Parameter Identification in Classes of Hereditary Systems of Neutral Type

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ABSTRACT

In this paper we prove theoretical convergence for a variety of parameter identification schemes, based on approximations by equations with piecewise constant arguments, for classes of neutral differential equations.

1. INTRODUCTION AND PROBLEM STATEMENT

Identification of unknown parameters in various classes of differential equations, and in particular in delay differential equations, has been studied by many authors (see e.g. [1], [2], [3] and [13] and the references therein). All of these papers follow the same "general method", which we briefly describe below.

Consider e.g., the initial value problem (IVP) for the nonlinear delay system with time-dependent delays

$$\dot{x}(t) = f\Big(t, x(t), x(t-\sigma(t))\Big), \qquad t \geq 0 \tag{1}$$

with initial condition

$$x(t) = \varphi(t), \qquad t \in [-r, 0], \tag{2}$$

containing "unknown" parameters, γ. In the initial value problem (IVP) defined by (1)-(2), the parameters, γ, are not known explicitly, but some

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information is available via measurements $(X_0, X_1, \ldots, X_l)$ of the solution, $x(t)$, at discrete time values $(t_0, t_1, \ldots, t_l)$. The goal is to find the parameter value, which minimizes the least squares fit-to-data criterion $J(\gamma) = \sum_{i=0}^{l} |x(t_i;\gamma) - X_i|^2$, where γ belongs to an admissible set Δ contained in the parameter space Γ. (Denote this problem by $\mathcal{P}$). The general method consists of the following steps:

Step 1) First take finite dimensional approximations of the parameters, γ^N, (i.e., $\gamma^N \in \Delta^N \subset \Gamma^N \subset \Gamma$, $\dim \Gamma^N < \infty$, $\gamma^N \to \gamma$ as $N \to \infty$).

Step 2) Consider a sequence of approximate initial value problems corresponding to a discretization of IVP (1)-(2) for some fixed parameter $\gamma^N \in \Gamma^N$ with solutions $y^M(\cdot;\gamma^N)$ satisfying $y^M(t,\gamma^N) \to x(t,\gamma)$ as $N, M \to \infty$, uniformly on compact time intervals, and $\gamma^N \in \Delta^N$.

Step 3) Define the least square minimization problems ($\mathcal{P}^{N,M}$): for each $N, M = 1, 2, \ldots$, i.e., find $\gamma^{N,M} \in \Delta^N \subset \Gamma^N$, which minimizes the least squares fit-to-data criterion $J^{N,M}(\gamma^N) = \sum_{i=0}^{l} |y^M(t_i;\gamma^N) - X_i|^2$, $\gamma^N \in \Delta^N$. Often Δ^N is the projection of Δ to Γ^N, and we restrict our discussion to this case.

Step 4) Assuming that Δ is a compact subset of Γ, argue, that the sequence of solutions, $\gamma^{N,M}$ $(N, M = 1, 2, \ldots)$, of the finite dimensional minimization problems $\mathcal{P}^{N,M}$, has a convergent subsequence with limit $\bar{\gamma} \in \Gamma$.

Step 5) Show that $\bar{\gamma}$ is the solution of the minimization problem $\mathcal{P}$.

Note that Steps 4 and 5 can be argued without using the particular approximation method, using only a compactness argument and Step 2 above (see [13]).

In this paper we apply this general framework for identifying parameters in IVPs corresponding to neutral functional differential equations (NFDEs) of the form

$$\frac{d}{dt}\Big(x(t) + q(t)x(t-\tau(t))\Big) = f\Big(t, x(t), x(t-\sigma(t))\Big), \qquad t \in [0,T], \quad (3)$$

and initial condition (2).

Our goal is the identification of q, τ in (3), and the associated initial function, φ. Note that other parameters in the right hand side of (3) can be identified in a similar way, but for simplicity in the discussion we restrict our attention to these three parameters.

Assuming that the parameters q, τ and φ are continuous functions, we use $\Gamma \equiv C([0,T];\mathbb{R}) \times C([0,T];\mathbb{R}) \times C([-r,0];\mathbb{R}^n)$ as our parameter space. To follow the general identification method described above, first we approximate q, τ and φ (in supremum norm) by (finite dimensional) functions q^N, τ^N and φ^N. In practice (see [9] and [10]) we use linear spline approximations (see e.g. [14]).

