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RESEARCH ARTICLE

Explicit Asymptotic Limits for a Class of Discrete Volterra Equations

István Győri[a] and David W. Reynolds[b] *

[a] *Department of Mathematics, University of Pannonia, H-8200 Veszprém, Egyetem u. 10, Hungary.*; [b]*School of Mathematical Sciences, Dublin City University, Dublin 9, Ireland.*

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This paper presents explicit asymptotic limits for some solutions of a special kind of Volterra difference equations, and exhibits a class of equations having asymptotic equilibrium.

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1. Introduction

This paper considers the asymptotic limits of solutions of

$$z(n+1;\xi) = g(n) + \xi + \sum_{j=0}^{n} H(n,j)z(j;\xi), \quad n \in \mathbb{Z}^+; \qquad z(0;\xi) = \xi, \tag{1}$$

where g is a fixed function for which $\lim_{n\to\infty} g(n)$ exists. The solution of (1) can be expressed as $z(n;\xi) = w(n) + y(n;\xi)$ where w solves $w(n+1) = g(n) + \sum_{j=0}^{n} H(n,j)w(j)$ for $n \in \mathbb{Z}^+$ and $w(0) = 0$. It is assumed that $\lim_{n\to\infty} w(n)$ exists (sufficient conditions on H which would ensure this can be found for example in [4, 6, 7]). Then investigation of $\lim_{n\to\infty} z(n;\xi)$ reduces to studying the asymptotic limit of $y(\cdot;\xi)$, the solution of the homogeneous problem

$$y(n+1;\xi) = \xi + \sum_{j=0}^{n} H(n,j)y(j;\xi), \quad n \in \mathbb{Z}^+; \qquad y(0;\xi) = \xi. \tag{2}$$

Equation (2) is said to have *asymptotic equilibrium* if $\lim_{n\to\infty} y(n;\xi)$ exists for every vector ξ, and $\lim_{n\to\infty} y(n;\xi) \neq 0$ whenever $\xi \neq 0$. Clearly the equation has asymptotic equilibrium if and only if $M := \lim_{n\to\infty} Y(n)$ exists and M is invertible, where $Y(n+1) = I + \sum_{j=0}^{n} H(n,j)Y(j)$ for $n \in \mathbb{Z}^+$ and $Y(0) = I$.

In this note, an explicit formula for $\lim_{n\to\infty} y(n;\xi)$ is derived if the kernel has the form $H(n,j) = \sum_{k=1}^{\infty} h_k(n,j)Q^k$, and the initial condition satisfies $Q^p\xi \neq 0$ but

*Corresponding author. Email: david.reynolds@dcu.ie

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$Q^{p+1}\xi = 0$. In particular if Q is nilpotent, the limiting matrix M can be explicitly determined and shown to be invertible, so that (2) has asymptotic equilibrium.

One motivation for this work was to explore the connection between the approaches used in [1, 2] and [5], for studying asymptotically periodic solutions of discrete Volterra equations.

2. Main Result

Let $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ denote the set of non-negative integers, and $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$. If $a = (a_i)$ is a vector in $\mathbb{K}^d$, $\|a\| = \sqrt{\sum_{i=0}^{d} |a_i|^2}$. $\mathbb{K}^{d\times d}$ is the space of all $d \times d$ matrices with entries in $\mathbb{K}$, and the zero and identity matrices are denoted by 0 and I respectively. If $A = (A_{ij}) \in \mathbb{K}^{d\times d}$, $\|A\| = \sqrt{\sum_{i=0}^{d}\sum_{j=0}^{d} |A_{ij}|^2}$.

Also $\ell^\infty(\mathbb{Z}^+; \mathbb{K}^d)$ denotes the set of all bounded $f : \mathbb{Z}^+ \to \mathbb{K}^d$, and $\ell_c^\infty(\mathbb{Z}^+; \mathbb{K}^d)$ the set of all $f \in \ell^\infty(\mathbb{Z}^+; \mathbb{K}^d)$ for which $\lim_{n\to\infty} f(n) =: f(\infty)$ exists. We employ some terminology for Volterra kernels which is similar to that used in [3, Ch. 9].

