3)$ and $\theta\in\mathcal{B}_\Theta(\widehat\theta;\delta_3)$.

Define the sets

$$M_2:=\overline{\mathcal{B}}_{\mathbb{R}^n}(w_1;\varepsilon_1),\quad M_3:=\overline{\mathcal{B}}_\Theta(\widehat\theta;\delta_3),\quad M_4:=\overline{\mathcal{B}}_\Xi(\widehat\xi;\delta_3)$$

and

$$M_5:=\overline{\mathcal{B}}_{\mathbb{R}^n}(w_2;\varepsilon_1),\quad M_6:=\overline{\mathcal{B}}_\Lambda(\widehat\lambda;\delta_3),\quad M_7:=\overline{\mathcal{B}}_X(\widehat\chi;\delta_3).$$

Throughout this proof the extension of the function $\psi\in C$ to the interval $[-r,\infty)$ by the constant value $\psi(0)$ will be denoted by

$$\widetilde\psi(t):=\begin{cases}\psi(t), & t\in[-r,0],\\ \psi(0), & t\ge 0.\end{cases}$$

We define the following constants and sets

$$\begin{aligned}
K_2 &:= |f(0,\widehat{\varphi},\widehat{\varphi}(-\tau(0,\widehat{\varphi},\widehat{\xi})),\widehat{\theta})| + \varepsilon_0,\\
\beta_1 &:= \frac{\delta_3}{3},\\
\delta &:= \min\Big\{\frac{\delta_3}{3},\,\frac{\varepsilon_1}{2}\Big\},\\
a_0 &:= |\widehat{\varphi}|_{W^{1,\infty}} + \delta,\\
M_{1,0} &:= \{\psi\in W^{1,\infty}\colon\ |\psi-\widehat{\varphi}|_C\le\delta_3,\ |\dot{\psi}|_{L^\infty}\le a_0\}.
\end{aligned}$$

It is easy to check that $M_{1,0}$ is closed in C and it is bounded in $W^{1,\infty}$, so let

$$\begin{aligned}
L_{3,0} &:= L_3(T_2,M_{1,0},M_5,M_6) \text{ be the Lipschitz constant defined by (A3) (ii),}\\
L_{6,0} &:= L_6(T_2,M_{1,0},M_7) \text{ be the Lipschitz constant defined by (A4) (ii),}\\
K_{1,1} &:= L_{3,0}(1+a_0(2+L_{6,0}(1+a_0))),\\
a_1 &:= \max\{a_0, K_{1,1}+K_2\},\\
\alpha_1 &:= \min\Big\{\frac{\beta_1}{a_1},\,\frac{\varepsilon_1}{2a_0},\,T_2,\,r_0\Big\},\\
E_1 &:= \Big\{y\in C([-r,\alpha_1],\mathbb{R}^n)\colon\ y(s)=0,\ \ s\in[-r,0] \text{ and } |y(s)|\le\beta_1,\ \ s\in[0,\alpha_1]\Big\}.
\end{aligned}$$

We have $|\dot{\varphi}|_{L^\infty}\le|\varphi|_{W^{1,\infty}}\le|\widehat{\varphi}|_{W^{1,\infty}}+|\varphi-\widehat{\varphi}|_{W^{1,\infty}}\le a_0$ for $\varphi\in\mathcal{B}_{W^{1,\infty}}(\widehat{\varphi};\delta)$, and so $\mathcal{B}_{W^{1,\infty}}(\widehat{\varphi};\delta)\subset M_{1,0}$. Then for $y\in E_1$, $\varphi\in\mathcal{B}_{W^{1,\infty}}(\widehat{\varphi};\delta)$, $t\in[0,\alpha_1]$ and $\zeta\in[-r,0]$ we get

$$\begin{aligned}
|y(t+\zeta)+\widetilde{\varphi}(t+\zeta)-\widehat{\varphi}(\zeta)| &\le |y(t+\zeta)|+|\widetilde{\varphi}(t+\zeta)-\widetilde{\varphi}(\zeta)|+|\varphi(\zeta)-\widehat{\varphi}(\zeta)|\\
&< \beta_1+t|\dot{\varphi}|_{L^\infty}+\delta\\
&\le \beta_1+\alpha_1 a_0+\delta\\
&\le \delta_3,
\end{aligned}\tag{3.10}$$

and hence $|y_t+\widetilde{\varphi}_t-\widehat{\varphi}|_C<\delta_3$. Consequently, $y_t+\widetilde{\varphi}_t\in\mathcal{B}_C(\widehat{\varphi};\delta_3)\subset\Omega_1$, and so

$$\Big|f\Big(t,y_t+\widetilde{\varphi}_t,y(t-\tau(t,y_t+\widetilde{\varphi}_t,\xi))+\widetilde{\varphi}(t-\tau(t,y_t+\widetilde{\varphi}_t,\xi)),\theta\Big)\Big|\le K_1,$$

and $\psi=y_t+\widetilde{\varphi}_t$ satisfies (3.9) for $y\in E_1$, $\varphi\in\mathcal{B}_{W^{1,\infty}}(\widehat{\varphi};\delta)$, $\xi\in\mathcal{B}_\Xi(\widehat{\xi};\delta)$, $\theta\in\mathcal{B}_\Theta(\widehat{\varphi};\delta)$ and $t\in[0,\alpha_1]$. Therefore the definitions of M_2, M_5 and (3.9) yield

$$(y_t+\widetilde{\varphi}_t)(-\tau(t,\psi,\xi))\in M_2,\qquad (y_t+\widetilde{\varphi}_t)(-\rho(t,\psi,\chi))\in M_5\tag{3.11}$$

for $t\in[0,\alpha_1]$, $y\in E_1$, $\varphi\in\mathcal{B}_{W^{1,\infty}}(\widehat{\varphi};\delta)$, $\chi\in\mathcal{B}_X(\widehat{\chi};\delta)$ and $\xi\in\mathcal{B}_\Xi(\widehat{\xi};\delta)$.

Fix $\gamma=(\varphi,\theta,\xi,\lambda,\chi)\in\mathcal{B}_\Gamma(\bar{\gamma};\delta)$. Then $\varphi\in\mathcal{B}_{W^{1,\infty}}(\widehat{\varphi};\delta)$, $\theta\in\mathcal{B}_\Theta(\widehat{\theta};\delta)$, $\chi\in\mathcal{B}_X(\widehat{\chi};\delta)$, $\lambda\in\mathcal{B}_\Lambda(\widehat{\lambda};\delta)$ and $\chi\in\mathcal{B}_X(\widehat{\chi};\delta)$. We can use the method of steps to show that the IVP (3.1)-(3.2) corresponding to γ has a solution. First note that a solution will satisfy $x_t(\zeta)=x(t+\zeta)=\varphi(t+\zeta)=\widetilde{\varphi}_t(\zeta)$ for $t\in[0,r_0]$ and $\zeta\in[-r,-r_0]$. We have $t-\rho(t,\widetilde{\varphi}_t,\chi)\le t-r_0\le0$ for $t\in[0,r_0]$, so $y_t(-\rho(t,\widetilde{\varphi}_t,\chi))=0$ for $t\in[0,r_0]$. Hence (3.11) yields that $\varphi[t-\rho(t,\widetilde{\varphi}_t,\chi)]\in M_5$

for $t \in [0, r_0]$. An estimate similar to (3.10) gives $|\widetilde{\varphi}_t - \widehat{\varphi}|_C < \delta_3$ for $t \in [0, r_0]$. Therefore, the function

$$\mu^1(t) := g\{t, \widetilde{\varphi}_t, \varphi[t - \rho(t, \widetilde{\varphi}_t, \chi)], \lambda\}, \qquad t \in [0, r_0] \tag{3.12}$$

is well-defined. Then (A3) (ii), (A4) (ii), Lemma 2.3, $|\dot{\varphi}|_{L^\infty} \le a_0$, $\widetilde{\varphi}_t \in M_{1,0}$ for $t \in [0, r_0]$, and the definition of $K_{1,1}$ yield

$$\begin{aligned} |\mu^1(t) - \mu^1(\bar{t})| &\le L_{3,0}\Big\{|t - \bar{t}| + \max_{\zeta \in [-r, -r_0]} |\varphi(t + \zeta) - \varphi(\bar{t} + \zeta)| \\ &\qquad + \Big|\varphi[t - \rho(t, \widetilde{\varphi}_t, \chi)] - \varphi[\bar{t} - \rho(\bar{t}, \widetilde{\varphi}_{\bar{t}}, \chi)]\Big|\Big\} \\ &\le L_{3,0}\Big\{|t - \bar{t}| + |\dot{\varphi}|_{L^\infty}|t - \bar{t}| + |\dot{\varphi}|_{L^\infty}[1 + L_{6,0}(1 + |\dot{\varphi}|_{L^\infty})]|t - \bar{t}|\Big\} \\ &\le K_{1,1}|t - \bar{t}|, \qquad t, \bar{t} \in [0, r_0]. \end{aligned} \tag{3.13}$$

On the interval $[0, r_0]$ Equation (3.1) is equivalent to

$$\frac{d}{dt}\Big(x(t) - \mu^1(t)\Big) = f(t, x_t, x(t - \tau(t, x_t, \xi)), \theta), \qquad t \in [0, r_0].$$

Therefore, (3.1) is equivalent to

$$x(t) = \mu^1(t) + \varphi(0) - \mu^1(0) + \int_0^t f(s, x_s, x(s - \tau(s, x_s, \xi)), \theta)\, ds, \qquad t \in [0, r_0]. \tag{3.14}$$

We introduce the new variable $y(t) := x(t) - \widetilde{\varphi}(t)$, and we define the operator

$$T^1(y, \gamma)(t) := \begin{cases} \mu^1(t) - \mu^1(0) + \displaystyle\int_0^t f\Big(s, y_s + \widetilde{\varphi}_s, (y + \widetilde{\varphi})(s - \tau(s, y_s + \widetilde{\varphi}_s, \xi)), \theta\Big) ds, & t \in [0, \alpha_1], \\ 0, & t \in [-r, 0]. \end{cases}$$

Then in the new variable y, on the interval $[-r, \alpha_1]$ the IVP (3.1)-(3.2) is equivalent to the fixed point problem

$$y = T^1(y, \gamma).$$

It is easy to check that $T^1(\cdot, \gamma)$ maps the closed, bounded and convex subset E_1 of C into E_1 for all $\gamma \in \mathcal{B}_\Gamma(\widehat{\gamma}; \delta)$. Therefore, Schauder's Fixed Point Theorem yields the existence of a fixed point $y = y(\cdot, \gamma)$ of $T^1(\cdot, \gamma)$, and therefore, (3.1) has a solution $x = x(\cdot, \gamma) = y(\cdot, \gamma) + \widetilde{\varphi}$ on the interval $[-r, \alpha_1]$. Estimate (3.13) yields that μ^1 is Lipschitz continuous, and therefore, it is a.e. differentiable, and $|\dot{\mu}^1(t)| \le K_{1,1}$ for a.e. $t \in [0, \alpha_1]$. Hence y, and so, x is also a.e. differentiable on $t \in [-r, \alpha_1]$, and (3.14) implies $|\dot{x}(t)| = |\dot{y}(t)| \le K_{1,1} + K_2$ for a.e. $t \in [0, \alpha_1]$, and so $|\dot{x}(t)| \le a_1$ for a.e. $t \in [-r, \alpha_1]$.