In Section 2 we define several Euler-type approximation schemes (see (7)-(8), (7)-(26) and (28)-(8) below) for NFDEs of the form (3), using equations with piecewise constant arguments (EPCAs), and show that each scheme satisfies the convergence property required in Step 2 of the general method. For corresponding numerical studies we refer to [9] and [10], where several examples illustrate the applicability of the identification method defined in this paper. See also [4] and [11] for identification methods based on Euler-type approximation schemes, and [12] and the references therein for numerical approximation methods for NFDEs.

Note that EPCAs were used first in [5] to obtain numerical approximation schemes and to prove the convergence of the approximation method for linear delay and neutral differential equations with constant delays, and later in [6] for nonlinear delay equations with state-dependent delays. An EPCA-based identification scheme was studied in [8] for a class of delay equations with state-dependent delays.

2. CONVERGENCE RESULTS

Consider the vector NFDE

$$\frac{d}{dt}\Big(x(t)+q(t)x(t-\tau(t))\Big)=f\Big(t,x(t),x(t-\sigma(t))\Big), \qquad t\in[0,T] \tag{4}$$

with initial condition

$$x(t)=\varphi(t), \qquad t\in[-r,0], \tag{5}$$

We make the following assumptions:

(H1) $T>0$, $q\in C([0,T];\mathbb{R})$, $\sigma\in C([0,T];[0,\infty))$, $\varphi\in C([-r,0];\mathbb{R}^n)$,

(H2) $f\in C([0,T]\times\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$ is locally Lipschitz-continuous in its second and third arguments, i.e., for every $M>0$ there exists $L=L(M)\geq 0$ such that $|f(t,x,y)-f(t,\bar{x},\bar{y})|\leq L\Big(|x-\bar{x}|+|y-\bar{y}|\Big)$, for $t\in[0,T]$, $x,\bar{x},y,\bar{y}\in\mathbb{R}^n$, $|x|$, $|\bar{x}|$, $|y|$, $|\bar{y}|\leq M$,

(H3) $\tau\in C([0,T];\mathbb{R})$ is such that $0<\tau(t)$ for $t\in[0,T]$,

(H4) r is a positive constant satisfying $\tau(t)\leq r$ and $\sigma(t)\leq r$ for $t\in[0,T]$.

Here, and throughout this paper $|\cdot|$ denotes a vector norm on $\mathbb{R}^n$, and $|q|_C$, $|\tau|_C$ and $|\varphi|_C$ denote the respective supremum norms on $C([0,T];\mathbb{R})$, $C([0,T];\mathbb{R})$ and $C([-r,0];\mathbb{R}^n)$.

Under assumptions (H1)–(H4) IVP (4)-(5) has a unique solution (see [7]).

Throughout this paper we shall use the notation $[t]_h \equiv [t/h]h$, where $[\cdot]$ is the greatest integer function. For later reference we mention an elementary property of this function:

$$t - h < [t]_h \le t. \tag{6}$$

Let h be a positive number, q^N, τ^N and φ^N are continuous functions (approximating q, τ and φ as $N \to \infty$), and $0 < \tau^N(t) \le r$ for $t \in [0,T]$. Following the ideas of [5], we associate the following NFDE with piecewise constant arguments to (4) and to the parameters q^N, τ^N and φ^N:

$$\begin{aligned}\frac{d}{dt}\Big(y_{h,N}(t) + q^N([t]_h)y_{h,N}(t - [\tau^N([t]_h)]_h)\Big)& \\ = f\Big([t]_h, y_{h,N}([t]_h), y_{h,N}([t]_h - [\sigma([t]_h)]_h)\Big)&\end{aligned} \tag{7}$$

for $t \in [0,T]$, with the initial condition

$$y_{h,N}(t) = \varphi^N(t), \qquad t \in [-r, 0]. \tag{8}$$

The subscript h and N of $y_{h,N}(t)$ emphasizes that $y_{h,N}(t)$ is the solution of (7) corresponding to the discretization parameter h and parameter values q^N, τ^N and φ^N. By a solution of IVP (7)-(8) we mean a function $y_{h,N} : [-r,T] \to \mathbb{R}^n$, which is defined on $[-r,0]$ by (8), such that the function $t \mapsto y_{h,N}(t) + q^N([t]_h)y_{h,N}(t - [\tau^N([t]_h)]_h)$ is continuous on $[0,T]$, and its derivative exists at each point $t \in [0,T)$, with the possible exception of the points kh $(k = 0,1,2,\ldots)$ where finite one-sided derivatives exist, and the function $y_{h,N}$ satisfies (7) on each interval $[kh,(k+1)h) \cap [0,T]$ $(k = 0,1,2,\ldots)$. Note that $y_{h,N}(t)$ is, in general, only right-continuous at positive mesh points.