Definition 2.1. A mapping $H : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{K}^{d\times d}$ is a *Volterra kernel* H if $H(n, j) = 0$ for all $j > n \geq 0$. H is called a Volterra kernel of *type* ℓ_c^∞ if in addition

(i) $\sup_{n\geq 0} \sum_{j=0}^{n} \|H(n, j)\|$ is finite;
(ii) $L_H := \lim_{n\to\infty} \sum_{j=0}^{n} H(n, j)$ exists;
(iii) $\lim_{n\to\infty} H(n, j) = H_\infty(j)$ exists for all $j \in \mathbb{Z}^+$.

It is convenient to use the notation $(H \star f)(n) := \sum_{j=0}^{n} H(n, j)f(j)$ for $n \geq 0$. It is shown in [7, Thm 3.7] that if H is of type ℓ_c^∞ and $f \in \ell_c^\infty(\mathbb{Z}^+; \mathbb{K}^d)$, then

$$\lim_{n\to\infty} (H \star f)(n) = L_H f(\infty) + \sum_{j=0}^{\infty} H_\infty(j)[f(j) - f(\infty)]. \tag{3}$$

THEOREM 2.2. *Suppose that*

(a) $\xi \in \mathbb{K}^d$, $Q \in \mathbb{K}^{d\times d}$, $Q^p\xi \neq 0$ *and* $Q^{p+1}\xi = 0$ *for some integer* $p \geq 1$,
(b) $\{h_k\}$ *is a sequence of Volterra kernels* $h_k : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{K}$,
(c) for each $(n, j) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, $H(n, j) = \sum_{k=1}^{\infty} h_k(n, j)Q^k$ *converges absolutely,*
(d) $\lim_{n\to\infty} \sum_{j=0}^{n} h_1(n, j)$ *exists,*
(e) if $p \geq 2$, h_k *is of type* ℓ_c^∞ *for all* $2 \leq k \leq p$.

Then the solution of (2) *has the form* $y(n; \xi) = \xi + \sum_{i=1}^{p} \eta_i(n)Q^i\xi$ *for all* $n \geq 0$, *where* η_i *can be calculated recursively and* $\lim_{n\to\infty} \eta_i(n) =: \eta_i(\infty)$ *exists, for each* $i \in \{1, \ldots, p\}$.

Proof. It is convenient to set $h_0(n, j) = 1$ if $j = 0$, and $h_0(n, j) = 0$ if $1 \leq j \leq n$. Substitution of this and (c) into (2) leads to $y(n + 1; \xi) = \sum_{j=0}^{n}\sum_{k=0}^{\infty} h_k(n, j)Q^k y(j; \xi)$ for all $n \geq 0$. It is easily shown by induction that the solution can be expressed in the required form: indeed if $y(j) = \sum_{s=0}^{p} \eta_s(j)Q^s\xi$ with $\eta_0(j) = 1$ for all $0 \leq j \leq n$,

$$y(n + 1; \xi) = \sum_{j=0}^{n}\sum_{k=0}^{\infty}\sum_{s=0}^{p} h_k(n, j)\eta_s(j)Q^{k+s}\xi.$$

Because of the absolute convergence of $\sum_{k=1}^{\infty} h_k(n,j)Q^k$,

$$\sum_{s=0}^{p}\sum_{k=0}^{\infty}|h_k(n,j)||\eta_s(j)|\|Q^{k+s}\xi\| \le \sum_{s=0}^{p}\Big(\sum_{k=0}^{\infty}|h_k(n,j)|\|Q^k\|\Big)|\eta_s(j)|\|Q^s\xi\|$$

is finite. Thus we can interchange the order of summation in

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{s=0}^{p} h_k(n,j)\eta_s(j)Q^{k+s}\xi &= \sum_{s=0}^{p}\sum_{k=0}^{\infty} h_k(n,j)\eta_s(j)Q^{k+s}\xi \\ &= \sum_{s=0}^{p}\sum_{k=0}^{p-s} h_k(n,j)\eta_s(j)Q^{k+s}\xi = \sum_{s=0}^{p}\sum_{r=s}^{p} h_{r-s}(n,j)\eta_s(j)Q^r\xi.\end{aligned}$$