(ii) Next we show by iteration that the solution obtained in part (i) of the proof can be extended to a larger interval so that estimate (3.7) remains to hold with some N independent of the selection of γ from $\mathcal{B}_\Gamma(\widehat{\gamma}; \delta)$. Let $j := 2$, and let $x = x(\cdot, \gamma)$ be the solution of (3.1)-(3.2) on $[-r, \alpha_{j-1}]$, $\varphi^j := x_{\alpha_{j-1}}$ and

$$\mu^j(t) := g\Big(t + \alpha_{j-1}, \widetilde{\varphi^j_t}, \varphi^j[t - \rho(t + \alpha_{j-1}, \widetilde{\varphi^j_t}, \chi)], \lambda\Big), \qquad t \in [0, r_0],$$

where $\widetilde{\varphi^j_t}$ denotes the segment function of $\widetilde{\varphi^j}$ at t. If $\alpha_{j-1} < T_2$, repeating the first part of the proof, we are looking for an extension of the solution of the IVP (3.1)-(3.2) by solving the fixed point equation

$$y = T^j(y, \gamma),$$

where $y(t) := x(t + \alpha_{j-1}) - \widetilde{\varphi^j}(t)$, and

$$T^j(y,\gamma)(t) := \begin{cases} \mu^j(t) - \mu^j(0) \\ \quad + \displaystyle\int_0^t f(s + \alpha_{j-1}, y_s + \widetilde{\varphi^j_s}, (y + \widetilde{\varphi^j})(s - \tau(s + \alpha_{j-1}, y_s + \widetilde{\varphi^j_s}, \xi)), \theta)\, ds, & \\ & t \in [0, \Delta\alpha_j], \\ 0, & t \in [-r, 0] \end{cases}$$

for some $\Delta\alpha_j \in (0, T_2 - \alpha_{j-1}]$. Relation (3.10) yields that $|\varphi^j - \widehat{\varphi}|_C < \delta_3$. Therefore, there exists $\varepsilon_j > 0$ such that $\overline{\mathcal{B}}_C(\varphi^j; \varepsilon_j) \subset \mathcal{B}_C(\widehat{\varphi}; \delta_3)$. Define the constants and sets

$$\begin{aligned}
\beta_j &:= \frac{\varepsilon_j}{2}, \\
M_{1,j-1} &:= \{\psi \in W^{1,\infty}\colon\ |\psi - \widehat{\varphi}|_C \le \delta_3,\ |\dot{\psi}|_{L^\infty} \le a_{j-1}\}, \\
L_{3,j-1} &:= L_3(T_2, M_{1,j-1}, M_5, M_6) \text{ be the Lipschitz constant defined by (A3) (ii)}, \\
L_{6,j-1} &:= L_6(T_2, M_{1,j-1}, M_7) \text{ be the Lipschitz constant defined by (A4) (ii)}, \\
K_{1,j} &:= L_{3,j-1}(1 + a_{j-1}(2 + L_{6,j-1}(1 + a_{j-1}))), \\
a_j &:= \max\{a_{j-1}, K_{1,j} + K_2\}, \\
\Delta\alpha_j &:= \min\Big\{\frac{\beta_j}{a_j}, \frac{\varepsilon_j}{2a_{j-1}}, T_2 - \alpha_{j-1}, r_0\Big\}, \\
\alpha_j &:= \alpha_{j-1} + \Delta\alpha_j
\end{aligned}$$

and

$$E_j := \Big\{y \in C([-r, \Delta\alpha_j], \mathbb{R}^n)\colon\ y(s) = 0,\ s \in [-r, 0] \text{ and } |y(s)| \le \beta_j,\ s \in [0, \Delta\alpha_j]\Big\}.$$

Since $|\dot{\varphi}^j|_{L^\infty} \le a_{j-1}$, it is easy to check that $|y_t + \widetilde{\varphi^j_t} - \varphi^j|_C \le \varepsilon_j$ for $t \in [0, \Delta\alpha_j]$, $y \in E_j$, and hence $\alpha_j \le T_2$ and (3.9) imply $(y_t + \widetilde{\varphi^j_t})(-\tau(t + \alpha_{j-1}, y_t + \widetilde{\varphi}^j_t, \xi)) \in M_2$ and $(y_t + \widetilde{\varphi^j_t})(-\rho(t + \alpha_{j-1}, y_t + \widetilde{\varphi^j_t}, \chi)) \in M_5$ for $t \in [0, \Delta\alpha_j]$, $y \in E_j$. Also, one can check that $|\mu^j(t) - \mu^j(\bar{t})| \le K_{1,j}|t - \bar{t}|$ for $t, \bar{t} \in [0, r_0]$, and the operator $T^j(\cdot, \gamma)$ maps E_j into E_j for all $\gamma \in \mathcal{B}_\Gamma(\widehat{\gamma}; \delta)$. Hence Schauder's Fixed Point Theorem yields the existence of a fixed point y of $T^j(\cdot, \gamma)$ in E_j, and hence the function $x(t) := y(t - \alpha_{j-1}) + \widetilde{\varphi}^j(t - \alpha_{j-1})$, $t \in [\alpha_{j-1}, \alpha_j]$ gives an extension of the solution of the IVP (3.1)-(3.2) from $[-r, \alpha_{j-1}]$ to the interval $[-r, \alpha_j]$. Moreover, for the extended solution we have $|\dot{x}(t)| \le a_j$ for a.e. $t \in [-r, \alpha_j]$. If $\alpha_j < T_2$, by repeating the previous iteration, we can extend the solution to a larger interval. In case of an infinite iteration, we stop it after finitely many steps to guarantee the boundedness of the sequence a_j. Suppose we repeat the iteration k times. Then let $\alpha := \alpha_k$. This completes the proof of the existence of a solution $x = x(\cdot, \gamma)$ of the IVP (3.1)-(3.2) on $[-r, \alpha]$ for any $\gamma \in \mathcal{B}_\Gamma(\widehat{\gamma}; \delta)$, which satisfies $|\dot{x}(t)| \le a_k$ for a.e. $t \in [-r, \alpha]$. The estimate

$$|x(t)| \le |\varphi(0)| + \int_0^t |\dot{x}(s)|\, ds \le a_0 + a_k \alpha, \qquad t \in [0, \alpha]$$

yields that x satisfies (3.7) with $N := \max\{a_k, a_0 + a_k\alpha\}$. Define the set

$$M_1 := M_{1,k} = \Big\{ \psi \in W^{1,\infty} \colon\ |\psi - \widehat{\varphi}|_C \le \delta_3, \quad |\dot{\psi}|_{L^\infty} \le a_k \Big\}.$$

Then $M_{1,j} \subset M_1$ for all $j = 0, \dots, k$, and $x_t \in M_1$ for $t \in [0, \alpha]$. The Arzelà-Ascoli Theorem implies that M_1 is a compact subset of C, and hence the solution $x = x(\cdot, \gamma)$ constructed by the above argument satisfies (3.6) for $t \in [0, \alpha]$ and $\gamma \in \mathcal{B}_\Gamma(\widehat{\gamma}; \delta)$.

(iii) The uniqueness of the solution will follow from (3.8). To show (3.8) suppose $\gamma = (\varphi, \xi, \theta, \lambda, \chi)$ and $\bar{\gamma} = (\bar{\varphi}, \bar{\xi}, \bar{\theta}, \bar{\lambda}, \bar{\chi})$ are fixed parameters in $\mathcal{B}_\Gamma(\widehat{\gamma}; \delta)$, and let x be *any* fixed solution of the IVP (3.1)-(3.2) corresponding to γ, and let $\bar{x} := x(\cdot; \bar{\gamma})$ be the solution of the IVP (3.1)-(3.2) obtained by the argument of part (i) of the proof on the interval $[-r, \alpha]$. Then part (i) of the proof yields $|\bar{x}_t|_{W^{1,\infty}} \le N$ and

$$|\bar{x}_t - \widehat{\varphi}|_C < \delta_3, \quad |\bar{x}(t - \tau(t, \bar{x}_t, \bar{\xi})) - w_1| < \varepsilon_1, \quad |\bar{x}(t - \rho(t, \bar{x}_t, \bar{\chi})) - w_2| < \varepsilon_1 \tag{3.15}$$

for $t \in [0, \alpha]$, and therefore $\bar{x}(t - \tau(t, \bar{x}_t, \bar{\xi})) \in M_2$ and $\bar{x}(t - \rho(t, \bar{x}_t, \bar{\chi})) \in M_5$ for $t \in [0, \alpha]$. Since $\gamma \in \mathcal{B}_\Gamma(\widehat{\gamma}; \delta)$, it follows that $\varphi \in \mathcal{B}_{W^{1,\infty}}(\widehat{\varphi}; \delta)$, $\xi \in \mathcal{B}_\Xi(\widehat{\xi}; \delta)$, $\theta \in \mathcal{B}_\Theta(\widehat{\theta}; \delta)$, $\lambda \in \mathcal{B}_\Lambda(\widehat{\lambda}; \delta)$ and $\chi \in \mathcal{B}_X(\widehat{\chi}; \delta)$. Hence $\delta < \delta_3$ and (3.9) yield $|\varphi - \widehat{\varphi}|_C < \delta_3$, $|\varphi(-\tau(0, \varphi, \xi)) - w_1| < \varepsilon_1$ and $|\varphi(-\rho(0, \varphi, \chi)) - w_2| < \varepsilon_1$. Therefore the continuity of x implies that the above inequalities are preserved for small t. Let $\alpha_\gamma \in (0, \alpha]$ be the largest number for which

$$|x_t - \widehat{\varphi}|_C < \delta_3, \qquad |x(t - \tau(t, x_t, \xi)) - w_1| < \varepsilon_1, \qquad |x(t - \rho(t, x_t, \chi)) - w_2| < \varepsilon_1 \tag{3.16}$$

hold for $t \in [0, \alpha^\gamma)$. Then $x(t - \tau(t, x_t, \xi)) \in M_2$ and $x(t - \rho(t, x_t, \chi)) \in M_5$ also hold for $t \in [0, \alpha^\gamma]$.