It is easy to see (by using the method of steps) that IVP (7)-(8) has a unique solution defined on $[0,T]$. Introduce the notations $a(k) \equiv y_h(kh)$ and $b(k) \equiv \lim_{s\to kh-} y_{h,N}(t)$ for the value of the solution and its left-sided limit at mesh points, respectively. Integrating (7) from kh to t and then taking the limit as $t \to (k+1)h^+$ yields the recursive formula:

$$\begin{aligned} a(k+1) &= a(k) + q^N(kh)a(k - [\tau^N(kh)/h]) \\ &\quad - q^N((k+1)h)a(k+1-[\tau^N((k+1)h)/h]) \qquad (9)\\ &\quad + hf(kh, a(k), a(k - [\sigma(kh)/h])), \quad \text{for } k = 0,1,\ldots, \\ a(k) &= \varphi^N(kh), \qquad \text{for } k = 0,-1,\ldots. \quad -r \le kh \le 0. \qquad (10)\end{aligned}$$

The continuity of $y_{h,N}(t) + q^N([t]_h)y_{h,N}(t - [\tau^N([t]_h)]_h)$ and φ imply

$$\begin{aligned} b(k+1) &= a(k+1) - q^N(kh)b\big(k+1-\big[\tau^N(kh)/h\big]\big) \qquad (11)\\ &\quad + q^N((k+1)h)a\big(k+1-\big[\tau^N((k+1)h)/h\big]\big)\,,\ k = 0,1,\dots,\\ b(k) &= \varphi^N(kh), \qquad \text{for } k = 0,-1,\dots, \quad -r \le kh \le 0. \qquad (12) \end{aligned}$$

Note that if $q(t)$ and $\tau(t)$ are constant functions, then $a(k) = b(k)$, i.e., $y_{h,N}(t)$ is continuous at mesh points.

These formulas show that the computation of the solution of IVP (7)-(8) at mesh points is an easy numerical task, moreover the recursive formulas use values of the solution and the initial function only at mesh points. Since the right hand side of (7) is piecewise constant, the function $y_{h,N}(t)+q^N([t]_h)y_{h,N}(t-[\tau^N([t]_h)]_h)$ is piecewise linear, but unfortunately $y_{h,N}(t)$, in general, is not linear between mesh points, therefore computing the solution of IVP (7)-(8) between mesh points is numerically difficult. Consequently, this scheme is recommended for use here if we need the solution only at mesh points, i.e., if the measurements are taken at mesh points. Note that in many cases this is not a restrictive assumption.

Next we state a slightly generalized version of Lemma 3.2 from [5].

Lemma 1. *Let $a > 0$, $b \ge 0$, $\alpha \ge 0$, $\beta > 0$, $\gamma \equiv \max\{\alpha,\beta\}$, and $g : [0,T] \to [0,\infty)$ be continuous and nondecreasing. Let $u : [-\gamma,T] \to [0,\infty)$ be continuous except at finite many points $0 < t_1 < t_2 < \ldots < t_m \le T$, where finite one-sided limits exist, and satisfy the inequality*

$$u(t) \le g(t) + bu(t-\beta) + a\int_0^t u(s-\alpha)\,ds, \qquad t \in [0,T].$$

Then $u(t) \le d(t)e^{ct}$ for $t \in [0,T]$, where c is the unique positive solution of $cbe^{-c\beta} + ae^{-c\alpha} = c$, and

$$d(t) \equiv \max\left\{\frac{g(t)}{1-be^{-c\beta}},\ \max_{-\gamma\le s\le 0} e^{-cs}u(s)\right\}, \qquad t \in [0,T].$$

Note that in [5] this result was formulated for the case when $u(t)$ is continuous. The proof for this case is an obvious modification of that of the continuous case, and therefore omitted here.

The next theorem shows that the approximation scheme (7)-(8) has the uniform convergence property under a double limiting process, as required in Step 2 of the general identification method.