Because the last summation is over a finite range, we see that

$$y(n+1;\xi) = \sum_{r=0}^{p}\Big(\sum_{s=0}^{r}\sum_{j=0}^{n} h_{r-s}(n,j)\eta_s(j)\Big)Q^r\xi.$$

Since $\{\xi, Q\xi, \dots, Q^p\xi\}$ is a linearly independent subset of $\mathbb{K}^d$, we deduce that $y(n+1) = \sum_{r=0}^{p}\eta_r(n+1)Q^r\xi$, with $\eta_r(n+1) = \sum_{s=0}^{r}\sum_{j=0}^{n} h_{r-s}(n,j)\eta_s(j)$. Thus $\eta_0(n+1) = \sum_{j=0}^{n} h_0(n,j)\eta_0(j) = 1$, as it should be. For $r \ge 1$, we obtain

$$\eta_r(n+1) = h_0(n,0)\eta_r(0) + \sum_{s=0}^{r-1}\sum_{j=0}^{n} h_{r-s}(n,j)\eta_s(j),$$

so that $\eta_1(n+1) = 1 + \sum_{j=0}^{n} h_1(n,j)$ and

$$\eta_r(n+1) = 1 + \sum_{j=0}^{n} h_r(n,j) + \sum_{s=1}^{r-1}(h_{r-s}\star\eta_s)(n), \quad r \ge 2.$$

Thus once the limits $\eta_1(\infty), \dots, \eta_{r-1}(\infty)$ have been determined, we can deduce an expression for $\eta_r(\infty)$ using (3). □

Corollary 2.3. *Suppose that hypotheses* (b), (c), (d), *and* (e) *in Theorem 2.2 are true, and*

(a′) $\xi \in \mathbb{K}^d$, $Q \in \mathbb{K}^{d\times d}$, $Q^p \ne 0$ *and* $Q^{p+1} = 0$ *for some integer* $p \ge 1$.

Then the solution of (2) *has the form* $y(n;\xi) = \xi + \sum_{i=1}^{p}\eta_i(n)Q^i\xi$ *for all* $n \ge 0$, *where* η_i *can be calculated recursively and* $\lim_{n\to\infty}\eta_i(n) =: \eta_i(\infty)$ *exists, for each* $i \in \{1,\dots,p\}$. *Also* M *defined by* $M\xi := \lim_{n\to\infty} y(n;\xi)$, *exists, is invertible and given by* $M = I + \sum_{i=1}^{p}\eta_i(\infty)Q^i$.

Corollary 2.4. *Suppose that hypotheses* (b), (c), *and* (d) *in Theorem 2.2 are true, and*

(a″) $\xi \in \mathbb{K}^d$, $Q \in \mathbb{K}^{d\times d}$, $Q \ne 0$ *and* $Q^2 = 0$ *for some integer* $p \ge 1$.

Then the solution of (2) *has the form* $y(n;\xi) = \xi + \sum_{j=0}^{n-1} h(n-1,j)Q\xi$ *for all* $n \ge 0$, $\lim_{n\to\infty} y(n;\xi) = M\xi$, *where* $M := I + \lim_{n\to\infty}\sum_{j=0}^{n} h(n,j)Q$ *is invertible.*

3. Examples

In this section we examine some examples of the form

$$z(n+1;\xi) = \xi + \beta(n)e + \sum_{j=0}^{n} h(n,j)Qz(j;\xi), \quad n \in \mathbb{Z}^+; \qquad z(0;\xi) = \xi, \tag{4}$$

It is supposed that $Qe = 0$, in which case the solution has the form $z(n;\xi) = \beta(n)e + y(n;\xi)$ for $n \geq 1$, where $y(\cdot;\xi)$ is the solution of the equation

$$y(n+1;\xi) = \xi + \sum_{j=0}^{n} h(n,j)Qy(j;\xi), \quad n \in \mathbb{Z}^+; \qquad y(0;\xi) = \xi, \tag{5}$$

which is a special case of (2). We also assume that β is in $\ell_c^\infty(\mathbb{Z}^+;\mathbb{K})$, and that h is a Volterra kernel for which $\lim_{n\to\infty}\sum_{j=0}^{n} h(n,j)$ exists.