Next we show that $x_t \in M_1$ for $t \in [0, \alpha^\gamma]$. It is enough to show that $|\dot{x}_t|_{L^\infty} \le a_k$ for a.e. $t \in [0, \alpha^\gamma]$. Let $m = [\alpha^\gamma / r_0]$, where here $[\cdot]$ is the greatest integer part function. Note that $m \le k$ since $m r_0 \le \alpha^\gamma \le \alpha = \alpha_k \le k r_0$. Let $t_j := j r_0$ for $j = 0, \dots, m$, and $t_{m+1} := \alpha^\gamma$. Suppose first that $t_0 \le \bar{t} \le t \le t_1$. Then integrating (3.1) from $\bar{t}$ to t and using (A3) (ii), (A4) (i), (ii), (3.16), $|\dot{\varphi}|_{L^\infty} \le a_0$ and the definitions of $L_{3,0}$, $L_{6,0}$, K_2, $K_{1,1}$ and a_1 we get

$$\begin{aligned} |x(t) - x(\bar{t})| &\le |g(t, x_t, x(t - \rho(t, x_t, \chi)), \lambda) - g(\bar{t}, x_{\bar{t}}, x(\bar{t} - \rho(\bar{t}, x_{\bar{t}}, \chi)), \lambda)| \\ &\qquad + \int_{\bar{t}}^t |f(s, x_s, x(s - \tau(s, x_s, \xi)), \theta)|\, ds \\ &= |g(t, \widetilde{\varphi}_t, \varphi(t - \rho(t, \widetilde{\varphi}_t, \chi)), \lambda) - g(\bar{t}, \widetilde{\varphi}_{\bar{t}}, \varphi(\bar{t} - \rho(\bar{t}, \widetilde{\varphi}_{\bar{t}}, \chi)), \lambda)| \\ &\qquad + \int_{\bar{t}}^t |f(s, x_s, x(s - \tau(s, x_s, \xi)), \theta)|\, ds \end{aligned}$$

$$
\begin{aligned}
&\le L_{3,0}\Big(|t-\bar t| + \max_{\zeta\in[-r,-r_0]} |\varphi(t+\zeta)-\varphi(\bar t+\zeta)| \\
&\qquad + |\varphi(t-\rho(t,\widetilde\varphi_t,\chi)) - \varphi(\bar t-\rho(\bar t,\widetilde\varphi_{\bar t},\chi))|\Big) + K_2|t-\bar t| \\
&\le \Big(L_{3,0}(1+a_0(2+L_{6,0}(1+a_0))) + K_2\Big)|t-\bar t| \\
&\le a_1|t-\bar t|, \qquad t,\bar t\in[t_0,t_1].
\end{aligned}
$$

Then $a_0 \le a_1$ implies $|x(t)-x(\bar t)| \le a_1|t-\bar t|$ for $t,\bar t\in[-r,t_1]$.

Suppose now that $|x(t)-x(\bar t)| \le a_j|t-\bar t|$ holds for $t,\bar t\in[-r,t_j]$ for some $j\le m$. Then for $t,\bar t\in[-r,t_{j+1}]$ we get easily that

$$
\begin{aligned}
|x(t)-x(\bar t)| &\le \Big(L_{3,j}(1+a_j(2+L_{6,j}(1+a_j))) + K_2\Big)|t-\bar t| \\
&\le a_{j+1}|t-\bar t|, \qquad t,\bar t\in[t_0,t_{j+1}].
\end{aligned}
$$

This shows that $|x(t)-x(\bar t)| \le a_k|t-\bar t|$ for $t,\bar t\in[-r,\alpha^\gamma]$, hence $|\dot x_t|_{L^\infty}\le a_k$ for $t\in[0,\alpha^\gamma]$, and therefore $x_t\in M_1$ for $t\in[0,\alpha^\gamma]$.

Let $L_1 = L_1(\alpha,M_1,M_2,M_3)$, $L_2 = L_2(\alpha,M_1,M_4)$, $L_3 = L_3(\alpha,M_1,M_5,M_6)$ and $L_6 = L_6(\alpha,M_1,M_7)$ be the Lipschitz constants from (A1) (ii), (A2) (ii), (A3) (ii) and (A4) (ii), respectively. Integrating (3.1) from 0 to t we get for $t\in[0,\alpha^\gamma]$

$$
\begin{aligned}
&|x(t)-\bar x(t)| \\
&\quad\le |g(t,x_t,x(t-\rho(t,x_t,\chi)),\lambda) - g(t,\bar x_t,\bar x(t-\rho(t,\bar x_t,\bar\chi)),\bar\lambda)| + |\varphi(0)-\bar\varphi(0)| \\
&\qquad + |g(0,\varphi,\varphi(-\rho(0,\varphi,\chi)),\lambda) - g(0,\bar\varphi,\bar\varphi(-\rho(0,\bar\varphi,\bar\chi)),\bar\lambda)| \\
&\qquad + \int_0^t \Big| f(s,x_s,x(s-\tau(s,x_s,\xi)),\theta) - f(s,\bar x_s,\bar x(s-\tau(s,\bar x_s,\bar\xi)),\bar\theta)\Big|\,ds \\
&\quad\le L_3\Big(\max_{\zeta\in[-r,-r_0]} |x(t+\zeta)-\bar x(t+\zeta)| + |x(t-\rho(t,x_t,\chi)) - \bar x(t-\rho(t,\bar x_t,\bar\chi))| \\
&\qquad\quad + |\lambda-\bar\lambda|\Big) + |\varphi-\bar\varphi|_C \\
&\qquad + L_3(|\varphi-\bar\varphi|_C + |\varphi(-\rho(0,\varphi,\chi)) - \bar\varphi(-\rho(0,\bar\varphi,\bar\chi))| + |\lambda-\bar\lambda|) \\
&\qquad + L_1\int_0^t \Big(|x_s-\bar x_s|_C + |x(s-\tau(s,x_s,\xi)) - \bar x(s-\tau(s,\bar x_s,\bar\xi))| + |\theta-\bar\theta|_\Theta\Big)\,ds.
\end{aligned}
$$

Lemma 2.3, $|\bar x_t|_{W^{1,\infty}}\le N$ for $t\in[0,\alpha]$ and (A2) (ii) yield

$$
\begin{aligned}
&|x(s-\tau(s,x_s,\xi)) - \bar x(s-\tau(s,\bar x_s,\bar\xi))| \\
&\quad\le |\bar x(s-\tau(s,x_s,\xi)) - \bar x(s-\tau(s,\bar x_s,\bar\xi))| + |x(s-\tau(s,x_s,\xi)) - \bar x(s-\tau(s,x_s,\xi))| \\
&\quad\le N|\tau(s,x_s,\xi)-\tau(s,\bar x_s,\bar\xi)| + |x_s-\bar x_s|_C \\
&\quad\le L_2N(|x_s-\bar x_s|_C + |\xi-\bar\xi|_\Xi) + |x_s-\bar x_s|_C, \qquad s\in[0,\alpha^\gamma]. \qquad (3.17)
\end{aligned}
$$

Define $\mu(t) := \max\{|x(s)-\bar x(s)| : -r\le s\le t\}$ for $t\in[0,\alpha^\gamma]$. Assumption (A4) (i), Lemma 2.3, $|\bar{x}_t|_{W^{1,\infty}} \le N$ for $t \in [0,\alpha]$ and (A4) (ii) imply

$$\begin{aligned} &|x(t-\rho(t,x_t,\chi)) - \bar{x}(t-\rho(t,\bar{x}_t,\bar{\chi}))| \\ &\quad\le |x(t-\rho(t,x_t,\chi)) - \bar{x}(t-\rho(t,x_t,\chi))| + |\bar{x}(t-\rho(t,x_t,\chi)) - \bar{x}(t-\rho(t,\bar{x}_t,\bar{\chi}))| \\ &\quad\le \mu(t-r_0) + N|\rho(t,x_t,\chi) - \rho(t,\bar{x}_t,\bar{\chi})| \\ &\quad\le (1+NL_6)\mu(t-r_0) + NL_6|\chi-\bar{\chi}|_X, \qquad t \in [0,\alpha^\gamma]. \end{aligned}$$

Similarly, $|\varphi(-\rho(0,\varphi,\chi)) - \bar{\varphi}(-\rho(0,\bar{\varphi},\bar{\chi}))| \le (1+NL_6)|\varphi-\bar{\varphi}|_C + NL_6|\chi-\bar{\chi}|_X$. Therefore

$$\begin{aligned} |x(t)-\bar{x}(t)| \;\le\;& K_3\mu(t-r_0) + (K_3+1)|\varphi-\bar{\varphi}|_{W^{1,\infty}} + 2L_3|\lambda-\bar{\lambda}| + 2NL_3L_6|\chi-\bar{\chi}|_X \\ &+ L_1\int_0^t \Big((2+L_2N)\mu(s) + L_2N|\xi-\bar{\xi}|_\Xi + |\theta-\bar{\theta}|_\Theta\Big)\,ds, \quad t\in[0,\alpha^\gamma], \end{aligned}$$

where $K_3 := L_3(2+NL_6)$. Lemma 2.1 yields

$$\mu(t) \le K_3\mu(t-r_0) + K_4|\gamma-\bar{\gamma}|_\Gamma + K_5\int_0^t \mu(s)\,ds, \quad t\in[0,\alpha^\gamma],$$

where $K_4 := K_3 + 1 + 2L_3 + 2NL_3L_6 + L_1(L_2N+1)\alpha$ and $K_5 := L_1(2+L_2N)$. Applying Lemma 2.2 we get

$$|x(t)-\bar{x}(t)| \le \mu(t) \le de^{ct}, \qquad t\in[-r,\alpha^\gamma], \tag{3.18}$$

where $c>0$ is the solution of $cK_3e^{-cr_0} + K_5 = c$, and $d = d(\gamma,\bar{\gamma})$ is defined by

$$d := \max\left\{\frac{K_4|\gamma-\bar{\gamma}|_\Gamma}{1-K_3e^{-cr_0}}, e^{cr}|\varphi-\bar{\varphi}|_C\right\}.$$

Therefore there exists $K_6>0$ such that $d(\gamma,\bar{\gamma}) \le K_6|\gamma-\bar{\gamma}|_\Gamma$, so, combining this with (3.18), we get

$$|x(t)-\bar{x}(t)| \le L|\gamma-\bar{\gamma}|_\Gamma, \qquad t\in[-r,\alpha^\gamma], \quad \gamma\in\mathcal{B}_\Gamma(\bar{\gamma};\delta), \tag{3.19}$$

where $L = K_6e^{c\alpha}$. Note that the Lipschitz-constant L is independent of the selection of $\gamma,\bar{\gamma}\in P$. This concludes the proof of (3.8) on $[-r,\alpha^\gamma]$.

Hence if $\gamma=\bar{\gamma}$, then (3.19) yields that $x(t)=\bar{x}(t)$ for $t\in[0,\alpha^\gamma]$. But then (3.15) and the definition of α^γ yield that $\alpha^\gamma=\alpha$. This concludes the proof of the uniqueness of the solution of the IVP (3.1)-(3.2) on the interval $[-r,\alpha]$ for all $\gamma\in\mathcal{B}_\Gamma(\widehat{\gamma};\delta)$. This completes the proof of part (iv) of the theorem.

(iv) Fix $\gamma := (\varphi,\xi,\theta,\lambda,\chi)\in P\cap\mathcal{M}$, and suppose (A3) (iii) and (A4) (iii) hold. In particular, we get $\varphi\in C^1$. Define the operator

$$\mathcal{F}\colon C\to C, \quad (\mathcal{F}\psi)(s) := \begin{cases} \psi(s), & s\in[-r,-r_0], \\ \psi(-r_0), & s\in(-r_0,0]. \end{cases}$$

It follows from (A3) (ii) and (A4) (ii), respectively, that $g(t,\psi,u,\lambda) = g(t,\mathcal{F}\psi,u,\lambda)$, $\rho(t,\psi,\chi) = \rho(t,\mathcal{F}\psi,\chi)$, and hence $D_2g(t,\psi,u,\lambda) = D_2g(t,\mathcal{F}\psi,u,\lambda)$, $D_2\rho(t,\psi,\chi) = D_2\rho(t,\mathcal{F}\psi,\chi)$ for all $t\in[0,\alpha]$, $\psi\in M_1$, $u\in M_5$, $\lambda\in M_6$ and $\chi\in M_7$.