Theorem 1. *Assume (H1)–(H4) and let $|q^N - q|_C \to 0$, $|\tau^N - \tau|_C \to 0$ and $|\varphi^N \to \varphi|_C \to 0$ as $N \to \infty$. Then the solution, $y_{h,N}$, of IVP (7)-(8)*

converges uniformly on $[0,T]$ *to the solution,* x*, of IVP (4)-(5) as* $h \to 0^+$ *and* $N \to \infty$*, i.e.,* $\lim\limits_{\substack{h\to 0+\\ N\to\infty}} \max\limits_{0\le t\le T} |x(t) - y_{h,N}(t)| = 0.$

Proof. Define the constants $M \equiv \max\Big\{\max_{-r\le t\le T} |x(t)|, |q|_C\Big\} + 1$, and $\bar\tau \equiv \min_{0\le t\le T} \tau(t)$. (Note that $\bar\tau > 0$.) Without loss of generality we can assume that $|q^N - q|_C < 1$, $|\varphi^N - \varphi|_C < 1$, and $|\tau^N - \tau|_C < \bar\tau/2$ for all N. Let $0 < T_{h,N} \le T$ be the largest number such that $|y_{h,N}(t)| \le M$ for $t \in [0, T_{h,N})$. ($T_{h,N}$ is well-defined since $|\varphi^N(0)| < M$ by our assumptions.) Integrating (4) and (7) from 0 to t and using the respective initial conditions we get

$$x(t)+q(t)x(t-\tau(t)) = \varphi(0)+q(0)\varphi(-\tau(0))+\int_0^t f\Big(s,x(s),x(s-\sigma(s))\Big)\,ds, \tag{13}$$

and

$$\begin{aligned} y_{h,N}(t) + q^N([t]_h)y_{h,N}(t-[\tau^N([t]_h)]_h) &= \varphi^N(0) + q^N(0)\varphi^N(-[\tau^N(0)]_h) \\ &+ \int_0^t f\Big([s]_h, y_{h,N}([s]_h), y_{h,N}([s]_h - [\sigma([s]_h)]_h)\Big)\,ds. \end{aligned} \tag{14}$$

Using elementary estimates and (H2), with $L = L(M)$, (13) and (14) imply for $t \in [0, T_{h,N}]$,

$$\begin{aligned} &|x(t) - y_{h,N}(t)| \\ &\quad\le\ |\varphi(0) - \varphi^N(0)| + |q^N(0)||\varphi(-\tau(0)) - \varphi^N(-[\tau^N(0)]_h)| \\ &\qquad + |q(0) - q^N(0)||\varphi(-\tau(0))| + |q(t) - q^N([t]_h)||x(t-\tau(t))| \\ &\qquad + |q^N([t]_h)|x(t-\tau(t)) - y_{h,N}(t-[\tau^N([t]_h)]_h)| \\ &\qquad + L\int_0^t \Big(|s-[s]_h| + |x(s) - y_{h,N}([s]_h)|\Big)ds \\ &\qquad + L\int_0^t |x(s-\sigma(s)) - y_{h,N}([s]_h - [\sigma([s]_h)]_h)|\,ds. \end{aligned} \tag{15}$$

We introduce the following notations for $u \ge 0$:

$$\begin{aligned} \omega_x(u) &\equiv \sup\{|x(s_1) - x(s_2)| \ :\ |s_1 - s_2| \le u, \quad s_1, s_2 \in [-r, T]\}, \\ \omega_\tau(u) &\equiv \sup\{|\tau(s_1) - \tau(s_2)| \ :\ |s_1 - s_2| \le u, \quad s_1, s_2 \in [0, T]\}, \\ \omega_q(u) &\equiv \sup\{|q(s_1) - q(s_2)| \ :\ |s_1 - s_2| \le u, \quad s_1, s_2 \in [0, T]\}. \end{aligned}$$

Using these notations and (6) we have the following estimates for $t \in [0, T]$:

$$|q(t) - q^N([t]_h)| \ \le\ \omega_q(h) + |q - q^N|_C, \tag{16}$$

and

$$|\tau(t) - [\tau^N([t]_h)]_h| \quad \le \quad \omega_\tau(h) + |\tau - \tau^N|_C + h. \tag{17}$$

Similarly, the definition of ω_x and inequalities (6) and (17) imply

$$|\varphi(-\tau(0)) - \varphi^N(-[\tau^N(0)]_h)| \le \omega_x\Big(\omega_\tau(h) + |\tau - \tau^N|_C + h\Big) + |\varphi - \varphi^N|_C, \tag{18}$$

and

$$\begin{aligned} |x(t-\tau(t)) - y_{h,N}(t - [\tau^N([t]_h)]_h)| \;&\le\; \omega_x\Big(\omega_\tau(h) + |\tau - \tau^N|_C + h\Big) \\ &+ |x(t - [\tau^N([t]_h)]_h) - y_{h,N}(t - [\tau^N([t]_h)]_h)|. \end{aligned} \tag{19}$$