The particular examples in this section have been constructed by applying [5, Lemma 3] to examples of discrete Volterra equations discussed in [1, 2].

3.1 Examples with $Q \neq 0$ and $Q^2 = 0$

Assume in addition that $Q \neq 0$, $Q^2 = 0$. Then Corollary 2.4 provides the exact solution $z(n;\xi) = \beta(n)e + \xi + \sum_{j=0}^{n-1} h(n-1,j)Q\xi$ for $n \geq 1$ and the asymptotic limit $\lim_{n\to\infty} z(n;\xi) = \beta(\infty)e + M\xi$, where $M = I + \lim_{n\to\infty}\sum_{j=0}^{n} h(n,j)Q$ is invertible.

After transforming the example in [1], one obtains (4) with

$$h(n,j) = \frac{1}{7}\frac{(-1)^{j-1}}{2^{j-1}}\left\{1 - \left(\frac{-1}{6}\right)^{n-j+1}\right\}, \qquad \beta(n) = -\frac{3}{7}\left[1 - \left(\frac{-1}{6}\right)^{n}\right], \tag{6}$$

$$Q = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{7}$$

Here $\beta(\infty) = -3/7$ and $\lim_{n\to\infty}\sum_{j=0}^{n} h(n,j) = -4/21$, so that

$$M = \frac{1}{21}\begin{pmatrix} 17 & -4 \\ 4 & 25 \end{pmatrix}.$$

The particular solution considered in [1] corresponds to $\xi = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$ and $z(\infty;\xi) = M\xi - \frac{3}{7}e = \begin{pmatrix} 1 & 2 \end{pmatrix}^T$.

A similar example can be derived from [2, Example 1]). Here e and Q remain as in (7), but

$$h(n,j) = \frac{1}{14}\frac{(-1)^{j-1}}{2^{j-1}}\left\{1 - \left(\frac{-1}{6}\right)^{n-j+1}\right\}, \qquad \beta(n) = -\frac{3}{14}\left[1 - \left(\frac{-1}{6}\right)^{n}\right],$$

which are just half the expressions in (6). Clearly $\beta(\infty) = -3/14$ and $\lim_{n\to\infty}\sum_{j=0}^{n} h(n,j) = -2/21$. Hence

$$M = \frac{1}{21}\begin{pmatrix} 19 & -2 \\ 21 & 23 \end{pmatrix}.$$

The particular solution considered in [2, Example 1] corresponds to $\xi = \begin{pmatrix}3/2 \; 3/2\end{pmatrix}^T$ and $z(\infty;\xi)) = M\xi - \frac{3}{14}e = \begin{pmatrix}1 \; 2\end{pmatrix}^T$.

3.2 Example with $Q^2 = Q^3 \neq 0$

It is now assumed that $Q \neq 0$ and $Q^2 = Q^3 \neq 0$. Then Theorem 2.2 says that $z(n;\xi) \to \beta(\infty)e + \xi + \lim_{n\to\infty} \sum_{j=0}^{n} h(n,j)Q\xi$ as $n \to \infty$, provided $Q^2\xi = 0$.

The transformation of [2, Example 3] yields the problem of h and β being as in (6), and

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad e = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

so that $Qe = 0$ and

$$Q^2 = Q^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

We note that $\lim_{n\to\infty} z(n;\xi) = -3/7e + M_0\xi$ provided $Q^2\xi = 0$ (which is equivalent to $\xi_1 = \xi_3$), with

$$M_0 = \frac{1}{21}\begin{pmatrix} 17 & -4 & 0 \\ 4 & 25 & 0 \\ 0 & -4 & 17 \end{pmatrix}$$

The particular solution considered in [2, Example 3] corresponds to $\xi = \begin{pmatrix}2 \; 1 \; 2\end{pmatrix}^T$ and $z(\infty;\xi) = M_0\xi - \frac{3}{7}e = \begin{pmatrix}1 \; 2 \; 1\end{pmatrix}^T$; also $Q^2\xi = 0$.