Next we show that the map $[0, r_0] \ni t \mapsto \mathcal{F}\widetilde{\varphi}_t \in C$ is continuously differentiable, and its derivative is $\mathcal{F}\widetilde{\dot{\varphi}}_t$. Here $\widetilde{\varphi}_t$ and $\widetilde{\dot{\varphi}}_t$ denote the segment functions of $\widetilde{\varphi}$ and $\widetilde{\dot{\varphi}}$ at t, respectively. Let $t \in [0, r_0]$, and h be such that $t + h \in [0, r_0]$. Then $t + \zeta \in [-r, 0]$ and $t + h + \zeta \in [-r, 0]$ for $\zeta \in [-r, -r_0]$, and hence

$$\begin{aligned} |\mathcal{F}\widetilde{\varphi}_{t+h} - \mathcal{F}\widetilde{\varphi}_t - h\mathcal{F}\widetilde{\dot{\varphi}}_t|_C &= \max_{\zeta\in[-r,0]} |(\mathcal{F}\widetilde{\varphi}_{t+h})(\zeta) - (\mathcal{F}\widetilde{\varphi}_t)(\zeta) - h(\mathcal{F}\widetilde{\dot{\varphi}}_t)(\zeta)| \\ &= \max_{\zeta\in[-r,-r_0]} |\widetilde{\varphi}_{t+h}(\zeta) - \widetilde{\varphi}_t(\zeta) - h\widetilde{\dot{\varphi}}_t(\zeta)| \\ &= \max_{\zeta\in[-r,-r_0]} |\varphi(t+h+\zeta) - \varphi(t+\zeta) - h\dot{\varphi}(t+\zeta)| \\ &\le |h| \max\{|\dot{\varphi}(u) - \dot{\varphi}(\bar{u})| \colon\ u, \bar{u} \in [-r, 0],\ |u - \bar{u}| \le |h|\}. \end{aligned}$$

This proves the differentiability of $\mathcal{F}\widetilde{\varphi}_t$ wrt t on $[0, r_0]$, using that $\varphi \in C^1$, so $\dot{\varphi}$ is uniformly continuous. A similar argument shows that $|\mathcal{F}\widetilde{\dot{\varphi}}_t - \mathcal{F}\widetilde{\dot{\varphi}}_{\bar{t}}|_C \to 0$ as $t \to \bar{t}$ for $t, \bar{t} \in [0, r_0]$.

Then assumptions (A3) (iii) and (A4) (iii) yield that the function μ^1 defined in (3.12) is continuously differentiable on $[0, r_0]$. Therefore (3.14) implies that x is continuously differentiable on $[0, r_0]$, and the compatibility condition in the definition of $\mathcal{M}$ yields $\varphi(0-) = x(0+)$, so x is continuously differentiable on $[-r, r_0]$. Hence $g(t, x_t, x(t - \rho(t, x_t, \chi)), \lambda)$ is differentiable wrt t for $t \in [0, r_0]$, and therefore on $[0, r_0]$ the IVP (3.1)-(3.2) is equivalent to

$$\begin{aligned} \dot{x}(t) &= D_1 g(t, x_t, x(v(t)), \lambda) + D_2 g(t, x_t, x(v(t)), \lambda)\dot{x}_t + D_3 g(t, x_t, x(v(t)), \lambda) \\ &\quad \times \dot{x}(v(t))\{1 - D_1\rho(t, x_t, \chi) - D_2\rho(t, x_t, \chi)\dot{x}_t\} + f(t, x_t, x(u(t)), \theta), \end{aligned} \tag{3.20}$$

where $v(t) := t - \rho(t, x_t, \chi)$ and $u(t) := t - \tau(t, x_t, \xi)$. (A1)–(A4) imply that the right-hand side of (3.20) is continuous in t, therefore the definition of $\mathcal{M}$ yields that $\dot{x}$ is continuous on $[-r, r_0]$. Now the continuity of $\dot{x}$ follows from (3.20) and the definition of $\mathcal{M}$, using the method of steps with the intervals $[ir_0, (i+1)r_0]$, $i = 0, 1, 2, \ldots$. □

4 Differentiability wrt parameters

In this section we study differentiability of solutions of the IVP (3.1)-(3.2) wrt the initial function, φ, the parameters ξ, θ, λ and χ of the functions τ, f, g and ρ, respectively.

Let the positive constants α and δ, the parameter set P, and the compact and convex sets M_1, M_2 and M_5 be defined by Theorem 3.2. Let

$$M_3 := \overline{\mathcal{B}}_\Theta(\widehat{\theta}; \delta), \quad M_4 := \overline{\mathcal{B}}_\Xi(\widehat{\xi}; \delta), \quad M_6 := \overline{\mathcal{B}}_\Lambda(\widehat{\lambda}; \delta) \quad \text{and} \quad M_7 := \overline{\mathcal{B}}_X(\widehat{\chi}; \delta), \tag{4.1}$$

as in the proof of Theorem 3.2.

First we define a few notations will be used throughout this section. Introduce

$$\begin{aligned} \omega_f(t, \bar{\psi}, \bar{u}, \bar{\theta}, \psi, u, \theta) &:= f(t, \psi, u, \theta) - f(t, \bar{\psi}, \bar{u}, \bar{\theta}) - D_2 f(t, \bar{\psi}, \bar{u}, \bar{\theta})(\psi - \bar{\psi}) \\ &\quad - D_3 f(t, \bar{\psi}, \bar{u}, \bar{\theta})(u - \bar{u}) - D_4 f(t, \bar{\psi}, \bar{u}, \bar{\theta})(\theta - \bar{\theta}) \end{aligned}$$

for $t \in [0,T]$, $\bar\psi,\psi \in M_1$, $\bar u, u \in M_2$, and $\bar\theta,\theta \in M_3$. Assumption (A1) (iii) and the convexity of M_1, M_2 and M_3 yield

$$\begin{aligned}
&|\omega_f(t,\bar\psi,\bar u,\bar\theta,\psi,u,\theta)| \\
&\quad\le \sup_{0<\nu<1}\Big(\Big|D_2 f(t,\bar\psi+\nu(\psi-\bar\psi),\bar u+\nu(u-\bar u),\bar\theta+\nu(\theta-\bar\theta)) \\
&\qquad\qquad -D_2 f(t,\bar\psi,\bar u,\bar\theta)\Big|_{\mathcal{L}(C,\mathbb{R}^n)}|\psi-\bar\psi|_C \\
&\qquad + \Big|D_3 f(t,\bar\psi+\nu(\psi-\bar\psi),\bar u+\nu(u-\bar u),\bar\theta+\nu(\theta-\bar\theta)) - D_3 f(t,\bar\psi,\bar u,\bar\theta)\Big||u-\bar u| \\
&\qquad + \Big|D_4 f(t,\bar\psi+\nu(\psi-\bar\psi),\bar u+\nu(u-\bar u),\bar\theta+\nu(\theta-\bar\theta)) \\
&\qquad\qquad -D_4 f(t,\bar\psi,\bar u,\bar\theta)\Big|_{\mathcal{L}(\Theta,\mathbb{R}^n)}|\theta-\bar\theta|_\Theta\Big)
\end{aligned}$$

for $t \in [0,\alpha]$, $\psi,\bar\psi \in M_1$, $u,\bar u \in M_2$ and $\theta,\bar\theta \in M_3$. Then

$$|\omega_f(t,\bar\psi,\bar u,\bar\theta,\psi,u,\theta)| \le \Omega_f\Big(|\psi-\bar\psi|_C+|u-\bar u|+|\theta-\bar\theta|_\Theta\Big)\Big(|\psi-\bar\psi|_C+|u-\bar u|+|\theta-\bar\theta|_\Theta\Big) \quad (4.2)$$

for $t \in [0,\alpha]$, $\psi,\bar\psi \in M_1$, $u,\bar u \in M_2$ and $\theta,\bar\theta \in M_3$, where

$$\begin{aligned}
\Omega_f(\varepsilon) \;:=\; \sup\Big\{\max\Big(&|D_2 f(t,\psi,u,\theta) - D_2 f(t,\bar\psi,\bar u,\bar\theta)|_{\mathcal{L}(C,\mathbb{R}^n)}, \\
&|D_3 f(t,\psi,u,\theta) - D_3 f(t,\bar\psi,\bar u,\bar\theta)|, \\
&|D_4 f(t,\psi,u,\theta) - D_4 f(t,\bar\psi,\bar u,\bar\theta)|_{\mathcal{L}(\Theta,\mathbb{R}^n)}\Big): \\
&\quad |\psi-\bar\psi|_C+|u-\bar u|+|\theta-\bar\theta|_\Theta \le \varepsilon, \\
&\quad t\in[0,\alpha],\ \psi,\bar\psi\in M_1,\ u,\bar u\in M_2,\ \theta,\bar\theta\in M_3\Big\}.
\end{aligned}$$

Similarly, we define

$$\omega_\tau(t,\bar\psi,\bar\xi,\psi,\xi) := \tau(t,\psi,\xi)-\tau(t,\bar\psi,\bar\xi)-D_2\tau(t,\bar\psi,\bar\xi)(\psi-\bar\psi)-D_3\tau(t,\bar\psi,\bar\xi)(\xi-\bar\xi)$$

for $t \in [0,\alpha]$, $\psi,\bar\psi \in M_1$ and $\xi,\bar\xi \in M_4$. Then Assumption (A2) (iii) gives that

$$|\omega_\tau(t,\bar\psi,\bar\xi,\psi,\xi)| \le \Omega_\tau(|\psi-\bar\psi|_C+|\xi-\bar\xi|)(|\psi-\bar\psi|_C+|\xi-\bar\xi|) \quad (4.3)$$

for $t \in [0,\alpha]$, $\psi,\bar\psi \in M_1$ and $\xi,\bar\xi \in M_4$, where

$$\begin{aligned}
\Omega_\tau(\varepsilon) \;:=\; \sup\Big\{\max\Big(&|D_2(t,\psi,\xi)-D_2(t,\bar\psi,\bar\xi)|_{\mathcal{L}(C,\mathbb{R}^n)}, \\
&|D_3(t,\psi,\xi)-D_3(t,\bar\psi,\bar\xi)|_{\mathcal{L}(\Xi,\mathbb{R}^n)}\Big): \\
&\quad t\in[0,\alpha],\ \psi,\bar\psi\in M_1,\ \xi,\bar\xi\in M_4,\ |\psi-\bar\psi|_C+|\xi-\bar\xi|\le\varepsilon\Big\}.
\end{aligned}$$

We introduce the function

$$\begin{aligned}
\omega_g(t,\bar\psi,\bar u,\bar\lambda,\psi,u,\lambda) \;:=\;& g(t,\psi,u,\lambda)-g(t,\bar\psi,\bar u,\bar\lambda)-D_2 g(t,\bar\psi,\bar u,\bar\lambda)(\psi-\bar\psi) \\
&- D_3 g(t,\bar\psi,\bar u,\bar\lambda)(u-\bar u)-D_4 g(t,\bar\psi,\bar u,\bar\lambda)(\lambda-\bar\lambda)
\end{aligned}$$

for $t \in [0,\alpha]$, $\bar\psi, \psi \in M_1$, $\bar u, u \in M_5$, $\bar\lambda, \lambda \in M_6$, and let $L_4 = L_4(\alpha, M_1, M_5, M_6)$ be the Lipschitz constant from (A3) (iv). Then (A4) (iii) yields

$$|\omega_g(t, \bar\psi, \bar u, \bar\lambda, \psi, u, \lambda)| \le L_4\big(\max_{\zeta\in[-r,-r_0]} |\psi(\zeta) - \bar\psi(\zeta)| + |u - \bar u| + |\lambda - \bar\lambda|_\Lambda\big)^2, \qquad (4.4)$$

for $t \in [0,\alpha]$, $\bar\psi, \psi \in M_1$, $u, \bar u \in M_5$, $\bar\lambda, \lambda \in M_6$.