Inequality (15), together with relations (6), (16) – (19), and definition of M yield for $t \in [0, T_{h,N}]$:

$$\begin{aligned} &|x(t) - y_{h,N}(t)| \\ &\quad \le \;(M+1)|\varphi - \varphi^N|_C + 2M|q - q^N|_C + M\omega_q(h) \\ &\qquad + 2M\omega_x\Big(\omega_\tau(h) + |\tau - \tau^N|_C + h\Big) \\ &\qquad + M|x(t - [\tau^N([t]_h)]_h) - y_{h,N}(t - [\tau^N([t]_h)]_h)| + LTh \\ &\qquad + L\int_0^t \Big(|x(s) - x([s]_h)| + |x([s]_h) - y_{h,N}([s]_h)|\Big)\,ds \\ &\qquad + L\int_0^t |x(s - \sigma(s)) - x([s]_h - [\sigma([s]_h)]_h)|\,ds \\ &\qquad + L\int_0^t |x([s]_h - [\sigma([s]_h)]_h) - y_{h,N}([s]_h - [\sigma([s]_h)]_h)|\,ds. \end{aligned} \tag{20}$$

Define the functions $z_{h,N}(t) \equiv \max_{-r\le s\le t}|x(s) - y_{h,N}(s)|$ and $g_{h,N}(t) \equiv (M+1)|\varphi - \varphi^N|_C + 2M|q - q^N|_C + M\omega_q(h) + 2M\omega_x\Big(\omega_\tau(h) + |\tau - \tau^N|_C + h\Big) + LTh + L\int_0^t\Big(|x(s) - x([s]_h)| + |x(s-\sigma(s)) - x([s]_h - [\sigma([s]_h)]_h)|\Big)ds$ for $t \in [-r, T]$ and $t \in [0, T]$, respectively. Using this notation (20) implies

$$|x(t) - y_{h,N}(t)| \le g_{h,N}(t) + Mz_{h,N}(t - [\tau^N([t]_h)]_h) + 2L\int_0^t z_{h,N}(s)\,ds \tag{21}$$

for $t \in [0, T_{h,N}]$. Assumption (H3), the assumed inequality $|\tau - \tau^N|_C \le \bar\tau/2$, and relation (6) yield

$$t - [\tau^N([t]_h)]_h \le t - \tau^N([t]_h) + h \le t - \bar\tau/2 + h. \tag{22}$$

Combining (21), (22) and the inequality $z_{h,N}(t) \le |\varphi - \varphi^N|_C$ for $t \in [-r, 0]$, we get

$$z_{h,N}(t) \le g_{h,N}(t) + M z_{h,N}(t - \bar{\tau}/2 + h) + 2L \int_0^t z_{h,N}(s)\, ds, \qquad t \in [0, T_{h,N}].$$

Applying Lemma 1 (for $h < \bar{\tau}/2$), we obtain the estimate

$$z_{h,N}(t) \quad \le \quad d_{h,N}(T) e^{c_h T}, \qquad t \in [0, T_{h,N}], \tag{23}$$

where $c_h > 0$ is the unique positive solution of $c_h M e^{-c_h(\bar{\tau}/2 - h)} + 2L = c_h$, and

$$d_{h,N}(t) \equiv \max \left\{ \frac{g_{h,N}(t)}{1 - M e^{-c_h(\bar{\tau}/2 - h)}},\ \max_{-r \le s \le 0} e^{-c_h s} z_{h,N}(s) \right\}, \qquad t \in [0, T].$$

Relation (23) establishes the theorem if we show that (i) c_h converges to a limit, $c > 0$, as $h \to 0^+$, and (ii) $\lim_{\substack{h \to 0+ \\ N \to \infty}} d_{h,N}(T) = 0$. This follows since (23) now yields that $|x(t) - y_{h,N}(t)| < 1$, and hence $|y_{h,N}(t)| < M$ for $t \in [0, T_{h,N}]$ and for small enough $h > 0$ and large enough N. Consequently this implies that $T_{h,N} = T$ for such h and N. It is easy to see that the definition of c_h yields $c_h \to c$, as $h \to 0^+$, where $c > 0$ is the unique positive solution of $cMe^{-c\bar{\tau}/2} + 2L = c$. Using the convergence of c_h, (ii) follows if we show that

$$\lim_{\substack{h \to 0+ \\ N \to \infty}} g_{h,N}(T) = 0 \tag{24}$$

and

$$\lim_{\substack{h \to 0+ \\ N \to \infty}} \max_{-r \le s \le 0} z_{h,N}(s) = 0. \tag{25}$$