What happens for initial values obeying $Q^2\xi \neq 0$? Since its minimal polynomial is cubic, $\{I, Q, Q^2\}$ is a linearly independent set in $\mathbb{R}^{3\times 3}$. We follow the method used to prove Theorem 2.2, and look for a solution $Y(n) = \eta_0(n)I + \eta_1(n)Q + \eta_2(n)Q^2$ for $n \in \mathbb{Z}^+$ of the matrix equation $Y(n+1) = I + \sum_{j=0}^{n} h(n,j)Q$ with $Y(0) = I$. By substituting the solution into this,

$$Y(n+1) = I + \sum_{j=0}^{n} h(n,j)\eta_0(j)Q + \sum_{j=0}^{n} h(n,j)[\eta_1(j) + \eta_2(j)]Q^2,$$

and the initial conditions $\eta_0(0) = 1$, $\eta_1(0) = 0$, $\eta_2(0) = 0$. Hence $\eta_0(n) \equiv 1$, $\eta_1(n+1) = \sum_{j=0}^{n} h(n,j)$ with $\eta_1(0) = 0$, and

$$\eta_2(n+1) = \sum_{j=0}^{n} h(n,j)\eta_1(j) + \sum_{j=0}^{n} h(n,j)\eta_2(j), \qquad \eta_2(0) = 0. \tag{8}$$

Though η_1 is explicitly known, this is not true of either η_2 or

$$Y(n) = \begin{pmatrix} 1+\eta_1(n) & \eta_1(n) & 0 \\ -\eta_1(n) & 1-\eta_1(n) & 0 \\ -\eta_2(n) & \eta_1(n) & 1+\eta_1(n)+\eta_2(n) \end{pmatrix}.$$

It is easily checked that h is of type ℓ_c^∞ and

$$\limsup_{n\to\infty}\sum_{j=m}^{n}|h(n,j)|\le\frac{7}{6}\limsup_{n\to\infty}\sum_{j=m+1}^{n+1}\frac{1}{2^j}=\frac{7}{6}\frac{\left(\frac{1}{2}\right)^{m+1}}{1-\frac{1}{2}}\xrightarrow{m\to\infty}0.$$

Hence it can be concluded from [4, Thm 3.1] that $\lim_{n\to\infty}\eta_2(n)=:\eta_2(\infty)$ exists. Though we are not able to compute its value, we infer that $M=\lim_{n\to\infty}Y(n)$ exists with

$$M=M_0+\eta_2(\infty)Q^2=\frac{1}{21}\begin{pmatrix}17 & -4 & 0\\ 4 & 25 & 0\\ -21\eta_2(\infty) & -4 & 21\eta_2(\infty)+17\end{pmatrix}.$$

from which we deduce that $\det M=\frac{17}{21}+\eta_2(\infty)$. Hence M would be invertible if $\eta_2(\infty)\neq-\frac{17}{21}$.

In order to ascertain whether M is invertible or not, it may be easier to consider $p(n)=1+\eta_1(n)+\eta_2(n)$ for $n\ge 0$: p indeed satisfies the initial-value problem $p(n+1)=1+\sum_{j=0}^{n}h(n,j)p(j)$ and $p(0)=1$. Again [4, Thm 3.1] implies that $p(\infty):=\lim_{n\to\infty}p(n)$ exists. M is invertible if and only if $p(\infty)\neq 0$.

4. Conclusions

Motivated by the study of asymptotic periodic solutions, this note attempts to investigate systematically a special class of discrete Volterra equations which have asymptotic equilibrium. To our knowledge, there is no such systematic investigation for a wider class of discrete Volterra equations. It seems to be a challenging project to study the asymptotic equlibria of a general discrete Volterra system and to utilize those results to analyse the existence of asymptotic periodic solutions.

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