Let $\bar\gamma = (\bar\varphi, \bar\xi, \bar\theta, \bar\lambda, \bar\chi) \in P \cap \mathcal{M}$, and $x(t) := x(t, \bar\gamma)$ be the corresponding solution of the IVP (3.1)-(3.2) on $[-r, \alpha]$. Note that Theorem 3.2 yields that x is continuously differentiable on $[-r, \alpha]$. Fix $h = (h^\varphi, h^\xi, h^\theta, h^\lambda, h^\chi) \in \Gamma$, and consider the variational equation

$$\begin{aligned}
&\frac{d}{dt}\Big(z(t) - D_2 g(t, x_t, x(t-\rho(t,x_t,\bar\chi)), \bar\lambda) z_t - D_3 g(t, x_t, x(t-\rho(t,x_t,\bar\chi)), \bar\lambda)\\
&\qquad \times\Big[-\dot x(t-\rho(t,x_t,\bar\chi))\Big\{D_2\rho(t,x_t,\bar\chi)z_t + D_3\rho(t,x_t,\bar\chi)h^\chi\Big\} + z(t-\rho(t,x_t,\bar\chi))\Big]\\
&\qquad -D_4 g(t, x_t, x(t-\rho(t,x_t,\bar\chi)), \bar\lambda)h^\lambda\Big)\\
&\quad= D_2 f(t, x_t, x(t-\tau(t,x_t,\bar\xi)), \bar\theta) z_t + D_3 f(t, x_t, x(t-\tau(t,x_t,\bar\xi)), \bar\theta)\\
&\qquad \times\Big[-\dot x(t-\tau(t,x_t,\bar\xi))\Big\{D_2\tau(t,x_t,\bar\xi)z_t + D_3\tau(t,x_t,\bar\xi)h^\xi\Big\} + z(t-\tau(t,x_t,\bar\xi))\Big]\\
&\qquad +D_4 f(t, x_t, x(t-\tau(t,x_t,\bar\xi)), \theta)h^\theta, \qquad t\in[0,\alpha] \qquad (4.5)\\
&z(t) = h^\varphi(t), \qquad t \in [-r, 0]. \qquad (4.6)
\end{aligned}$$

This is an inhomogeneous linear time-dependent but state-independent NFDE for z with continuous coefficients, therefore this IVP has a unique solution, $z(t) = z(t, \bar\gamma, h)$, which depends linearly on h. The boundedness of the map $\Gamma \to \mathbb{R}^n$, $h \mapsto z(t, \bar\gamma, h)$ for each $t \in [0, \alpha]$ follows from Theorem 4.1 below.

For a fixed $t \in [0,\alpha]$ we introduce the linear operator $L(t,x)\colon C \times \Xi \times \Theta \to \mathbb{R}^n$ defined by

$$\begin{aligned}
&L(t,x)(\psi, h^\xi, h^\theta)\\
&\quad := D_2 f(t, x_t, x(t-\tau(t,x_t,\bar\xi)), \bar\theta)\psi + D_3 f(t, x_t, x(t-\tau(t,x_t,\bar\xi)), \bar\theta)\\
&\qquad \times\Big[-\dot x(t-\tau(t,x_t,\bar\xi))\Big\{D_2\tau(t,x_t,\bar\xi)\psi + D_3\tau(t,x_t,\bar\xi)h^\xi\Big\} + \psi(-\tau(t,x_t,\bar\xi))\Big]\\
&\qquad +D_4 f(t, x_t, x(t-\tau(t,x_t,\bar\xi)), \bar\theta)h^\theta \qquad (4.7)
\end{aligned}$$

and the linear operator $G(t,x)\colon C \times \Lambda \times X \to \mathbb{R}^n$ defined by

$$\begin{aligned}
&G(t,x)(\psi, h^\lambda, h^\chi)\\
&\quad := D_2 g(t, x_t, x(t-\rho(t,x_t,\bar\chi)), \bar\lambda)\psi + D_3 g(t, x_t, x(t-\rho(t,x_t,\bar\chi)), \bar\lambda)\\
&\qquad \times\Big[-\dot x(t-\rho(t,x_t,\bar\chi))\Big\{D_2\rho(t,x_t,\bar\chi)\psi + D_3\rho(t,x_t,\bar\chi)h^\chi\Big\} + \psi(-\rho(t,x_t,\bar\chi))\Big]\\
&\qquad +D_4 g(t, x_t, x(t-\rho(t,x_t,\bar\chi)), \bar\lambda)h^\lambda. \qquad (4.8)
\end{aligned}$$

With these notations (4.5) can be rewritten as

$$\frac{d}{dt}\Big(z(t) - G(t,x)(z_t, h^\lambda, h^\chi)\Big) = L(t,x)(z_t, h^\xi, h^\theta), \qquad t \in [0,\alpha]. \qquad (4.9)$$

Let $L_1 = L_1(\alpha, M_1, M_2, M_3)$ and $L_2 = L_2(\alpha, M_1, M_4)$ be the Lipschitz constants from (A1) (ii) and (A2) (ii), respectively. Then (A1) (ii), (A2) (ii) and (3.7) yield

$$
\begin{aligned}
&|L(t,x)(\psi, h^\xi, h^\theta)| \\
&\quad \leq \; L_1|\psi|_C + L_1\Big(NL_2(|\psi|_C + |h^\xi|_\Xi) + |\psi|_C\Big) + L_1|h^\theta|_\Theta \\
&\quad \leq \; N_0\Big(|\psi|_C + |h^\xi|_\Xi + |h^\theta|_\Theta\Big), \qquad t \in [0,\alpha],\ \psi \in C,\ h^\xi \in \Xi,\ h^\theta \in \Theta, \qquad (4.10)
\end{aligned}
$$

where $N_0 := L_1(2NL_2 + 2)$.

Let $L_3 = L_3(\alpha, M_1, M_5, M_6)$, $L_6 = L_6(\alpha, M_1, M_7)$ be defined by (A3) (ii) and (A4) (ii), respectively. Then we have by (A3) (ii) and (A4) (ii) that

$$
|G(t,x)(\psi, h^\lambda, h^\chi)| \leq N_1\Big(\max_{\zeta\in[-r,-r_0]}|\psi(\zeta)| + |h^\lambda|_\Lambda + |h^\chi|_X\Big), \qquad t \in [0,\alpha], \qquad (4.11)
$$

for $\psi \in C,\ h^\lambda \in \Lambda,\ h^\chi \in X$, where $N_1 := L_3(2NL_6 + 2)$.

Theorem 4.1 *Assume (A1) (i)–(iii), (A2) (i)–(iii), (A3) (i)–(iv) and (A4) (i)–(iv), let $\alpha > 0$ and $P \subset \Pi$ be defined by Theorem 3.2. There exists $N_2 \geq 0$ such that the solution of the IVP* (4.5)-(4.6) *satisfies*

$$
|z(t,\gamma,h)| \leq N_2|h|_\Gamma, \qquad t \in [-r,\alpha], \quad h \in \Gamma, \quad \gamma \in P \cap \mathcal{M}. \qquad (4.12)
$$

Moreover, for $\bar\gamma \in P \cap \mathcal{M}$ there exists a monotone increasing function $A = A(\bar\gamma)$ such that $A\colon [0,\infty) \to [0,\infty)$, $A(u) \to 0$ as $u \to 0$, and

$$
|z(t,\bar\gamma,h) - z(\bar t,\bar\gamma,h)| \leq A(|t-\bar t|)|h|_\Gamma, \qquad t, \bar t \in [-r,\alpha], \quad h \in \Gamma. \qquad (4.13)
$$

Proof (i) Let $\gamma \in P\cap\mathcal{M}$ be fixed. For simplicity we use the notations $h = (h^\varphi, h^\xi, h^\theta, h^\lambda, h^\chi) \in \Gamma$, $x(t) := x(t,\gamma)$ and $z(t) := z(t,\gamma,h)$. Let δ, M_1, M_2 and M_5 be defined by Theorem 3.2, M_3, M_4, M_6 and M_7 be defined by (4.1), $L_1,\ldots,L_8$ be the corresponding Lipschitz constants form (A1)–(A4), and let N_0 and N_1 be corresponding constants defined by (4.10) and (4.11), respectively. Integrating (4.9) from 0 to t we get

$$
|z(t)| \leq |G(t,x)(z_t, h^\lambda, h^\chi)| + |h^\varphi(0)| + |G(0,x)(h^\varphi, h^\lambda, h^\chi)| + \int_0^t |L(s,x)(z_s, h^\xi, h^\theta)|\,ds
$$

for $t \in [0,\alpha]$, and therefore (4.10) and (4.11) yield

$$
\begin{aligned}
|z(t)| \;&\leq\; N_1 \max_{\zeta\in[-r,-r_0]}|z(t+\zeta)| + (1+N_1)|h^\varphi|_C + 2N_1(|h^\lambda|_\Lambda + |h^\chi|_X) \\
&\quad + N_0\int_0^t (|z_s|_C + |h^\xi|_\Xi + |h^\theta|_\Theta)\,ds, \qquad t \in [0,\alpha].
\end{aligned}
$$

An application of Lemma 2.1 implies

$$
\mu(t) \leq N_1\mu(t-r_0) + K_7|h|_\Gamma + N_0\int_0^t \mu(s)\,ds, \qquad t \in [0,\alpha],
$$

where $\mu(t) := \max\{|z(s)|\colon\ s \in [-r,t]\}$ and $K_7 := \max\{N_0\alpha, 1+N_1, 2N_1\}$. Then Lemma 2.2 yields

$$|z(t)| \le \mu(t) \le N_2|h|_\Gamma, \qquad t \in [0,\alpha],$$

where

$$N_2 := \max\left\{\frac{K_7}{1-N_1e^{-cr_0}}, e^{cr}\right\} e^{c\alpha}$$

and c is the positive solution of $cN_1e^{-cr_0} + N_0 = c$. Moreover, $\mu(0) \le N_2|h|_\Gamma$ yields that (4.12) holds for $t \in [-r,0]$, as well. This concludes the proof of (4.12).