The definition of $g_{h,N}(t)$ implies (24), since x, q and φ are uniformly continuous functions on $[-r, T]$, $[0, T]$ and $[-r, 0]$, respectively. Therefore, $\omega_x(h) \to 0$, $\omega_q(h) \to 0$ and $\omega_\tau(h) \to 0$ as $h \to 0^+$, and (6) implies that $[s]_h \to s$, $[\sigma([s]_h)]_h \to \sigma(s)$, as $h \to 0^+$. The Lebesgue's Dominated Convergence Theorem implies that the integral in $g_{h,N}(T)$ converges to zero as $h \to 0^+$. Finally, the inequality $0 \le z_{h,N}(s) \le |\varphi - \varphi^N|_C$, $s \in [-r, 0]$ yields (25), which completes the proof of the theorem. ∎

The difficulty in computing the solution of the IVP (7)-(8) is a consequence of the fact, that $y_{h,N}(t) + q^N([t]_h) y_{h,N}(t - [\tau^N([t]_h)]_h)$ is piecewise linear but the initial function, φ^N is not necessarily linear between mesh points. However, if we use linear interpolation for values of the initial function between mesh points, then the linearity of the initial function will be preserved for $y_{h,N}(t)$ for positive t. Let $\psi\ :\ [r,0]\to\mathbb{R}^n$ be a continuous function, $0<h<r$, and define

$$(\vartheta_h\psi)(s)\equiv\begin{cases}\psi(0), & s=0,\\ \psi([s]_h)\frac{[s]_h+h-s}{h}+\psi([s]_h+h)\frac{s-[s]_h}{h}, & s\in[[s]_h,[s]_h+h),\\ \psi(-r)\frac{-[r]_h-s}{r-[r]_h}+\psi(-[r]_h)\frac{s+r}{r-[r]_h}, & s\in[-r,-[r]_h),\end{cases}$$

i.e., $\vartheta_h\psi$ denotes the linear interpolate of the function ψ, using mesh points $-r,-[r]_h,-[r]_h+h,\ldots,-h,0$. We consider the modified initial condition

$$y_{h,N}(t)=(\vartheta_h\varphi^N)(t),\qquad t\in[-r,0]. \tag{26}$$

Clearly, IVP (7)-(26) has a unique solution, where the values of the solution at mesh points and its left-sided limits, $a(k)$ and $b(k)$, are defined by the recursive formulas (9)-(12), (i.e., they are identical to those of IVP (7)-(8)), and the solution, $y_{h,N}$, is linear on the intervals $[kh,(k+1)h)$. Note that the values of the initial function between mesh points are actually not used in computing the sequences $a(k)$ and $b(k)$. If approximate solution values between mesh points are not needed, then it is enough to generate the sequence $a(k)$, which can be done without generating $b(k)$. The following theorem shows that this modified scheme has the same uniform convergence property as scheme (7)-(8).

THEOREM 2. *Assume (H1)–(H4) and let* $|q^N-q|_C\to0$, $|\tau^N-\tau|_C\to0$ *and* $|\varphi^N\to\varphi|_C\to0$ *as* $N\to\infty$. *Then the solution,* $y_{h,N}$, *of IVP (7)-(26) converges uniformly on* $[0,T]$ *to the solution,* x, *of IVP (4)-(5) as* $h\to0^+$ *and* $N\to\infty$, *i.e.,* $\lim\limits_{\substack{h\to0+\\ N\to\infty}}\max\limits_{0\le t\le T}|x(t)-y_{h,N}(t)|=0$.

PROOF. Following the steps of the proof of Theorem 1, one can obtain an inequality almost identical to (15) (just replacing φ^N by $\vartheta_h\varphi^N$). In fact, it is identical to (15), since the arguments of $\vartheta_h\varphi^N$ are mesh points, hence $\vartheta_h\varphi^N$ can be replaced by φ^N. Therefore the proof of Theorem 1 can be used to complete the proof of this result. The only statement that needs a different proof here, (because of the different initial condition) is (25). Let $t\in[-[r]_h,0)$ (the case $t\in[-r,-[r]_h)$ can be proved similarly). The definitions of ϑ_h and ω_φ, and elementary manipulations yield

$$\begin{aligned}&z_{h,N}(t)\\ &\le\max_{-r\le s\le t}\Big\{\Big|\varphi(s)-\varphi([s]_h)\Big|\frac{[s]_h+h-s}{h}+\Big|\varphi(s)-\varphi([s]_h+h)\Big|\frac{s-[s]_h}{h}\\ &\qquad+\Big|\varphi([s]_h)-\varphi^N([s]_h)\Big|\frac{[s]_h+h-s}{h}\end{aligned}$$

$$
\begin{aligned}
& \quad + \left|\varphi([s]_h + h) - \varphi^N([s]_h + h)\right| \frac{s - [s]_h}{h} \Bigg\} \\
\le & \quad 2\omega_\varphi(h) + 2|\varphi - \varphi^N|_C, \qquad (27)
\end{aligned}
$$

which implies (25). ∎

As pointed out in [10], scheme (7)-(8), (and also (7)-(26)) is not appropriate in practice for identifying the delay function τ since the solution, $y_{h,N}$, of the corresponding approximate IVP does not depend continuously on τ^N. Continuity, and for some numerical minimization methods, even differentiability of the objective function is required to guarantee convergence of a minimization algorithm. One can avoid discretization of the value of τ^N by replacing $y_{h,N}(t - [\tau^N([t]_h)]_h)$ in (7) by $y_{h,N}(t - \tau^N([t]_h))$ or $y_{h,N}(t - \tau^N(t))$. It is easy to modify the proof of Theorem 1 for the corresponding schemes, and show uniform convergence of the respective solutions. But these schemes are not useful in practice since the solution, in general, is not piecewise linear. Therefore, evaluating the solution between mesh points (which is needed in the scheme) is difficult. We can get a numerically simpler approximating scheme by modifying (7) as follows. Consider

$$
\begin{aligned}
& \frac{d}{dt}\Big(y_{h,N}(t) + q^N([t]_h)\tilde{\vartheta}_h y_{h,N}(t - \tau^N(t))\Big) \\
& \qquad = f\Big([t]_h, y_{h,N}([t]_h), y_{h,N}([t]_h - [\sigma([t]_h)]_h)\Big) \qquad (28)
\end{aligned}
$$

for $t \in [0, T]$ with the associated initial condition (8), where $\tilde{\vartheta}_h y_{h,N}(t - \tau^N(t))$ denotes the linear interpolate of the composite function $y_{h,N}(\cdot - \tau^N(\cdot))$ using mesh points $-r, -[r]_h, \ldots, 0, h, 2h, \ldots$. We denote the left-sided limit of $y_{h,N}(t)$ at t by $\widehat{y}_{h,N}(t)$. As a result of the linearization of $y_{h,N}(t - \tau^N(t))$, the solution of this IVP is the unique piecewise continuous function satisfying

$$
\begin{aligned}
y_{h,N}(t) &= a(k)\frac{t - kh}{h} + b(k+1)\frac{(k+1)h - t}{h}, \quad t \in [kh, (k+1)h), \qquad (29) \\
y_{h,N}(t) &= \varphi^N(t), \qquad t \in [-r, 0], \qquad (30) \\
a(k+1) &= a(k) + q^N(kh) y_{h,N}(kh - \tau^N(kh)) \\
& \quad - q^N((k+1)h) y_{h,N}\Big((k+1)h - \tau^N((k+1)h)\Big) \qquad (31) \\
& \quad + hf\left(kh, y_{h,N}(kh), y_{h,N}\left(kh - [\sigma(kh)]_h\right)\right), \quad k = 0, 1, \ldots, \\
b(k+1) &= a(k+1) - q^N(kh)\widehat{y}_{h,N}\big((k+1)h - \tau^N((k+1)h)\big) \qquad (32) \\
& \quad + q^N((k+1)h) y_{h,N}\big((k+1)h - \tau^N((k+1)h)\big), \ k = 0, 1, \ldots
\end{aligned}
$$

The next theorem shows that this scheme preserves the uniform convergence property of (7)-(8) and (7)-(26).

THEOREM 3. *Assume (H1)–(H4) and let* $|q^N - q|_C \to 0$, $|\tau^N - \tau|_C \to 0$ *and* $|\varphi^N \to \varphi|_C \to 0$ *as* $N \to \infty$. *Then the solution,* $y_{h,N}$, *of IVP (28)-(8) converges uniformly on* $[0,T]$ *to the solution,* x, *of IVP (4)-(5) as* $h \to 0^+$ *and* $N \to \infty$, *i.e.,* $\lim\limits_{\substack{h\to 0^+\\ N\to\infty}} \max\limits_{0\le t\le T} |x(t) - y_{h,N}(t)| = 0$.