(ii) Let $\bar\gamma = (\bar\varphi, \bar\xi, \bar\theta, \bar\lambda, \bar\chi) \in P \cap \mathcal{M}$, $x(t) := x(t,\bar\gamma)$, $h = (h^\varphi, h^\xi, h^\theta, h^\lambda, h^\chi) \in \Gamma$, $z(t) := z(t,\bar\gamma,h)$, $v(t) := t - \rho(t,x_t,\bar\chi)$. Let $t,\bar t \in [0,\alpha]$, and consider

$$\begin{aligned}
&G(t,x)(z_t,h^\lambda,h^\chi) - G(\bar t,x)(z_{\bar t},h^\lambda,h^\chi)\\
&\quad= D_2g(t,x_t,x(v(t)),\bar\lambda)z_t - D_2g(\bar t,x_{\bar t},x(v(\bar t)),\bar\lambda)z_t + D_2g(\bar t,x_{\bar t},x(v(\bar t)),\bar\lambda)(z_t - z_{\bar t})\\
&\qquad+\Big[D_3g(t,x_t,x(v(t)),\bar\lambda) - D_3g(\bar t,x_{\bar t},x(v(\bar t)),\bar\lambda)\Big]\\
&\qquad\quad\times\Big[-\dot x(v(t))\Big\{D_2\rho(t,x_t,\bar\chi)z_t + D_3\rho(t,x_t,\bar\chi)h^\chi\Big\} + z(v(t))\Big]\\
&\qquad+D_3g(\bar t,x_{\bar t},x(v(\bar t)),\bar\lambda)\Big[-\Big(\dot x(v(t)) - \dot x(v(\bar t))\Big)\Big\{D_2\rho(t,x_t,\bar\chi)z_t + D_3\rho(t,x_t,\bar\chi)h^\chi\Big\}\Big]\\
&\qquad-D_3g(\bar t,x_{\bar t},x(v(\bar t)),\bar\lambda)\dot x(v(\bar t))\Big[D_2\rho(t,x_t,\bar\chi)z_t - D_2\rho(\bar t,x_{\bar t},\bar\chi)z_t\\
&\qquad\quad+D_2\rho(\bar t,x_{\bar t},\bar\chi)(z_t - z_{\bar t}) + D_3\rho(t,x_t,\bar\chi)h^\chi - D_3\rho(\bar t,x_{\bar t},\bar\chi)h^\chi\Big]\\
&\qquad+D_3g(\bar t,x_{\bar t},x(v(\bar t)),\bar\lambda)\Big[z(v(t)) - z(v(\bar t))\Big]\\
&\qquad+\Big[D_4g(t,x_t,x(v(t)),\bar\lambda) - D_4g(\bar t,x_{\bar t},x(v(\bar t)),\bar\lambda)\Big]h^\lambda.
\end{aligned}\tag{4.14}$$

Let N be defined by (3.7), $L_6 = L_6(\alpha,M_1,M_7)$, $L_7 = L_7(\alpha,M_1,M_7)$ and $L_8 = L_8(\alpha,M_1,M_7)$ be the Lipschitz constants defined by (A4) (ii) and (iv), respectively. Then (A4) (ii) and (3.7) yield

$$\begin{aligned}
|v(t) - v(\bar t)| &= |\rho(t,x_t,\bar\chi) - \rho(\bar t,x_{\bar t},\bar\chi)|\\
&\le L_6(|t-\bar t| + |x_t - x_{\bar t}|_C)\\
&\le L_6(1+N)|t-\bar t|, \qquad t,\bar t \in [0,\alpha],
\end{aligned}\tag{4.15}$$

and hence

$$|x(v(t)) - x(v(\bar t))| \le NL_6(1+N)|t-\bar t|, \qquad t,\bar t \in [0,\alpha]. \tag{4.16}$$

Define the function

$$\Omega_{\dot x}(\varepsilon) := \sup\Big\{|\dot x(u) - \dot x(\bar u)|\colon\ |u-\bar u| \le \varepsilon, \quad u,\bar u \in [-r,\alpha]\Big\}. \tag{4.17}$$

Since $\bar\gamma \in \mathcal{M}$, x is continuously differentiable on $[-r,\alpha]$, hence $\Omega_{\dot x}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Therefore (A3) (ii), (iv), (A4) (ii) and (3.7) imply for $t, \bar{t} \in [0, \alpha]$

$$\begin{aligned}
&|G(t,x)(z_t, h^\lambda, h^\chi) - G(\bar{t},x)(z_{\bar{t}}, h^\lambda, h^\chi)| \\
&\quad \le L_4\Big(|t-\bar{t}| + |x_t - x_{\bar{t}}|_C + |x(v(t)) - x(v(\bar{t}))|\Big)|z_t| \\
&\qquad + L_4 \max\{|z(t+\zeta) - z(t+\bar{\zeta})| \colon \zeta, \bar{\zeta} \in [-r, -r_0],\ |\zeta - \bar{\zeta}| \le L_5|t-\bar{t}|\} \\
&\qquad + L_3 \max_{\zeta\in[-r,-r_0]} |z(t+\zeta) - z(\bar{t}+\zeta)|
\end{aligned}$$

$$\begin{aligned}
&+L_4\Big(|t-\bar{t}| + |x_t - x_{\bar{t}}|_C + |x(v(t)) - x(v(\bar{t}))|\Big)\Big(NL_6(|z_t|_C + |h^\chi|_X) + |z(v(t))|\Big) \\
&+L_3\Omega_{\dot{x}}(|v(t) - v(\bar{t})|)L_6(|z_t|_C + |h^\chi|_X) + L_3N\Big(L_7(|t-\bar{t}| + |x_t - x_{\bar{t}}|_C)|z_t|_C \\
&+L_7 \max\{|z(t+\zeta) - z(t+\bar{\zeta})| \colon \zeta, \bar{\zeta} \in [-r, -r_0],\ |\zeta - \bar{\zeta}| \le L_8|t-\bar{t}|\} \\
&+L_6 \max_{\zeta\in[-r,-r_0]} |z(t+\zeta) - z(\bar{t}+\zeta)| + L_7(|t-\bar{t}| + |x_t - x_{\bar{t}}|_C)|h^\chi|_X\Big) \\
&+L_3|z(v(t)) - z(v(\bar{t}))| + L_4\Big(|t-\bar{t}| + |x_t - x_{\bar{t}}|_C + |x(v(t)) - x(v(\bar{t}))|\Big)|h^\lambda|_\Lambda.
\end{aligned}$$

Let

$$w(t,\varepsilon) := \max\{|z(s) - z(\bar{s})| \colon s, \bar{s} \in [-r, t],\ |s - \bar{s}| \le \varepsilon\}, \quad t \in [0,\alpha],\ \varepsilon \in [0,\infty).$$

Note that $w(t_1, \varepsilon_1) \le w(t_2, \varepsilon_2)$ for $0 \le t_1 \le t_2 \le \alpha$ and $0 \le \varepsilon_1 \le \varepsilon_2$. Then using (3.7), (4.12), (4.15), (4.16) and the definition of w we get for $0 \le \bar{t} \le t \le \alpha$

$$\begin{aligned}
&|G(t,x)(z_t, h^\lambda, h^\chi) - G(\bar{t},x)(z_{\bar{t}}, h^\lambda, h^\chi)| \\
&\quad \le L_4(1 + N + NL_6(1+N))N_2|t-\bar{t}||h|_\Gamma + L_4 w(t - r_0, L_5|t-\bar{t}|) \\
&\qquad + L_3 w(t - r_0, |t-\bar{t}|) \\
&\qquad + L_4(1 + N + NL_6(1+N))(NL_6(N_2+1) + N_2)|t-\bar{t}||h|_\Gamma \\
&\qquad + L_3\Omega_{\dot{x}}\Big(L_6(1+N)|t-\bar{t}|\Big)L_6(N_2+1)|h|_\Gamma + L_3N\Big(L_7(1+N)N_2|t-\bar{t}||h|_\Gamma \\
&\qquad + L_7 w(t - r_0, L_8|t-\bar{t}|) + L_6 w(t - r_0, |t-\bar{t}|) + L_7(1+N)|t-\bar{t}||h|_\Gamma\Big) \\
&\qquad + L_3 w(t - r_0, L_6(1+N)|t-\bar{t}|) + L_4(1 + N + NL_6(1+N))|t-\bar{t}||h|_\Gamma \\
&\quad \le a^0(|t-\bar{t}|)|h|_\Gamma + K_{11} w(t - r_0, K_{12}|t-\bar{t}|), \qquad (4.18)
\end{aligned}$$

where $a^0(u) := K_8 u + K_9\Omega_{\dot{x}}(K_{10}u)$ with appropriate nonnegative constants K_8, K_9, K_{10}, K_{11}, and $K_{12} := \max\{1, L_5, L_8, L_6(1+N)\}$.

Integrating (4.9) from $\bar{t}$ to t we get

$$z(t) - z(\bar{t}) = G(t,x)(z_t, h^\lambda, h^\chi) - G(\bar{t},x)(z_{\bar{t}}, h^\lambda, h^\chi) + \int_{\bar{t}}^t L(s,x)(z_s, h^\xi, h^\theta)\, ds.$$

Hence (4.10), (4.12) and (4.18) yield for $0 \le \bar{t} \le t \le \alpha$

$$|z(t) - z(\bar{t})| \le a^1(|t-\bar{t}|)|h|_\Gamma + K_{11} w(t - r_0, K_{12}|t-\bar{t}|) \qquad (4.19)$$

with $a^1(u) := a^0(u) + N_0(N_2+1)u$.

Let $m := [\alpha/r_0]$ (here $[\cdot]$ denotes the greatest integer part), and $t_j := jr_0$, $j = 0, 1, \ldots, m$, $t_{m+1} := \alpha$. First suppose $t, \bar{t} \in [t_0, t_1]$. Then $|\dot{h}^\varphi|_{L^\infty} \le |h^\varphi|_{W^{1,\infty}} \le |h|_\Gamma$ and Lemma 2.3 yield

$$|z(t) - z(\bar{t})| = |h^\varphi(t) - h^\varphi(\bar{t})| \le |t - \bar{t}||h|_\Gamma, \qquad t, \bar{t} \in [-r, 0].$$

Therefore (4.19) and the definition of w imply for $t, \bar{t} \in [t_0, t_1]$

$$|z(t) - z(\bar{t})| \le a^1(|t - \bar{t}|)|h|_\Gamma + K_{11} w(t_0, K_{12}|t - \bar{t}|) \le a^1(|t - \bar{t}|)|h|_\Gamma + K_{11}K_{12}|t - \bar{t}||h|_\Gamma.$$

For $-r \le \bar{t} \le t_0 \le t \le t_1$ the above inequalities yield

$$\begin{aligned} |z(t) - z(\bar{t})| &\le |z(t) - z(t_0)| + |z(t_0) - z(\bar{t})| \\ &\le a^1(t)|h|_\Gamma + K_{11}K_{12} t |h|_\Gamma + |\bar{t}||h|_\Gamma \\ &\le a^1(|t - \bar{t}|)|h|_\Gamma + (1 + K_{11}K_{12})|t - \bar{t}||h|_\Gamma. \end{aligned} \tag{4.20}$$

But now it is easy to see that (4.20) holds for all $-r \le \bar{t} \le t \le t_1$, and therefore,

$$w(t_1, \varepsilon) \le a^1(\varepsilon)|h|_\Gamma + (1 + K_{11}K_{12})\varepsilon |h|_\Gamma, \qquad \varepsilon > 0. \tag{4.21}$$

If $t, \bar{t} \in [t_1, t_2]$, then (4.19) and (4.21) yield

$$\begin{aligned} |z(t) - z(\bar{t})| &\le a^1(|t - \bar{t}|)|h|_\Gamma + K_{11} w(t_1, K_{12}|t - \bar{t}|) \\ &\le a^1(|t - \bar{t}|)|h|_\Gamma + K_{11} a^1(K_{12}|t - \bar{t}|)|h|_\Gamma + (K_{11}K_{12} \\ &\quad + K_{11}^2 K_{12}^2)|t - \bar{t}||h|_\Gamma \\ &\le (1 + K_{11}) a^2(|t - \bar{t}|)|h|_\Gamma + (K_{11}K_{12} + K_{11}^2 K_{12}^2)|t - \bar{t}||h|_\Gamma \Big), \end{aligned}$$

where $a^2(u) := a^1(K_{12}u)$. But then for $-r \le \bar{t} \le t_1 \le t \le t_2$ we have

$$\begin{aligned} |z(t) - z(\bar{t})| &\le |z(t) - z(t_1)| + |z(t_1) - z(\bar{t})| \\ &\le (2 + K_{11}) a^2(|t - \bar{t}|)|h|_\Gamma + (1 + 2K_{11}K_{12} + K_{11}^2 K_{12}^2)|t - \bar{t}||h|_\Gamma. \end{aligned}$$

Again, it follows that the last inequality holds for all $t, \bar{t} \in [-r, t_2]$.