PROOF. Similar to (15), one obtains the estimate

$$
\begin{aligned}
&|x(t) - y_{h,N}(t)| \\
&\quad \le \quad |\varphi(0) - \varphi^N(0)| + |q^N(0)||\varphi(-\tau(0)) - \varphi^N(-\tau^N(0))| \\
&\qquad\quad + |q(0) - q^N(0)||\varphi(-\tau(0))| + |q(t) - q^N([t]_h)||x(t-\tau(t))| \\
&\qquad\quad + |q^N([t]_h)|x(t-\tau(t)) - \tilde\vartheta_h y_{h,N}(t-\tau^N(t))| \\
&\qquad\quad + L\int_0^t \Big(|s - [s]_h| + |x(s) - y_{h,N}([s]_h)|\Big)\,ds \\
&\qquad\quad + L\int_0^t |x(s-\sigma(s)) - y_{h,N}([s]_h - [\sigma([s]_h)]_h)|\,ds.
\end{aligned}
$$

The inequality

$$
|\varphi(-\tau(0)) - \varphi^N(-\tau^N(0))| \le \omega_x\Big(|\tau - \tau^N|_C\Big) + |\varphi - \varphi^N|_C
$$

replaces (18) in this proof. Define $z_{h,N}$ as in the proof of Theorem 1. We can modify inequality (19) for this case, using an estimate similar to (27), as follows:

$$
\begin{aligned}
&|x(t-\tau(t)) - \tilde\vartheta_h y_{h,N}(t-\tau^N(t))| \\
&\qquad \le \quad 2\omega_x\Big(\omega_\tau(h) + |\tau - \tau^N|_C + h\Big) + 2z_{h,N}(t - \bar\tau/2 + h).
\end{aligned}
$$

The rest of the proof can be finished as in the proof of Theorem 1. ■

We conclude this paper by noting that the results of this paper can be generalized in a straightforward manner to NFDEs of the form

$$
\frac{d}{dt}\Big(x(t) + \sum_{i=1}^m q_i(t)x(t-\tau_i(t))\Big) = f\Big(t, x(t), x(t-\sigma_1(t)), \ldots, x(t-\sigma_p(t))\Big).
$$

(See [5] for the generalization of Lemma 1 for the several delay case.)

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REFERENCES

1 H. T. Banks, J. A. Burns and E. M. Cliff, Parameter estimation and identification for systems with delays, *SIAM J. Control and Optimization*, 19(6):791–828 (1981).

2 H. T. Banks and K. Kunisch, *Estimation Techniques for Distributed Parameter Systems*, Birkhäuser, 1989.

3 H. T. Banks and P. K. Daniel Lamm, Estimation of delays and other parameters in nonlinear functional differential equations, *SIAM J. Control and Optimization*, 21(6):895–915 (1983).

4 J. A. Burns and P. D. Hirsch, A difference equation approach to parameter estimation for differential-delay equations, *Applied Mathematics and Computation*, 7:281–311 (1980).

5 I. Győri, On approximation of the solutions of delay differential equations by using piecewise constant arguments, *Internat. J. of Math. & Math. Sci.*, 14(1):111–126 (1991).

6 I. Győri, F. Hartung and J. Turi, On numerical approximations for a class of differential equations with time- and state-dependent delays, *Appl. Math. Letters*, 8(6):19–24 (1995).

7 J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Spingler-Verlag, New York, 1993.

8 F. Hartung, On classes of functional differential equations with state-dependent delays, Ph.D. Dissertation, University of Texas at Dallas, 1995.

9 F. Hartung, T. L. Herdman and J. Turi, Identifications of parameters in hereditary systems, Proceedings of ASME Fifteenth Biennial Conference on Mechanical Vibration and Noise, Boston, Massachusetts, September 1995, DE-Vol. 84-3, 1995 Design Engineering Technical Conferences, Vol 3 - Part C, ASME 1995, pp. 1061–1066.

10 F. Hartung, T. L. Herdman and J. Turi, Identifications of parameters in hereditary systems: a numerical study, Proceedings of the 3rd IEEE Mediterranean Symposium on New Directions in Control and Automation, Cyprus, July 1995, pp. 291–298.

11 P. D. Hirsch, Parameter estimation in differential delay models, M.S. Thesis, Virginia Tech, 1980.

12 Z. Jackiewicz, Fully implicit one-step methods for neutral functional-differential equations, in the Proceedings of the 27th Conference on Decision and Control, Austin, Texas, December 1988.

13 K. A. Murphy, Estimation of time- and state-dependent delays and other parameters in functional differential equations, *SIAM J. Appl. Math.*, 50(4):972–1000 (1990).

14 M. H. Schultz, *Spline Analysis*, Prentice-Hall, 1973.