Repeating the previous steps for the intervals $[-r, t_j]$ for $j = 2, \ldots, m+1$, we get that

$$|z(t) - z(\bar{t})| \le A(|t - \bar{t}|)|h|_\Gamma$$

for $t, \bar{t} \in [-r, \alpha]$ with an appropriate function A satisfying $A(s) \to 0$ as $s \to 0+$, which proves (4.13). $\square$

We need the following estimates in the proof of the next theorem.

Lemma 4.2 *Assume (A3) (i)–(iv), (A4) (i)–(iv). Suppose* $\bar\gamma = (\bar\varphi, \bar\xi, \bar\theta, \bar\lambda, \bar\chi) \in P \cap \mathcal{M}$, $h_k = (h_k^\varphi, h_k^\xi, h_k^\theta, h_k^\lambda, h_k^\chi) \in \Gamma$ *is such that* $\bar\gamma + h_k \in P$ *for* $k \in \mathbb{N}$, *and* $|h_k|_\Gamma \to 0$ *as* $k \to \infty$. *Let* $x(t) := x(t, \bar\gamma)$, $x^k(t) := x(t, \bar\gamma + h_k)$, $z^k(t) := z(t, \bar\gamma, h_k)$, $v^k(t) := t - \rho(t, x_t^k, \bar\chi + h_k^\chi)$ *and* $v(t) := t - \rho(t, x_t, \bar\chi)$. *Then there exist a nonnegative constant* N_4 *and a nonnegative sequence* $A_k = A_k(\bar\gamma, h_k)$ *such that* $A_k \to 0$ *as* $k \to \infty$, *and for* $k \in \mathbb{N}$

$$\begin{aligned} &|g(t, x_t^k, x^k(v^k(t)), \bar\lambda + h_k^\lambda) - g(t, x_t, x(v(t)), \bar\lambda) - G(t,x)(z_t^k, h_k^\lambda, h_k^\chi)| \\ &\quad\le A_k |h_k|_\Gamma + N_4 \max_{\zeta \in [-r, -r_0]} |x^k(t+\zeta) - x(t+\zeta) - z^k(t+\zeta)|, \quad t \in [0, \alpha]. \end{aligned} \tag{4.22}$$

Proof Let α, M_1 and M_5 be defined by Theorem 3.2, M_6 and M_7 be defined by (4.1), and $L_3, \dots, L_7$ be the corresponding Lipschitz constants from (A3)–(A4). Simple manipulations yield

$$\begin{aligned} &g(t, x_t^k, x^k(v^k(t)), \bar\lambda + h_k^\lambda) - g(t, x_t, x(v(t)), \bar\lambda) - G(t,x)(z_t^k, h_k^\lambda, h_k^\chi) \\ &= g(t, x_t^k, x^k(v^k(t)), \bar\lambda + h_k^\lambda) - g(t, x_t, x(v(t)), \bar\lambda) \\ &\quad - D_2 g(t, x_t, x(v(t)), \bar\lambda)(x_t^k - x_t) + D_2 g(t, x_t, x(v(t)), \bar\lambda)(x_t^k - x_t - z_t^k) \\ &\quad - D_3 g(t, x_t, x(v(t)), \bar\lambda)\Big[x^k(v^k(t)) - x(v(t))\Big] - D_4 g(t, x_t, x(v(t)), \bar\lambda) h_k^\lambda \\ &\quad + D_3 g(t, x_t, x(v(t)), \bar\lambda)\Big[x^k(v^k(t)) - x(v^k(t)) - z^k(v^k(t))\Big] \\ &\quad + D_3 g(t, x_t, x(v(t)), \bar\lambda)\Big[x(v^k(t)) - x(v(t)) - \dot x(v(t))(v^k(t) - v(t))\Big] \\ &\quad + D_3 g(t, x_t, x(v(t)), \bar\lambda)\dot x(v(t))\Big[v^k(t) - v(t) + D_2\rho(t, x_t, \bar\chi)(x_t^k - x_t) \\ &\qquad + D_3\rho(t, x_t, \bar\chi) h_k^\chi\Big] - D_3 g(t, x_t, x(v(t)), \bar\lambda)\dot x(v(t)) D_2\rho(t, x_t, \bar\chi)\Big[x_t^k - x_t - z_t^k\Big] \\ &\quad + D_3 g(t, x_t, x(v(t), \bar\lambda))\Big[z^k(v^k(t)) - z^k(v(t))\Big], \qquad t \in [0, \alpha], \quad k \in \mathbb{N}. \end{aligned} \tag{4.23}$$

Using the definition of ω_g, and applying (A3) (iv), (A4) (ii), (3.7), (3.8) and (4.4) we have

$$\begin{aligned} &|\omega_g(t, x_t, x(v(t)), \bar\lambda, x_t^k, x^k(v^k(t)), \bar\lambda + h_k^\lambda)| \\ &\quad\le L_4\Big(|x_t^k - x_t|_C + |x^k(v^k(t)) - x(v(t))| + |h_k^\lambda|_\Lambda\Big)^2 \\ &\quad\le L_4\Big(|x_t^k - x_t|_C + |x^k(v^k(t)) - x(v^k(t))| + |x(v^k(t)) - x(v(t))| + |h_k^\lambda|_\Lambda\Big)^2 \\ &\quad\le L_4\Big(2|x_t^k - x_t|_C + |\dot x_t|_{L^\infty}|v^k(t) - v(t)| + |h_k^\lambda|_\Lambda\Big)^2 \\ &\quad\le L_4\Big((2 + NL_6)|x_t^k - x_t|_C + NL_6|h_k^\chi|_X + |h_k^\lambda|_\Lambda\Big)^2 \\ &\quad\le L_4\Big((2 + NL_6)L + NL_6 + 1\Big)^2 |h_k|_\Gamma^2, \qquad t \in [0, \alpha], \quad k \in \mathbb{N}. \end{aligned}$$

Assumptions (A4) (iii), (iv) and (3.8) imply

$$\begin{aligned}
&|v^k(t) - v(t) + D_2\rho(t, x_t, \bar\chi)(x_t^k - x_t) + D_3\rho(t, x_t, \bar\chi)h_k^\chi| \\
&\quad\leq\ |x_t^k - x_t|_C \max_{0<\nu<1} |D_2\rho(t, x_t + \nu(x_t^k - x_t), \bar\chi) - D_2\rho(t, x_t, \bar\chi)|_{\mathcal{L}(C,\mathbb{R}^n)} \\
&\qquad +|h_k^\chi|_X \max_{0<\nu<1} |D_3\rho(t, x_t, \bar\chi + \nu h_k^\chi) - D_3\rho(t, x_t, \bar\chi)|_{\mathcal{L}(X,\mathbb{R}^n)} \\
&\quad\leq\ L_7|x_t^k - x_t|_C^2 + L_7|h_k^\chi|_X^2 \\
&\quad\leq\ L_7(L^2 + 1)|h_k|_\Gamma^2, \qquad t \in [0, \alpha], \quad k \in \mathbb{N}.
\end{aligned}$$

Relations (3.8), (4.13) and (A4) (ii) yield

$$\begin{aligned}
|z^k(v^k(t)) - z^k(v(t))| &\leq A\Big(|v^k(t) - v(t)|\Big)|h_k|_\Gamma \\
&\leq A\Big(L_6(|x_t^k - x_t|_C + |h_k^\chi|_X)\Big)|h_k|_\Gamma \\
&\leq A\Big(L_6(L + 1)|h_k|_\Gamma\Big)|h_k|_\Gamma, \qquad t \in [0, \alpha], \quad k \in \mathbb{N}.
\end{aligned}$$

Relations (A4) (ii), (3.8), (4.13) and (4.17) imply

$$\begin{aligned}
&|x(v^k(t)) - x(v(t)) - \dot x(v(t))(v^k(t) - v(t))| \\
&\quad\leq\ |v^k(t) - v(t)| \sup_{0<\nu<1} \{|\dot x(v(t) + \nu(v^k(t) - v(t))) - \dot x(v(t))|\} \\
&\quad\leq\ L_6(L + 1)|h_k|\Omega_{\dot x}\Big(L_6(L + 1)|h_k|_\Gamma\Big), \qquad t \in [0, \alpha], \quad k \in \mathbb{N}.
\end{aligned}$$

Combining the above estimates, $t - r \leq v^k(t) \leq t - r_0$ together with (4.23), we get (4.22) with $A_k := L_3L_6(L + 1)\Omega_{\dot x}\Big(L_6(L + 1)|h_k|\Big) + L_3A\Big(L_6(L + 1)|h_k|_\Gamma\Big) + K_{13}|h_k|_\Gamma$ and with appropriate constants N_4 and K_{13}. □

Lemma 4.3 *Suppose (A1) (i)–(iii), (A2) (i)–(iii), and let* $\bar\gamma = (\bar\varphi, \bar\xi, \bar\theta, \bar\lambda, \bar\chi) \in P \cap \mathcal{M}$, $h_k = (h_k^\varphi, h_k^\xi, h_k^\theta, h_k^\lambda, h_k^\chi) \in \Gamma$ *be such that* $\bar\gamma + h_k \in P$ *for* $k \in \mathbb{N}$ *and* $|h_k|_\Gamma \to 0$ *as* $k \to \infty$. *Let* $x(t) := x(t, \bar\gamma)$, $x^k(t) := x(t, \bar\gamma + h_k)$, $z^k(t) := z(t, \bar\gamma, h_k)$, $u(t) := t - \tau(t, x_t, \bar\xi)$, *and* $u^k(t) := t - \tau(t, x_t^k, \bar\xi + h_k^\xi)$. *Then there exist a nonnegative constant* N_5 *and a nonnegative sequence* $B_k = B_k(\bar\gamma, h_k)$ *such that* $B_k \to 0$ *as* $k \to \infty$, *and*

$$\begin{aligned}
&|f(s, x_s^k, x^k(u^k(s)), \bar\theta + h_k^\theta) - f(s, x_s, x(u(s)), \bar\theta) - L(s, x)(z_s^k, h_k^\xi, h_k^\theta)| \\
&\quad\leq\ B_k|h_k|_\Gamma + N_5|x_s^k - x_s - z_s^k|_C, \qquad t \in [0, \alpha], \quad k \in \mathbb{N}. \qquad (4.24)
\end{aligned}$$

Proof Let α, M_1 and M_2 be defined by Theorem 3.2, M_3 and M_4 be defined by (4.1), and L_1 and L_2 be the corresponding Lipschitz constants from (A1) (ii) and (A4) (ii), respectively.

The definitions of ω_f and ω_τ yield

$$
\begin{aligned}
& f(s,x_s^k,x^k(u^k(s)),\bar\theta+h_k^\theta)-f(s,x_s,x(u(s)),\bar\theta)-L(s,x)(z_s^k,h_k^\xi,h_k^\theta)\\
&= \ \omega_f(s,x_s,x(u(s)),\bar\theta,x_s^k,x^k(u^k(s)),\bar\theta+h_k^\theta)+D_2f(s,x_s,x(u(s)),\bar\theta)\Big[x_s^k-x_s-z_s^k\Big]\\
&\quad +D_3f(s,x_s,x(u(s)),\bar\theta)\Big\{x^k(u^k(s))-x(u^k(s))-z^k(u^k(s))\\
&\qquad +x(u^k(s))-x(u(s))-\dot x(u(s))(u^k(s)-u(s))-\dot x(u(s))\omega_\tau(s,x_s,\bar\xi,x_s^k,\bar\xi+h_k^\xi)\\
&\qquad +\dot x(u(s))D_2\tau(s,x_s,\bar\xi)\Big[x_s^k-x_s-z_s^k\Big]+z^k(u^k(s))-z^k(u(s))\Big\}.
\end{aligned}
$$

Using (3.17) we have that

$$
\begin{aligned}
&|x_s^k-x_s|_C+|x^k(u^k(s))-x(u(s))|+|h_k^\theta|_\Theta\\
&\quad\le\ 2|x_s^k-x_s|_C+L_2N(|x_s^k-x_s|_C+|h_k^\xi|_\Xi)+|h_k^\theta|_\Theta\\
&\quad\le\ K_{14}|h_k|_\Gamma,\qquad s\in[0,\alpha],\quad k\in\mathbb{N},
\end{aligned}
$$

where $K_{14}:=2L+L_2N(L+1)+1$. Hence (4.2) implies

$$|\omega_f(s,x_s,x(u(s)),\bar\theta,x_s^k,x^k(u^k(s)),\bar\theta+h_k^\theta)|\le\Omega_f(K_{14}|h_k|_\Gamma)K_{14}|h_k|_\Gamma$$

for $s\in[0,\alpha]$ and $k\in\mathbb{N}$. Similarly,

$$|\omega_\tau(s,x_s,\bar\xi,x_s^k,\bar\xi+h_k^\xi)|\le\Omega_\tau((L+1)|h_k|_\Gamma)(L+1)|h_k|_\Gamma,\qquad s\in[0,\alpha],\ k\in\mathbb{N}.$$

Using (A2) (ii), (3.8) we get

$$|u^k(s)-u(s)|\le L_2\Big(|x_s^k-x_s|_C+|h_k^\xi|_\Xi\Big)\le L_2(L+1)|h_k|_\Gamma,$$

and therefore the definition of $\Omega_{\dot x}$ and (4.13) yield

$$|x(u^k(s))-x(u(s))-\dot x(u(s))(u^k(s)-u(s))|\le\Omega_{\dot x}\Big(L_2(L+1)|h_k|_\Gamma\Big)L_2(L+1)|h_k|_\Gamma$$

and

$$|z^k(u^k(s))-z^k(u(s))|\le A\Big(|u^k(s)-u(s)|\Big)|h_k|_\Gamma\le A\Big(L_2(L+1)|h_k|_\Gamma\Big)|h_k|_\Gamma$$

for $s\in[0,\alpha]$ and $k\in\mathbb{N}$. Therefore, combining the above estimates we get

$$
\begin{aligned}
& |f(s,x_s^k,x^k(u^k(s)),\bar\theta+h_k^\theta)-f(s,x_s,x(u(s)),\bar\theta)-L(s,x)(z_s^k,h_k^\xi,h_k^\theta)|\\
&\quad\le\ \Omega_f\Big(K_{14}|h_k|_\Gamma\Big)K_{14}|h_k|_\Gamma+L_1|x_s^k-x_s-z_s^k|_C\\
&\qquad +L_1\Big\{|x^k(u^k(s))-x(u^k(s))-z^k(u^k(s))|\\
&\qquad +\Omega_{\dot x}\Big(L_2(L+1)|h_k|_\Gamma\Big)L_2(L+1)|h_k|_\Gamma+N\Omega_\tau\Big((L+1)|h_k|_\Gamma\Big)(L+1)|h_k|_\Gamma\\
&\qquad +NL_2|x_s^k-x_s-z_s^k|_C+A\Big(L_2(L+1)|h_k|_\Gamma\Big)|h_k|_\Gamma\Big\}.
\end{aligned}
$$

Hence (4.24) holds with the sequence $B_k:=\Omega_f(K_{14}|h_k|_\Gamma)K_{14}+L_1\Omega_{\dot x}(L_2(L+1)|h_k|_\Gamma)L_2(L+1)+L_1N\Omega_\tau((L+1)|h_k|_\Gamma)(L+1)+L_1A(L_2(L+1)|h_k|_\Gamma)$ and with the constant $N_5:=L_1(2+NL_2)$. □

Next we study differentiability of the function $x(t,\gamma)$ wrt γ. We denote this differentiation by D_2x.

Theorem 4.4 *Assume (A1) (i)–(iii), (A2) (i)–(iii), (A3) (i)–(iv) and (A4) (i)–(iv), and let P and $\alpha > 0$ be defined by Theorem 3.2, $\bar\gamma \in P \cap \mathcal{M}$, and $x(t;\gamma)$ be the solution of the IVP* (3.1)-(3.2) *on $[-r,\alpha]$ for $\gamma \in \mathcal{B}_\Gamma(\bar\gamma;\delta)$. Then the function $x(t,\cdot) : \Gamma \supset P \to \mathbb{R}^n$ is differentiable at $\bar\gamma$ for $t \in [0,\alpha]$, and*

$$D_2x(t,\bar\gamma)h = z(t,\bar\gamma,h), \qquad h \in \Gamma, \quad t \in [0,\alpha],$$

where z is the solution of the IVP (4.5)-(4.6).

Proof Let $\bar\gamma = (\bar\varphi,\bar\xi,\bar\theta,\bar\lambda,\bar\chi) \in P$ be fixed, α, δ, M_1, M_2 and M_5 be defined by Theorem 3.2, M_3, M_4, M_6 and M_7 be defined by (4.1). Let $h_k = (h_k^\varphi, h_k^\xi, h_k^\theta, h_k^\lambda, h_k^\chi) \in \Gamma$ be a sequence such that $|h_k|_\Gamma \to 0$ as $k \to \infty$. We may assume that $|h_k|_\Gamma \le \delta$, hence $\bar\gamma + h_k \in P$ for $k \in \mathbb{N}$. For brevity, we use the notations $x(t) := x(t,\bar\gamma)$, $x^k(t) := x(t,\bar\gamma+h_k)$, $z^k(t) := z(t,\bar\gamma,h_k)$, $u(t) := t-\tau(t,x_t,\bar\xi)$, $u^k(t) := t-\tau(t,x_t^k,\bar\xi+h_k^\xi)$, $v(t) := t-\rho(t,x_t,\bar\chi)$ and $v^k(t) := t-\rho(t,x_t^k,\bar\chi+h_k^\chi)$.

Integrating (3.1) and (4.5) we get for $t \in [0,\alpha]$

$$\begin{aligned}
x^k(t) &= g\Big(t, x_t^k, x^k(v^k(t)), \bar\lambda + h_k^\lambda\Big) + \bar\varphi(0) + h_k^\varphi(0)\\
&\quad - g\Big(0, \bar\varphi + h_k^\varphi, \bar\varphi(v^k(0)) + h_k^\varphi(v^k(0)), \bar\lambda + h_k^\lambda\Big)\\
&\quad + \int_0^t f(s, x_s^k, x^k(u^k(s)), \bar\theta + h_k^\theta)\,ds,\\
x(t) &= g(t, x_t, x(v(t)), \bar\lambda) + \bar\varphi(0) - g(0,\bar\varphi,\bar\varphi(v(0)),\bar\lambda) + \int_0^t f(s, x_s, x(u(s)), \bar\theta)\,ds,\\
z^k(t) &= G(t,x)(z_t^k, h_k^\lambda, h_k^\chi) + h_k^\varphi(0) - G(0,x)(h_k^\varphi, h_k^\lambda, h_k^\chi)\\
&\quad + \int_0^t L(s,x)(z_s^k, h_k^\xi, h_k^\theta)\,ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&x^k(t) - x(t) - z^k(t)\\
&\quad = g(t, x_t^k, x^k(v^k(t)), \bar\lambda + h_k^\lambda) - g(t, x_t, x(v(t)), \bar\lambda) - G(t,x)(z_t^k, h_k^\lambda, h_k^\chi)\\
&\qquad - \Big[g\Big(0, \bar\varphi + h_k^\varphi, \bar\varphi(v^k(0)) + h_k^\varphi(v^k(0)), \bar\lambda + h_k^\lambda\Big) - g(0,\bar\varphi,\bar\varphi(-v(0)),\bar\lambda)\\
&\qquad\qquad - G(0,x)(h_k^\varphi, h_k^\lambda, h_k^\chi)\Big]\\
&\qquad + \int_0^t \Big[f(s, x_s^k, x^k(u^k(s)), \bar\theta + h_k^\theta) - f(s, x_s, x(u(s)), \bar\theta)\\
&\qquad\qquad - L(s,x)(z_s^k, h_k^\xi, h_k^\theta)\Big]\,ds.
\end{aligned}$$

Define the function $w^k(t) := x^k(t) - x(t) - z^k(t)$. Then Lemmas 4.2 and 4.3 yield for $t \in [0, \alpha]$

$$|w^k(t)| \;\leq\; C_k|h_k|_\Gamma + N_4 \max_{\zeta\in[-r,-r_0]} |w^k(t+\zeta)| + N_5 \int_0^t |w_s^k|_C \, ds, \tag{4.25}$$

where $C_k := 2A_k + B_k\alpha \to 0$ as $k \to \infty$. Let $\mu^k(t) := \max\{|w^k(s)| \colon -r \leq s \leq t\}$. We have $w^k(t) = 0$ for $t \in [-r, 0]$. Therefore Lemma 2.1 implies from (4.25) that

$$\mu^k(t) \;\leq\; C_k|h_k|_\Gamma + N_4\mu^k(t - r_0) + N_5 \int_0^t \mu^k(s)\, ds, \quad t \in [0, \alpha]. \tag{4.26}$$

Therefore Lemma 2.2 and $\mu^k(t) = 0$ for $t \in [-r, 0]$ yield

$$|x^k(t) - x(t) - z(t)| \leq \mu^k(t) \leq \frac{C_k}{1 - N_4 e^{-cr_0}} e^{c\alpha} |h_k|_\Gamma, \qquad t \in [0, \alpha], \tag{4.27}$$

where c is the unique positive solution of $cN_4e^{-cr_0} + N_5 = c$. Hence the claim of the theorem follows, since $C_k \to 0$ as $k \to \infty$.

The proof of the theorem is complete. □

The proof immediately implies differentiability of the parameter map in the C-norm:

Corollary 4.5 *Assume the conditions of Theorem 4.4. Then the function*

$$\Gamma \supset P \to C, \qquad \gamma \mapsto x(\cdot, \gamma)_t$$

is differentiable at $\bar\gamma \in P \cap \mathcal{M}$ *for* $t \in [0, \alpha]$, *and its derivative is given by*

$$D_2 x_t(\cdot, \bar\gamma)h = z_t(\cdot, \bar\gamma, h), \qquad h \in \Gamma, \quad t \in [0, \alpha].$$

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